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SUMS OF MULTINOMIAL COEFFICIENTS

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ABSTRACT. $\sum q!/(h_1! \dots h_n!)$ with $h_1 + \dots + h_n = q$, the first a h_j's odd and the rest even, is expressed in terms of values of Krawtchouk polynomials.

Let $n > 0$, $q > 0$ and $a \geqq 0$ be integers. Our aim is to give a formula for the sum of multinomial coefficients

$$C(n, q, a) = \sum \binom{q}{h_1 \dots h_n}$$

where the summation is over the nonnegative integers $h_1, \dots, h_n$ satisfying

(i) $h_1 + \dots + h_n = q$;

(ii) $h_1, \dots, h_a$ are odd;

and

(iii) $h_{a+1}, \dots, h_n$ are even.

Apart from theoretical applications, the formula is useful if n is given and it is required to compute $C(n, q, a)$ for several values of q and a.

Preliminaries. The following facts can be found in [3], Chapter 5, Section 7 of [2] and [1].

Let $\mathscr{S}_b^{(n)}$ be the elementary symmetric polynomial of degree b in n indeterminates: $\mathscr{S}_0^{(n)} = 1$ and for $1 \leqq b \leqq n$

$$\mathscr{S}_b^{(n)}(x_1, \dots, x_n) = \sum\nolimits_{1 \leqq j_1 < j_2 < \dots < j_b \leqq n} x_{j_1} x_{j_2} \dots x_{j_b}.$$

For $0 \leqq a \leqq n$, put

$$\mathscr{S}_{ab}^{(n)} = \mathscr{S}_b^{(n)}(-1, \dots, -1, 1, \dots, 1), \tag{1}$$

where the number of -1's is a. Then $\mathscr{S}_{ab}^{(n)} = K_b(a; n)$, where $K_b(x; n)$ is the *Krawtchouk polynomial* defined by

$$K_b(x; n) = \sum_{c=0}^{n} (-1)^c \binom{x}{c} \binom{n-x}{b-c}.$$

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For a given n, the matrix $\mathscr{S}^{(n)} = (\mathscr{S}^{(n)}_{ab})^n_{a,b=0}$ is easily computable recursively from

$$\mathscr{S}^{(n)}_{a0} = 1; \ \mathscr{S}^{(n)}_{0b} = \binom{n}{b}$$

and

$$\mathscr{S}^{(n)}_{ab} = \mathscr{S}^{(n)}_{a-1,b} - \mathscr{S}^{(n)}_{a-1,b-1} - \mathscr{S}^{(n)}_{a,b-1}, \qquad a, b = 1, \ldots, n.$$

This matrix satisfies

$$(2) \qquad (\mathscr{S}^{(n)})^2 = 2^n I_{n+1},$$

where I_{n+1} is the identity matrix of order $n + 1$.

For an integer h denote by $r(h)$ the remainder, 0 or 1, of h modulo 2. Let

$$f = \sum c_{h_1 \ldots h_n} x_1^{h_1} \ldots x_n^{h_n}$$

be a polynomial in the indeterminates $x_1, \ldots, x_n$ over a field of characteristic different from 2. It was shown in [1] that the *reduced polynomial*

$$Rf = \sum c_{h_1 \ldots h_n} x_1^{r(h_1)} \ldots x_n^{r(h_n)}$$

is the unique polynomial of degree not exceeding 1 in each indeterminate which coincides with f on $\{-1, 1\}^n$. The formula for Rf which is given in [1] is not required here, as the uniqueness statement suffices.

Reduction of symmetric polynomials.

LEMMA. Let f *as above be a symmetric polynomial. Then*

$$(3) \quad Rf = 2^{-n} \sum_{a=0}^{n} \left\{ \sum_{b=0}^{n} \mathscr{S}^{(n)}_{ab} f(-1, \ldots, -1, 1, \ldots, 1) \right\} \mathscr{S}^{(n)}_a(x_1 \ldots, x_n),$$

where the number of -1*'s in the* b'th *summand of the inner sum is* b.

PROOF. By the uniqueness property of Rf, it suffices to show that the right hand side of (3) coincides with f on $\{-1, 1\}^n$. Since both are symmetric, it is enough to verify this on the vectors $(-1, \ldots, -1, 1, \ldots, 1)$ of length n, where the number of -1's is c, $0 \leqq c \leqq n$. Substituting such a vector in the right hand side of (3), we get using (1)

$$2^{-n} \sum_{a=0}^{n} \left\{ \sum_{b=0}^{n} \mathscr{S}^{(n)}_{ab} f(-1, \ldots, -1, 1, \ldots, 1) \right\} \mathscr{S}^{(n)}_{ca}$$

$$= \sum_{b=0}^{n} \left\{ f(-1, \ldots, -1, 1, \ldots, 1) 2^{-n} \sum_{a=0}^{n} \mathscr{S}^{(n)}_{ca} \mathscr{S}^{(n)}_{ab} \right\}$$

$$= \sum_{b=0}^{n} f(-1, \ldots, -1, 1, \ldots, 1) \delta_{cb} \qquad \text{(by (2); using Kronecker's } \delta)$$

$$= f(-1, \ldots, -1, 1, \ldots, 1),$$

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where the number of -1's is c.

FORMULA. $C(n, q, a) = 2^{-n} \sum_{b=0}^{n} \mathcal{S}_{ab}^{(n)}(n - 2b)^q$.

PROOF. For a vector of integers $\mathbf{h} = (h_1, \ldots, h_n)$, put $r(\mathbf{h}) = (r(h_1), \ldots, r(h_n))$. The latter belongs to the set J_n of (0, 1) $-$ vectors of length n. The weight, $wt(\mathbf{j})$ of a vector $\mathbf{j} = (j_1, \ldots, j_n)$ of J_n is the number of nonzero coordinates of $\mathbf{j}$.

Let $f = (x_1 + \ldots + x_n)^q$. By the multinomial theorem

$$f = \sum_{h_1+\ldots+h_n=q} \binom{q}{h_1 \ldots h_n} x_1^{h_1} \ldots x_n^{h_n}.$$

Reducing, we obtain

$$(4) \qquad Rf = \sum_{a=0}^{n} \sum_{\{\mathbf{j}\in J_n: wt(\mathbf{j})=a\}} \left\{ \sum_{\Sigma h_i = q, r(\mathbf{h})=\mathbf{j}} \binom{q}{h_1 \ldots h_n} \right\} x_1^{j_1} \ldots x_n^{j_n}.$$

Since $\binom{q}{h_1 \ldots h_n}$ is symmetric in $h_1, \ldots, h_n$, the innermost sum of (4) depends only on n, q and $wt(\mathbf{j})$. If $wt(\mathbf{j}) = a$, then this sum equals $C(n, q, a)$. Therefore

$$(5) \qquad Rf = \sum_{a=0}^{n} C(n, q, a) \sum_{wt(\mathbf{j})=a} x_1^{j_1} \ldots x_n^{j_n} = \sum_{a=0}^{n} C(n, q, a) \mathcal{S}_a^{(n)}.$$

On the other hand, $f(-1, \ldots, -1, 1, \ldots, 1)$, where the number of -1's is b, equals $(-b + n - b)^q = (n - 2b)^q$. Therefore, by the lemma,

$$(6) \qquad Rf = 2^{-n} \sum_{a=0}^{n} \left\{ \sum_{b=0}^{n} \mathcal{S}_{ab}^{(n)}(n - 2b)^q \right\} \mathcal{S}_a^{(n)}.$$

Since the $\mathcal{S}_a^{(n)}$ are linearly independent, comparison of coefficients in (5) and (6) yields the desired formula.

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