ANOTHER LAW FOR THE 3-METABELIAN GROUPS

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The 3-metabelian groups are those groups in which every subgroup generated by three elements is metabelian. In [2] it was stated and in [3] it was proved that the variety of such groups may be defined by the one law

$$(1) \qquad [x, y; x, z] = 1;$$

(by $[a, b; c, d]$ we mean $[[a, b], [c, d]]$, and for other definitions and notation we refer to [1]). Recently Bachmuth and Lewin obtained in [1] the surprising and remarkable result that the same variety is defined by the law

$$(2) \qquad [x, y, z][y, z, x][z, x, y] = 1.$$

Now (2) is reminiscent of the relation

$$(3) \qquad [x, y, z^x][y, z, x^y][z, x, y^z] = 1$$

which holds in all groups and which is apparently due to Philip Hall. Using the identities $[x, y, [y, z]^x] = [y, z, x^y]^{-1}[y, z, x]$, etc., we find that (2) is equivalent to

$$(4) \qquad [x, y, [y, z]^x]^u [z, x, [x, y]^z]^v [y, z, [z, x]^y] = 1$$

where $u = [z, x, y]$ and $v = [y, z, x][z, x, y]$. Note that apart from certain displeasing conjugates (4) is curiously similar to both (1) and (2).

The question which naturally arises at this point is answered by

THEOREM. *The variety defined by the law*

$$(5) \qquad [x, y; y, z][y, z; z, x][z, x; x, y] = 1$$

is the variety of 3-*metabelian groups*.

PROOF. We calculate. In preparation to replacing y by xy in (5) we note that

$$[x, xy; xy, z] = [x, y; y, z][x, y, [x, z]^y]^{[y,z]},$$
$$[xy, z; z, x] = [[x, z]^y, [z, x]]^{[y,z]}[y, z; z, x].$$

Then replacement and another use of (5) for cancellation gives

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(6) $$[x, y, [x, z]^y][[x, z]^y, [z, x]] = 1.$$

Next we want to replace x by zx in (6). Since

$$[zx, y, [zx, z]^y] = [[z, y]^x, [x, z]^y]^{[x, y]}[x, y, [x, z]^y],$$

we eventually obtain

(7) $$[[z, y]^x \; [x, z]^y] = 1.$$

Replace y by yx in (7) and note that

$$[[z, yx]^x, [x, z]^{yx}] = [[z, x]^x, [x, z]^{yx}]^{[z, y]^{x^2}}[[z, y]^{x^2}, [x, z]^{yx}].$$

We find that

(8) $$[z, x, [x, z]^y] = 1.$$

By (6) and (8) we have

(9) $$[x, y, [x, z]^y] = 1,$$

a law which is equivalent to (1) since $[x, z]^y = [x, y]^{-1}[x, zy]$. Since on the other hand (1) clearly implies (5) the theorem has been proved.

References

[1] Seymour Bachmuth and Jacques Lewin, The Jacobi identity in groups, *Math. Zeit.* 83 (1964), 170–176.
[2] I. D. Macdonald, On certain varieties of groups, *Math. Zeit.* 76 (1961), 270–282.
[3] I. D. Macdonald, On certain varieties of groups. II, *Math. Zeit.* 78 (1962), 175–188.

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