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GENERALIZED INVERSES OF A SUM IN RINGS

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Abstract

We study properties of the Drazin index of regular elements in a ring with a unity 1. We give expressions for generalized inverses of $1 - ba$ in terms of generalized inverses of $1 - ab$. In our development we prove that the Drazin index of $1 - ba$ is equal to the Drazin index of $1 - ab$.

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1. Introduction

Let $\mathcal{R}$ be a ring with a unity 1. An element a is said to be regular if there is an element x such that $axa = a$. If it exists, then it is called an inner inverse of a (von Neumann inverse). We will denote by $a\{1\} = \{x \in \mathcal{R} \mid axa = a\}$ the set of all inner inverses of a and we will write a^- to designate a member of $a\{1\}$. A reflexive inverse a^+ of a is an inner and outer inverse of a, that is, $a^+ \in a\{1\}$ and $a^+aa^+ = a^+$.

An element a is said to be Drazin invertible provided there is a common solution for the equations

$$xax = x, \quad ax = xa, \quad a^kxa = a^k \text{ for some } k \geq 0.$$

If a common solution exists, then it is unique and it will be denoted by a^D (see [2]). The smallest integer k for which the above equations hold is called the Drazin index of a, denoted by $\operatorname{ind}(a)$.

The Drazin index can be characterized in terms of right and left ideals generated by a power of a as follows [6]: it is the case that $\operatorname{ind}(a) = k$ if and only if k is the smallest nonnegative integer for which $a^k\mathcal{R} = a^{k+1}\mathcal{R}$ and $\mathcal{R}a^k = \mathcal{R}a^{k+1}$, or equivalently, $a^k \in a^{k+1}\mathcal{R} \cap \mathcal{R}a^{k+1}$.

If a is Drazin invertible with $\operatorname{ind}(a) = 1$, then a is regular. In the former case the Drazin inverse of a is known as the group inverse of a, denoted by $a^\sharp$.

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It is well known that the smallest k for which $(a^k)^\sharp$ exists equals $\mathrm{ind}(a) = k$, and $a^D = (a^k)^\sharp a^{k-1} = a^{k-1}(a^k)^\sharp$.

If there exists an element $a^\pi \in \mathcal{R}$ such that a^π is idempotent, $aa^\pi = a^\pi a$, aa^π is nilpotent, and $a + a^\pi$ is nonsingular, then it is called a spectral idempotent of a; such an element is unique (if it exists). We know that a is Drazin invertible if and only the spectral idempotent of a exists. In this case we have $a^D = (a + a^\pi)^{-1}(1 - a^\pi)$ and $a^\pi = 1 - aa^D$. Characterizations of ring elements with related spectral idempotents are given in [4, 7].

Let $\mathcal{R}$ be a ring with an involution $x \to x^*$ such that $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, for all $x, y \in \mathcal{R}$. We say that a is Moore–Penrose invertible if the equations

$$bab = b, \quad aba = a, \quad (ab)^* = ab, \quad (ba)^* = ba$$

have a common solution; such a solution is unique if it exists (see [2, 5]), and it will be denoted by $a^\dagger$.

We say that an element a is EP if a is Moore–Penrose invertible and $aa^\dagger = a^\dagger a$. An element a is generalized EP if there exists $k \in \mathbb{N}$ such that a^k is EP.

Barnes [1] proved that the ascents (descents) of $I - RS$ and $I - SR$ are equal for bounded operators on Banach spaces $R \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, X)$. Consequently, the Drazin indices of $I - RS$ and $I - SR$ are equal. In this paper we deal with the Drazin index of $1 - ab$ and $1 - ba$ in rings, and therefore neither functional calculi nor operator theory can be used. Moreover, we provide a formula for the reflexive inverse, the group inverse and the Drazin inverse of $1 - ba$ in terms of the corresponding generalized inverse of $1 - ab$.

In our development, we extend the following characterization of the Drazin index given by Puystjens and Hartwig [10]: given a regular element $a \in \mathcal{R}$, then $\mathrm{ind}(a) \le 1$ if and only if $\mathrm{ind}(a + 1 - aa^-) = 0$, for one and hence all choices of $a^- \in a\{1\}$.

2. Auxiliary results

In this section we give some auxiliary lemmas. We start with an elementary known result.

LEMMA 2.1. *Let $a, b \in \mathcal{R}$. Then $1 - ab$ is invertible if and only if $1 - ba$ is invertible.*

LEMMA 2.2. *Let a be a regular element. Then, given a natural number n,*

$$(a + 1 - aa^-)^n = (a^2a^- + 1 - aa^-)^n + \sum_{i=1}^{n} a^i(1 - aa^-). \tag{2.1}$$

PROOF. The proof is by induction on n. Denote

$$z = a + 1 - aa^- \quad \text{and} \quad x = a^2a^- + 1 - aa^-.$$

It is clear that $z = x + a(1 - aa^-)$. Assuming (2.1) to hold for k, we will prove it for $k + 1$.

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We note that

$$zx = x^2 + a(1 - aa^-) \quad \text{and} \quad za = a^2.$$

Now, by the induction step,

$$\begin{aligned} z^{k+1} &= z\Big(x^k + \sum_{i=1}^{k} a^i(1 - aa^-)\Big) \\ &= x^{k+1} + a(1 - aa^-) + \sum_{i=1}^{k} a^{i+1}(1 - aa^-) \\ &= x^{k+1} + \sum_{i=1}^{k+1} a^i(1 - aa^-). \end{aligned}$$

□

LEMMA 2.3. *Let $a, b \in \mathcal{R}$. Then, given a natural number n,*

$$(1 - ba)^n = 1 - bra \quad \textit{and} \quad (1 - ab)^n = 1 - rab,$$

where

$$r = \sum_{j=0}^{n-1} (1 - ab)^j.$$

PROOF. This can be easily proved by induction on n. □

In [7] the authors give the following characterization of EP elements in a ring.

LEMMA 2.4. *Let $\mathcal{R}$ be a ring with an involution $x \to x^*$. For $a \in \mathcal{R}$ the following conditions are equivalent.*

(i) *a is EP.*
(ii) *a is Drazin and Moore–Penrose invertible and $a^D = a^\dagger$.*
(iii) *a is group invertible and $a^\pi = (a^*)^\pi$.*

3. Main results

The following theorem is an answer to a question raised by Patricio and Veloso in [8] about the equivalence between the conditions $\mathrm{ind}(a^2a^- + 1 - aa^-) = k$ and $\mathrm{ind}(a + 1 - aa^-) = k$, and provides a new characterization of the Drazin index.

THEOREM 3.1. *Let a be a regular noninvertible element. The following conditions are equivalent.*

(i) $\mathrm{ind}(a) = k + 1$.
(ii) $\mathrm{ind}(a^2a^- + 1 - aa^-) = k$, *for one and hence all choices of $a^- \in a\{1\}$.*
(iii) $\mathrm{ind}(a + 1 - aa^-) = k$, *for one and hence all choices of $a^- \in a\{1\}$.*

PROOF. The equivalence of (i) and (ii) is proved in [8, Theorem 2.1]. We proceed to show that (ii) ⇒ (iii). Denote

$$x = a^2a^- + 1 - aa^- \quad \text{and} \quad z = a + 1 - aa^-.$$

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Assume that $\operatorname{ind}(x) = k$, or equivalently, $\operatorname{ind}(a) = k+1$. Then $x^k = x^{k+1}\mathcal{R}$ and $a^{k+1} = a^{k+2}w$ for some $w \in \mathcal{R}$. By (2.1),

$$\begin{aligned}
z^k\mathcal{R} &= \Big(1 + \sum_{i=1}^{k} a^i(1-aa^-)\Big)x^k\mathcal{R} \\
&= \Big(1 + \sum_{i=1}^{k} a^i(1-aa^-)\Big)x^{k+1}\mathcal{R} \\
&= \Big(z^{k+1} - \sum_{i=1}^{k+1} a^i(1-aa^-) + \sum_{i=1}^{k} a^i(1-aa^-)\Big)\mathcal{R} \\
&= (z^{k+1} - a^{k+1}(1-aa^-))\mathcal{R} = (z^{k+1} - a^{k+2}w(1-aa^-))\mathcal{R} \\
&= z^{k+1}(1 - aw(1-aa^-))\mathcal{R} \subseteq z^{k+1}\mathcal{R}.
\end{aligned}$$

This gives $z^k\mathcal{R} = z^{k+1}\mathcal{R}$. On the other hand, since $\operatorname{ind}(x) = k$ we also have $x^k = ux^{k+1}$ for some $u \in \mathcal{R}$. By (2.1),

$$\begin{aligned}
\mathcal{R}z^k &= \mathcal{R}\Big(x^k + \sum_{i=1}^{k} a^i(1-aa^-)\Big) \\
&= \mathcal{R}\Big(ux^{k+1} + \sum_{i=1}^{k} a^i(1-aa^-)\Big) \\
&= \mathcal{R}\Big(u - u\sum_{i=1}^{k+1} a^i(1-aa^-) + \sum_{i=1}^{k} a^i(1-aa^-)\Big)z^{k+1} \subseteq \mathcal{R}z^{k+1}.
\end{aligned}$$

From this we conclude that $\mathcal{R}z^k = \mathcal{R}z^{k+1}$. Consequently, $\operatorname{ind}(z) \le k$.

By symmetrical arguments, we can show that $\operatorname{ind}(z) = k$ implies that $\operatorname{ind}(x) \le k$. Further, if $\operatorname{ind}(z) < k$, having $\operatorname{ind}(x) = k$, then we would get that $\operatorname{ind}(x) \le k-1$, and we would arrive at a contradiction. Therefore $\operatorname{ind}(z) = k$. □

We can state the symmetrical form of Theorem 3.1.

COROLLARY 3.2. *Let a be a regular noninvertible element. The following conditions are equivalent.*

(i) $\operatorname{ind}(a) = k+1$.
(ii) $\operatorname{ind}(a^-a^2 + 1 - a^-a) = k$, *for one and hence all choices of* $a^- \in a\{1\}$.
(iii) $\operatorname{ind}(a + 1 - a^-a) = k$, *for one and hence all choices of* $a^- \in a\{1\}$.

The following corollary is an extension of the analogous result for the Drazin index of a complex partitioned matrix over $\mathbb{C}$ [3, Theorem 7.7.5].

COROLLARY 3.3. *Let $\mathcal{R}$ be any ring with unity. If*

$$M = \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} \in \mathcal{R}_{n\times n},$$

where $A \in \mathcal{R}_{r\times r}$ is invertible, then $\operatorname{ind}(M) = \operatorname{ind}(A + BCA^{-1}) + 1$.

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PROOF. We have that $M^- = \begin{pmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{pmatrix}$ is an inner inverse of M and

$$M + I - MM^- = \begin{pmatrix} A + BCA^{-1} & 0 \\ C - CA^{-1}(I - BCA^{-1}) & I \end{pmatrix}.$$

Using the following known result for block triangular matrices,

$$\max\{\operatorname{ind}(I), \operatorname{ind}(A + BCA^{-1})\} \le \operatorname{ind}(M + I - MM^-) \le \operatorname{ind}(A + BCA^{-1}) + \operatorname{ind}(I),$$

we conclude that $\operatorname{ind}(M + I - MM^-) = \operatorname{ind}(A + BCA^{-1})$. Now we see that $\operatorname{ind}(M) = \operatorname{ind}(A + BCA^{-1}) + 1$ follows from Theorem 3.1. □

It is well known that $1 - ba$ is regular if and only if $1 - ab$ is regular. Moreover, if $(1 - ab)^-$ is an inner inverse of $1 - ab$ then $(1 - ba)^- = 1 + b(1 - ab)^- a$ is an inner inverse of $1 - ba$. In the following, we will extend the same reasoning to other generalized inverses, namely the reflexive, group and Drazin inverse.

THEOREM 3.4. *Let $a, b \in \mathcal{R}$. If $(1 - ab)^+$ is a reflexive inverse of $1 - ab$, then a reflexive inverse of $1 - ba$ is given by*

$$(1 - ba)^+ = 1 + b((1 - ab)^+ - pq)a,$$

where $p = 1 - (1 - ab)^+(1 - ab)$ and $q = 1 - (1 - ab)(1 - ab)^+$.

PROOF. Let $x = 1 + b((1 - ab)^+ - pq)a$. Then

$$(1 - ba)x = 1 - bqa.$$

Further,

$$(1 - ba)x(1 - ba) = 1 - ba - bqa(1 - ba)a = 1 - ba$$

and

$$\begin{aligned} x(1 - ba)x &= x - xbqa \\ &= x - bqa - b((1 - ab)^+ - pq)abqa \\ &= x, \end{aligned}$$

where we have simplified by writing $ab = 1 - (1 - ab)$ and using the relations

$$(1 - ab)(1 - ab)^+(1 - ab) = (1 - ab)$$

and

$$(1 - ab)^+(1 - ab)(1 - ab)^+ = (1 - ab)^+.$$ □

THEOREM 3.5. *Let $a, b \in \mathcal{R}$. If $1 - ab$ is group invertible, then $1 - ba$ is group invertible and*

$$(1 - ba)^\sharp = 1 + b((1 - ab)^\sharp - (1 - ab)^\pi)a,$$

where $(1 - ab)^\pi = 1 - (1 - ab)^\sharp(1 - ab)$.

PROOF. Let $x = 1 + b((1-ab)^\sharp - (1-ab)^\pi)a$. First, we note that $(1-ab)^\sharp$ is a reflexive inverse that commutes with $1-ab$. In view of the preceding theorem we have that x is a reflexive inverse of $1-ba$. Next, we will prove that x commutes with $1-ba$. We have

$$x(1-ba) = 1 - ba + b(1-ab)^\sharp(1-ab)a = 1 - b(1-ab)^\pi a$$

and, similarly, $(1-ba)x = 1 - b(1-ab)^\pi a$, which gives $x(1-ba) = (1-ba)x$. Therefore x satisfies the three equations involved in the definition of group inverse. □

THEOREM 3.6. *Let $a, b \in \mathcal{R}$. If $1-ab$ is Drazin invertible with $\operatorname{ind}(1-ab) = k$, then $1-ba$ is Drazin invertible with $\operatorname{ind}(1-ba) = k$ and*

$$(1-ba)^D = 1 + b((1-ab)^D - (1-ab)^\pi r)a,$$

where

$$r = \sum_{j=0}^{k-1}(1-ab)^j.$$

PROOF. Assume that $\operatorname{ind}(1-ab) = k \geq 2$. Then $(1-ab)^k$ is group invertible and Theorem 3.1 leads to

$$\operatorname{ind}(1-(1-(1-ab)^k)(1-ab)^k((1-ab)^k)^\sharp) = 0.$$

By Lemma 2.3,

$$1-(1-ab)^k = rab \quad \text{and} \quad 1-(1-ba)^k = bra, \tag{3.1}$$

where $r = \sum_{j=0}^{k-1}(1-ab)^j$. From the above relations, $1 - rab(1-ab)^k((1-ab)^k)^\sharp$ is invertible and by Lemma 2.1 we have that $1 - b(1-ab)(1-ab)^D ra$ is invertible. Further,

$$\begin{aligned}
&(1-b(1-ab)(1-ab)^D ra)(1-ba)^k\\
&\quad = (1-ba)^k - b(1-ab)(1-ab)^D ra(1-ba)^k\\
&\quad = (1-ba)^k - b(1-ab)^k ra\\
&\quad = (1-bra)(1-ba)^k = (1-ba)^{2k}.
\end{aligned}$$

From this it follows that

$$(1-ba)^k = (1-b(1-ab)(1-ab)^D ra)^{-1}(1-ba)^{2k} \in \mathcal{R}(1-ba)^{k+1}.$$

On the other hand,

$$\begin{aligned}
&(1-ba)^k(1-b(1-ab)(1-ab)^D ra)\\
&\quad = (1-ba)^k - (1-ba)^k b(1-ab)(1-ab)^D ra\\
&\quad = (1-ba)^k - b(1-ab)^k ra = (1-ba)^{2k}
\end{aligned}$$

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and hence

$$(1-ba)^k = (1-ba)^{2k}(1-b(1-ab)(1-ab)^D ra)^{-1} \in (1-ba)^{k+1}\mathcal{R}.$$

Therefore

$$(1-ba)^k \in \mathcal{R}(1-ba)^{k+1} \cap (1-ba)^{k+1}\mathcal{R},$$

which implies that $\operatorname{ind}(1-ba) \leq k$.

Further, analysis similar to that of the last part of the proof of Theorem 3.1 shows that $\operatorname{ind}(1-ab)=k$. Now,

$$(1-ba)^D = ((1-ba)^k)^\sharp (1-ba)^{k-1}.$$

In view of (3.1) and applying Theorem 3.5, it follows that

$$\begin{aligned}
((1-ba)^k)^\sharp = (1-bra)^\sharp &= 1 + b((1-rab)^\sharp - (1-rab)^\pi)ra \\
&= 1 + b(((1-ab)^k)^\sharp - ((1-ab)^k)^\pi)ra \\
&= 1 + b(((1-ab)^D)^k - (1-ab)^\pi)ra.
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-ba)^D &= (1 + b(((1-ab)^D)^k - (1-ab)^\pi)ra)(1-ba)^{k-1} \\
&= (1-ba)^{k-1} + b(((1-ab)^D)^k - (1-ab)^\pi)(1-ab)^{k-1}ra \\
&= 1 - br'a + b((1-ab)^D r - (1-ab)^\pi (1-ab)^{k-1})a \\
&= 1 + b((1-ab)^D - (1-ab)^\pi r' - (1-ab)^\pi (1-ab)^{k-1})a \\
&= 1 + b((1-ab)^D - (1-ab)^\pi r)a,
\end{aligned}$$

where $r' = \sum_{j=0}^{k-2}(1-ab)^j$, completing the proof. □

Let $\mathcal{R}_{n\times n}$ the ring of $n \times n$ matrices over $\mathcal{R}$. Any matrix $A \in \mathcal{R}_{r\times n}$ ($B \in \mathcal{R}_{n\times r}$) with $r < n$ may be enlarged to square $n \times n$ matrix $A'(B')$ by adding zeros. Then we can compute a generalized inverse of $I - BA = I - B'A'$ using preceding results in the ring $\mathcal{R}_{n\times n}$. Finally, we can rewrite the corresponding expression for the generalized inverse of $I - B'A'$ in terms of A and B, getting that formulas similar to that in the preceding theorems hold for rectangular matrices A and B.

EXAMPLE 3.7. We consider the following matrices with entries in the univariate polynomial ring in x over $\mathbb{Z}_8$, the ring of integers modulo 8:

$$A = \begin{pmatrix} x & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 7x \\ 2 \\ x^2+3 \end{pmatrix}.$$

Then

$$I - BA = \begin{pmatrix} x^2+1 & 2x & x \\ 6x & 5 & 6 \\ 7x^3+5x & 6x^2+2 & 7x^2+6 \end{pmatrix} \quad \text{and} \quad 1 - AB = 2.$$

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The zero-degree polynomial equal to 2 is nilpotent of index 3, and so $\operatorname{ind}(1-AB)=3$ and $(1-AB)^D=0$. Applying Theorem 3.6, we get

$$\begin{aligned}(I-BA)^D &= I+\begin{pmatrix}7x\\ 2\\ x^2+3\end{pmatrix}(0-1(1+2+2^2))\begin{pmatrix}x & 2 & 1\end{pmatrix}\\ &= \begin{pmatrix}7x^2+1 & 6x & 7x\\ 2x & 5 & 2\\ x^3+3x & 2x^2+6 & x^2+4\end{pmatrix}.\end{aligned}$$

We know that in general $1-ab$ is EP may not imply that $1-ba$ is EP. In the following result we give a necessary and sufficient condition for such an implication to hold.

COROLLARY 3.8. *Let $\mathcal{R}$ be a ring with an involution $x\to x^*$. If $1-ab$ is EP, then $1-ba$ is EP if and only if $a^*(1-ab)^\pi b^*=b(1-ab)^\pi a$. In this case,*

$$(1-ba)^\dagger = 1+b((1-ab)^\dagger-(1-(1-ab)(1-ab)^\dagger))a.$$

PROOF. Since $1-ab$ is EP, by Lemma 2.4 we have that $1-ab$ is group invertible and Moore–Penrose invertible and $(1-ab)^\sharp=(1-ab)^\dagger$. Now, from Theorem 3.5 it follows that $1-ba$ is also group invertible and

$$(1-ba)^\sharp = 1+b((1-ab)^\sharp-(1-ab)^\pi)a,$$

and consequently, $(1-ba)^\pi=b(1-ab)^\pi a$. Thus, by Lemma 2.4, $1-ba$ is EP if and only if $((1-ba)^*)^\pi=(1-ba)^\pi$, that is,

$$(b(1-ab)^\pi a)^*=b(1-ab)^\pi a.$$

Hence, using the fact that $((1-ab)^*)^\pi=(1-ab)^\pi$, the result follows. □

COROLLARY 3.9. *Let $\mathcal{R}$ be a ring with an involution $x\to x^*$. If $1-ab$ is generalized EP, then $1-ba$ is generalized EP if and only if $(ra)^*(1-ab)^\pi b^*=b(1-ab)^\pi ra$, where $r=\sum_{j=0}^{k-1}(1-ab)^j$ and $k=\operatorname{ind}(1-ab)$.*

PROOF. Since $1-ab$ is generalized EP then there exists a smallest integer $k\in\mathbb{N}$ such that $(1-ab)^k$ is EP. From Lemma 2.4 we can deduce that $\operatorname{ind}(1-ab)=k$. Now, by Lemma 2.3 we have $(1-ab)^k=1-rab$, where r is defined as in the statement of this corollary. By Corollary 3.8, $(1-ba)^k=1-bra$ is EP if and only if $(b(1-ab)^\pi ra)^*=b(1-ab)^\pi ra$, completing the proof. □

In this example we show that the existence of the Moore–Penrose inverse of $1-ab$ does not imply the existence of the Moore–Penrose inverse of $1-ba$.

EXAMPLE 3.10. Consider the following matrices over the field $\mathbb{C}$ of complex numbers, with the involution defined by $A^\star=A^T$:

$$A=\begin{pmatrix}0 & -i\\ 1 & 0\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}.$$

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Then

$$I - AB = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad I - BA = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

and, further,

$$(I - AB)^\star(I - AB) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (I - BA)^\star(I - BA) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\operatorname{rank}(I - AB) = 1$ and

$$\operatorname{rank}(I - AB)^\star(I - AB) = \operatorname{rank}(I - AB)(I - AB)^\star = 1$$

we conclude, applying [9, Theorem 1], that $I - AB$ is Moore–Penrose invertible. On the other hand, since $\operatorname{rank}(I - BA) = 1$ and $\operatorname{rank}(I - BA)^\star(I - BA) = 0$ we conclude that $I - BA$ is not Moore–Penrose invertible.

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