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Chasing Silver

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Abstract. We show that limits of CS iterations of the n-Silver forcing notion have the n-localization property.

1 Introduction

This paper is concerned with the n-localization property of the n-Silver forcing notion and countable support (CS) iterations of such forcings. The property of n-localization was introduced by Newelski and Rosłanowski [12, p. 826].

Definition 1.1 Let n be an integer greater than 1.

(i) A tree T is *an n-ary tree* provided that $(\forall s \in T)(|\mathrm{succ}_T(s)| \leq n)$.

(ii) A forcing notion $\mathbb{P}$ has the *n-localization property* if

$$\Vdash_{\mathbb{P}} \text{“ } (\forall f \in {}^{\omega}\omega)(\exists T \in \mathbf{V})(T \text{ is an } n\text{-ary tree and } f \in [T]) \text{ ”.}$$

Later the n-localization property, the σ-ideal generated by n-ary trees, and the n-Sacks forcing notion $\mathbb{D}_n$ (see Definition 2.1) were applied to problems on convexity numbers of closed subsets of $\mathbb{R}^n$, ([3–5]).

We do not yet have any result of the form “CS iteration of proper forcing notions with the n-localization property has the n-localization”. A somewhat uniform and general treatment of preserving the n-localization was recently presented in [15]. However, that treatment does not cover the n-Silver forcing notion $\mathbb{S}_n$ (see Definition 2.1). As a matter of fact, at one point it was not clear if $\mathbb{S}_n$ has the property at all. It was stated in [12, Theorem 2.3] that the same proof as for $\mathbb{D}_n$ works also for CS iterations and products of the n-Silver forcing notions $\mathbb{S}_n$ (see Definition 2.1(3)). Perhaps some old wisdom got lost, but it does not appear likely that *the same arguments work for the n-Silver forcing $\mathbb{S}_n$*. In the present paper we correct this gap and provide a full proof that CS iterations of $\mathbb{S}_n$ (and other forcings listed in Definition 2.1) have the n-localization property, see Corollary 2.6.

Our main result, Theorem 2.5, seems to be very $\mathbb{S}_n$-specific and it is not clear to what extent it may be generalized. In particular, the following general problem remains open.

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Problem 1.2 Do CS iterations of proper forcing notions with the n-localization property have the n-localization property? What if we restrict ourselves to (s)nep forcing notions (see Shelah [17]) or even Suslin$^+$ (see [6, 9])?

1.1 Notation

Our notation is rather standard and compatible with that of classical textbooks [7]. In forcing, however, we keep the older convention that *a stronger condition is the larger one*.

(i) n is our fixed integer, $n \geq 2$.

(ii) For two sequences η, ν we write $\nu \lhd \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \lhd \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.

(iii) A *tree* is a family of finite sequences closed under initial segments. For a tree T and $\eta \in T$ we define *the successors of* η *in* T and the *maximal points of* T by:

$$\mathrm{succ}_T(\eta) = \{\nu \in T : \eta \lhd \nu \text{ and } \neg(\exists \rho \in T)(\eta \lhd \rho \lhd \nu)\},$$
$$\max(T) = \{\nu \in T : \text{ there is no } \rho \in T \text{ such that } \nu \lhd \rho\}.$$

For a tree T the family of all ω-branches through T is denoted by $[T]$.

(iv) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$-names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below, *e.g.*, $\underset{\sim}{\tau}$, $\underset{\sim}{X}$.

Let us explain what is a possible problem with the n-Silver forcing; let us look at the "classical" Silver forcing $\mathbb{S}_2$. Given a Silver condition f such that $f \Vdash_{\mathbb{S}_2} \underset{\sim}{\tau} \in {}^{\omega}\omega$, standard arguments allow it to be assumed that the complement of the domain of f can be enumerated in the increasing order as $\{k_i : i < \omega\}$ and that for each $i \in \omega$ and $\rho \colon \{k_j : j < i\} \to 2$ the condition $f \cup \rho$ decides the value of $\underset{\sim}{\tau}{\restriction} i$, say $f \cup \rho \Vdash \underset{\sim}{\tau}{\restriction} i = \sigma_\rho$. Now one could take the tree

$$T^{\oplus} = \left\{\nu \in {}^{\omega>}\omega : (\exists i < \omega)(\exists \rho \in {}^{\{k_j : j<i\}}2)(\nu \trianglelefteq \sigma_\rho)\right\}.$$

Easily $p \Vdash \underset{\sim}{\tau} \in [T^{\oplus}]$, but $T^{\oplus}$ does not have to be a binary tree! (It could well be that $\sigma_\rho = \sigma^*$ for all ρ of length 100 and then $\sigma_{\rho'}$ for ρ' of length 101 are pairwise distinct.) So we would like to make sure that σ_ρ for ρ's of the same length are distinct, but this does not have to be possible. To show that $\mathbb{S}_2$ has the 2-localization property we have to be a little bit more careful. Let us give a combinatorial result which easily implies that $\mathbb{S}_2$ has the 2-localization property. Its proof is the heart of our proof of Theorem 2.5.

Fix $\Psi \colon {}^{\omega>}2 \to \omega$. We define $\Psi^* \colon {}^{\omega>}2 \to {}^{\omega>}\omega$ by induction. Let $\Psi^*(\langle\ \rangle) = \langle\ \rangle$ and define $\Psi^*(t^\frown\langle i\rangle) = \Psi^*(t)^\frown\langle\Psi(t^\frown\langle i\rangle)\rangle$. If ξ is a partial function from ω to 2 and $\ell \leq \omega$, define $W^\ell(\xi) = \{t \in {}^m 2 : m < \min(\ell+1, \omega) \text{ and } \xi{\restriction} m \subseteq t\}$ and then define $T^\ell(\xi) = \{\Psi^*(t) : t \in W^\ell(\xi)\}$, $T(\xi) = T^\omega(\xi)$.

Theorem 1.3 *For any $\Psi \colon {}^{\omega>}2 \to \omega$ there is a partial function $\xi \colon \omega \to 2$ with co-infinite domain such that $T(\xi)$ is a binary tree.*

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Proof To begin, two equivalence relations on ${}^{\omega>}2$ will be defined. First, define $s \equiv t$ if and only if $\Psi(t^\frown\theta) = \Psi(s^\frown\theta)$ for all $\theta \in {}^{\omega>}2$. Next, define $s \sim t$ if and only if $\Psi^*(s) = \Psi^*(t)$.

Now construct by induction on $m < \omega$ an increasing sequence

$$x_0 < x_1 < \cdots < x_m < N_m$$

and $\xi_m \colon N_m \setminus \{x_0, x_1, \ldots, x_m\} \to 2$ such that $T^{N_m}(\xi_m)$ is a binary branching tree and, moreover, if s and t are maximal elements of $W^{N_m}(\xi_m)$ and $t \sim s$, then $t \equiv s$. The induction starts with $x_0 = 0$. If the induction has been completed for m, then let $x_{m+1} = N_m$. Let $\Delta = \{d_0, d_1, \ldots, d_j\}$ be a set of maximal elements of $W^{N_m}(\xi_m)$ such that precisely one member of each $\sim$ equivalence class belongs to Δ. Now, by induction on $i \leq j$ define N^i and $\xi^i \colon N^i \setminus (N_m + 1) \to 2$ as follows. Let $N^0 = N_m + 1$ and let $\xi^0 = \varnothing$. Given N^i and ξ^i, if there is some $N > N^i$ and $\xi \supseteq \xi^i$ such that $d_i{}^\frown\langle 0\rangle^\frown\xi \equiv d_i{}^\frown\langle 1\rangle^\frown\xi$, then let $N^{i+1} = N$ and let $\xi^{i+1} = \xi$. Otherwise it must be the case that $d_i{}^\frown\langle 0\rangle^\frown\xi^i \not\equiv d_i{}^\frown\langle 1\rangle^\frown\xi^i$ and so it must be possible to find $N^{i+1} > N^i$ and $\xi^{i+1} \supseteq \xi^i$ such that $d_i{}^\frown\langle 0\rangle^\frown\xi^{i+1} \not\sim d_i{}^\frown\langle 1\rangle^\frown\xi^{i+1}$. Finally, let $N_{m+1} = N^j$ and $\xi_{m+1} = \xi_m \cup \xi^j$.

To see that this works, it must be shown that $T^{N_{m+1}}(\xi_{m+1})$ is a binary tree and that if s and t are maximal elements of $W^{N_{m+1}}(\xi_{m+1})$ and $t \sim s$, then $t \equiv s$. To check the first condition it suffices to take t a maximal element of $T^{N_m}(\xi_m)$ and check that the tree $T^{N_{m+1}}(\xi_{m+1})$ above t is binary. Then $t = \Psi^*(d_i)$ for some i, and the tree $T^{N_{m+1}}(\xi_{m+1})$ above t is generated by all $\Psi^*(d^\frown\langle a\rangle^\frown\xi^j)$ where $d \sim d_i$ and $a \in 2$. Note however that by the induction hypothesis, if $d \sim d_i$, then $d \equiv d_i$ and so

$$\Psi^*(d^\frown\langle a\rangle^\frown\xi^j) = \Psi^*(d_i{}^\frown\langle a\rangle^\frown\xi^j).$$

Therefore $\Psi^*(d^\frown\langle a\rangle^\frown\xi^j)$ depends only on a and not on d and so $T^{N_{m+1}}(\xi_{m+1})$ is binary above t.

To check the second condition, suppose that s and t are maximal elements of $W^{N_{m+1}}(\xi_{m+1})$ and $t \sim s$. This implies that $t{\restriction}N_m \sim s{\restriction}N_m$ and hence $t{\restriction}N_m \equiv s{\restriction}N_m$. Let i be such that $t{\restriction}N_m \sim s{\restriction}N_m \sim d_i$. If $t(N_m) = s(N_m) = y$, then $t = t{\restriction}N_m{}^\frown\langle y\rangle^\frown\xi^j$ and $s = s{\restriction}N_m{}^\frown\langle y\rangle^\frown\xi^j$ and, since $t{\restriction}N_m \equiv s{\restriction}N_m$, it is immediate that $t \equiv s$. So assume that $t(N_m) = 0$ and $s(N_m) = 1$. By the same argument it follows that $t \equiv d_i{}^\frown\langle 0\rangle^\frown\xi^j$ and $s \equiv d_i{}^\frown\langle 1\rangle^\frown\xi^j$. Hence it suffices to show that $d_i{}^\frown\langle 0\rangle^\frown\xi^j \equiv d_i{}^\frown\langle 1\rangle^\frown\xi^j$. Note that $d_i{}^\frown\langle 0\rangle^\frown\xi^j \sim d_i{}^\frown\langle 1\rangle^\frown\xi^j$ since $t \sim d_i{}^\frown\langle 0\rangle^\frown\xi^j$ and $s \sim d_i{}^\frown\langle 1\rangle^\frown\xi^j$. This means that it must have been possible to find ξ^i such that $d_i{}^\frown\langle 0\rangle^\frown\xi^i \equiv d_i{}^\frown\langle 1\rangle^\frown\xi^i$. It follows that $d_i{}^\frown\langle 0\rangle^\frown\xi^j \equiv d_i{}^\frown\langle 1\rangle^\frown\xi^j$.

After the construction is carried out we let $\xi = \bigcup_{m<\omega} \xi_m$. ■

2 The Result and Its Applications

Let us start by recalling the definitions of the forcing notions which have appeared in the literature in the context of the n-localization property.

Definition 2.1 (i) *The n-Sacks forcing notion* $\mathbb{D}_n$ consists of perfect trees $p \subseteq {}^{\omega>}n$ such that $(\forall \eta \in p)(\exists \nu \in p)(\eta \lhd \nu$ and $\mathrm{succ}_p(\eta) = n)$. The order of $\mathbb{D}_n$ is the

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reverse inclusion, *i.e.*, $p \le_{\mathbb{D}_n} q$ (q is $\mathbb{D}_n$-stronger than p) if and only if $q \subseteq p$. (See [12].)

(ii) *The uniform n-Sacks forcing notion* $\mathbb{Q}_n$ consists of perfect trees $p \subseteq {}^{\omega>}n$ such that $(\exists X \in [\omega]^{\omega})(\forall \eta \in p)(\mathrm{lh}(\eta) \in X \ \Rightarrow\ \mathrm{succ}_p(\nu) = n)$. The order of $\mathbb{Q}_n$ is the reverse inclusion, *i.e.*, $p \le_{\mathbb{Q}_n} q$ (q is $\mathbb{Q}_n$-stronger than p) if and only if $q \subseteq p$. (See [14].)

(iii) Let us assume that $G = (V, E)$ is a hypergraph on a Polish space V such that
- $E \subseteq [V]^{n+1}$ is open in the topology inherited from V^{n+1},
- $(\forall e \in E)(\forall v \in V \setminus e)(\exists w \in e)((e \setminus \{w\}) \cup \{v\} \in E)$,
- for every non-empty open subset U of V and every countable family $\mathcal{F}$ of subsets of U, either $\bigcup \mathcal{F} \neq U$ or $[F]^{n+1} \cap E \neq \varnothing$ for some $F \in \mathcal{F}$.

The Geschke forcing notion $\mathbb{P}_G$ *for* G consists of all closed sets $C \subseteq V$ such that the hypergraph $(C, E \cap [C]^{n+1})$ is uncountably chromatic on every non-empty open subset of C. The order of $\mathbb{P}_G$ is the inverse inclusion, *i.e.*, $C \le_{\mathbb{P}_G} D$ (D is $\mathbb{P}_G$-stronger than C) if and only if $D \subseteq C$. (See [3].)

Definition 2.2 (i) *The n-Silver forcing notion* $\mathbb{S}_n$ consists of partial functions f such that $\mathrm{Dom}(f) \subseteq \omega$, $\mathrm{Rng}(f) \subseteq n$ and $\omega \setminus \mathrm{Dom}(f)$ is infinite. The order of $\mathbb{S}_n$ is the inclusion, *i.e.*, $f \le_{\mathbb{S}_n} g$ (g is $\mathbb{S}_n$-stronger than f) if and only if $f \subseteq g$.

(ii) For an integer $i \in \omega$ and a condition $f \in \mathbb{S}_n$ we let $\mathrm{FP}_i(f)$ to be the unique element of $\omega \setminus \mathrm{Dom}(f)$ such that $|\mathrm{FP}_i(f) \setminus \mathrm{Dom}(f)| = i$. (The FP stands for *Free Point*.)

(iii) A binary relation $\le^*_i$ on $\mathbb{S}_n$ is defined by $f \le^*_i g$ if and only if ($f, g \in \mathbb{S}_n$ and) $f \le_{\mathbb{S}_n} g$ and $(\forall j \in \omega)(j < \lfloor i/4 \rfloor \ \Rightarrow\ \mathrm{FP}_j(f) = \mathrm{FP}_j(g))$.

(iv) For $f \in \mathbb{S}_n$ and $\sigma \colon N \to n$, $N < \omega$ we define $f * \sigma$ as the unique condition in $\mathbb{S}_n$ such that $\mathrm{Dom}(f * \sigma) = \mathrm{Dom}(f) \cup \{\mathrm{FP}_i(f) : i < N\}$, $f \subseteq f * \sigma$ and $f * \sigma(\mathrm{FP}_i(f)) = \sigma(i)$ for $i < N$.

The following properties of forcing notions were introduced in [15] to deal with the n-localization of CS iterations.

Definition 2.3 Let $\mathbb{P}$ be a forcing notion.

(i) For a condition $p \in \mathbb{P}$ we define a game $\eth^{\ominus}_n(p, \mathbb{P})$ of two players, *Generic* and *Antigeneric*. A play of $\eth^{\ominus}_n(p, \mathbb{P})$ lasts ω moves, and during it the players construct a sequence $\langle (s_i, \bar{\eta}^i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ as follows. At a stage $i < \omega$ of the play:

(α) First Generic chooses a finite n-ary tree s_i such that $|\max(s_0)| \le n$, and if $i = j + 1$, then s_j is a subtree of s_i such that

$$\big(\forall \eta \in \max(s_i)\big)\,\big(\exists \ell < \mathrm{lh}(\eta)\big)\,\big(\eta{\restriction}\ell \in \max(s_j)\big),$$

and

$$\big(\forall \nu \in \max(s_j)\big)\,\big(0 < \big|\{\eta \in \max(s_i) : \nu \lhd \eta\}\big| \le n\big).$$

(β) Next Generic picks an enumeration $\bar{\eta}^i = \langle \eta^i_\ell : \ell < k_i \rangle$ of $\max(s_i)$ (so $k_i < \omega$), and then the two players play a subgame of length k_i, choosing

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successive terms of a sequence $\langle p^i_{\eta^i_\ell}, q^i_{\eta^i_\ell} : \ell < k_i\rangle$. At a stage $\ell < k_i$ of the subgame:

$(\gamma)^i_\ell$ First Generic picks a condition $p^i_{\eta^i_\ell} \in \mathbb{P}$ such that If $j < i, \nu \in \max(s_j)$ and $\nu \lhd \eta^i_\ell$, then $q^j_\nu \leq p^i_{\eta^i_\ell}$ and $p \leq p^i_{\eta^i_\ell}$.

$(\delta)^i_\ell$ Then Antigeneric answers with a condition $q^i_{\eta^i_\ell}$ stronger than $p^i_{\eta^i_\ell}$.

After the subgame of this stage is over, the players put $\bar{p}^i = \langle p^i_{\eta^i_\ell} : \ell < k_i\rangle$ and $\bar{q}^i = \langle q^i_{\eta^i_\ell} : \ell < k_i\rangle$.

Finally, Generic wins the play $\langle (s_i, \bar{\eta}^i, \bar{p}^i, \bar{q}^i) : i < \omega\rangle$ if and only if

($\circledast$) there is a condition $q \geq p$ such that for every $i < \omega$ the family $\{q^i_\eta : \eta \in \max(s_i)\}$ is predense above q.

(ii) We say that $\mathbb{P}$ has the $\ominus_n$*-property* whenever Generic has a winning strategy in the game $\eth^{\ominus}_n(p, \mathbb{P})$ for any $p \in \mathbb{P}$.

(iii) Let $K \in [\omega]^\omega$, $p \in \mathbb{P}$. A strategy **st** for Generic in $\eth^{\ominus}_n(p, \mathbb{P})$ is K*-nice* whenever

($\boxtimes^K_{\text{nice}}$) if so far Generic used **st**, and s_i and $\bar{\eta}^i = \langle \eta^i_\ell : \ell < k\rangle$ are given to that player as innings at a stage $i < \omega$, then

- $s_i \subseteq \bigcup_{j \leq i+1} {}^j(n+1)$, $\max(s_i) \subseteq {}^{(i+1)}(n+1)$;
- if $\eta \in \max(s_i)$ and $i \notin K$, then $\eta(i) = n$;
- if $\eta \in \max(s_i)$ and $i \in K$, then $\mathrm{succ}_{s_i}(\eta{\restriction}i) = n$;
- if $i \in K$ and $\langle p^i_{\eta^i_\ell}, q^i_{\eta^i_\ell} : \ell < k\rangle$ is the result of the subgame of level i in which Generic uses **st**, then the conditions $p^i_{\eta^i_\ell}$ (for $\ell < k$) are pairwise incompatible.

(iv) We say that $\mathbb{P}$ has the *nice* $\ominus_n$*-property* if for every $K \in [\omega]^\omega$ and $p \in \mathbb{P}$, Generic has a K-nice winning strategy in $\eth^{\ominus}_n(p, \mathbb{P})$.

Theorem 2.4 **(See [15, 3.1, 1.6, 1.4])** *The limits of CS iterations of the forcing notions defined in Definitions* 2.1 *and* 2.2 *have the nice* $\ominus_n$*-property.*

Now we may formulate our main result.

Theorem 2.5 *Assume that* $\mathbb{P}$ *has the nice* $\ominus_n$*-property and the* n*-localization property. Let* $\underset{\sim}{\mathbb{S}}_n$ *be the* $\mathbb{P}$*-name for the* n*-Silver forcing notion. Then the composition* $\mathbb{P} * \underset{\sim}{\mathbb{S}}_n$ *has the* n*-localization property.*

The proof of Theorem 2.5 is presented in the following section. Let us note here that this theorem implies n-localization for CS iterations of the forcing notions mentioned earlier.

Corollary 2.6 *Let* $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma\rangle$ *be a CS iteration such that, for every* $\xi < \gamma$, $\underset{\sim}{\mathbb{Q}}_\xi$ *is a* $\mathbb{P}_\xi$*-name for one of the forcing notions defined in Definitions* 2.1 *and* 2.2*. Then* $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ *has the* n*-localization property.*

Proof By induction on γ.

If $\gamma = \gamma_0 + 1$ and $\underset{\sim}{\mathbb{Q}}_{\gamma_0}$ is a $\mathbb{P}_{\gamma_0}$-name for the n-Silver forcing notion, then Theorem 2.5 applies. (Note that $\mathbb{P}_{\gamma_0}$ has the nice $\ominus_n$-property by Theorem 2.4 and it has the n-localization property by the inductive hypothesis.)

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If $\gamma = \gamma_0 + 1$ and $\mathbb{Q}_{\gamma_0}$ is a $\mathbb{P}_{\gamma_0}$-name for $\mathbb{D}_n$ or $\mathbb{Q}_n$ or $\mathbb{P}_G$, then [15, Theorem 3.5(2)] applies. (Note that $\mathbb{P}_{\gamma_0}$ has the nice $\ominus_n$-property by Theorem 2.4 and it has the n-localization property by the inductive hypothesis.)

If γ is limit, then [15, Theorem 3.5(1)] applies. ■

The first immediate consequence of Corollary 2.6 is that if $n < m$, then the forcing notions $\mathbb{D}_n$ and $\mathbb{D}_m$ differ in a strong sense: CS iterations of the former forcing do not add generic objects for the latter forcing. A similar observation can be formulated for the Silver forcing notions, as in the following.

Corollary 2.7 *No CS iteration of $\mathbb{S}_2$ adds an $\mathbb{S}_4$-generic real.*

Another application of Corollary 2.6 and the CS iteration of the Silver forcing notion is related to covering numbers of some ideals.

Definition 2.8 Let $2 \leq m < \omega$.

(i) For a function $\varphi\colon {}^{<\omega}m \to m$, put

$$A_\varphi = \{c \in {}^{\omega}m : (\exists k < \omega)(\forall \ell \geq k)(c(\ell) \neq \varphi(c{\restriction}\ell))\}.$$

We let $\mathfrak{D}_m = \{A \subseteq {}^{\omega}m : A \subseteq A_\varphi \text{ for some function } \varphi\colon {}^{<\omega}m \to m\}$.

(ii) We define

$$\mathfrak{P}_m = \{A \subseteq {}^{\omega}m : (\forall K \in [\omega]^{\omega})(\exists f \in {}^{K}m)(\forall c \in A)(f \not\subseteq c)\},$$
$$\mathfrak{R}_m = \{A \subseteq {}^{\omega}m : (\forall f \in \mathbb{S}_m)(\exists g \geq_{\mathbb{S}_m} f)(\forall c \in A)(g \not\subseteq c)\}.$$

(iii) The covering number $\mathrm{cov}(\mathfrak{I})$ of an ideal $\mathfrak{I}$ of subsets of a space $\mathcal{X}$ is defined as

$$\mathrm{cov}(\mathfrak{I}) = \min\big(|\mathcal{B}| : \mathcal{B} \subseteq \mathfrak{I} \text{ and } \bigcup \mathcal{B} = \mathcal{X}\big).$$

Note that $\mathfrak{D}_{n+1}$ is a σ-ideal of subsets of ${}^{\omega}(n+1)$, moreover it is the σ-ideal generated by sets of the form $[T]$ for n-ary trees $T \subseteq {}^{<\omega}(n+1)$. The ideals $\mathfrak{D}_m$ appeared implicitly in Mycielski's proof of the determinacy of unsymmetrtic games on analytic sets in [10] and later were studied, for instance, in [4, 12, 13].

Also $\mathfrak{P}_n$ and $\mathfrak{R}_n$ are σ-ideals of subsets of ${}^{\omega}n$. The ideal $\mathfrak{P}_n$ is one of the ideals motivated by the Mycielski ideals of [11]. It was introduced in [13] and later it was studied, for example, in [1, 2, 8, 14, 16, 18]. Shelah and Steprāns [18] showed that $\mathrm{cov}(\mathfrak{P}_n) = \mathrm{cov}(\mathfrak{P}_{n+1})$, $\mathrm{cov}(\mathfrak{R}_n) \geq \mathrm{cov}(\mathfrak{R}_{n+1})$, and consistently the latter inequality is strict. The consistency result in [18] was actually much stronger and it was obtained by means of finite support iteration of ccc forcing notions. However, if we are interested in the consistency of "$\mathrm{cov}(\mathfrak{R}_n) > \mathrm{cov}(\mathfrak{R}_{n+1})$" only, then a CS iteration of $\mathbb{S}_n$ will do the following.

Corollary 2.9 *Assume CH. Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \omega_2\rangle$ be a countable support iteration such that $\Vdash_{\mathbb{P}_\alpha}$ "$\mathbb{Q}_\alpha = \mathbb{S}_n$" (for all $\alpha < \omega_2$). Then*

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{“} 2^{\aleph_0} = \mathrm{cov}(\mathfrak{R}_n) = \mathrm{cov}(\mathfrak{P}_n) = \mathrm{cov}(\mathfrak{P}_{n+1}) = \aleph_2, \text{ and } \mathrm{cov}(\mathfrak{R}_{n+1}) = \mathrm{cov}(\mathfrak{D}_{n+1}) = \aleph_1\text{”}.$$

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3 Proof of Theorem 2.5

Let $\underset{\sim}{\tau}$ be a $\mathbb{P} * \underset{\sim}{\mathbb{S}}_n$-name for a member of ${}^{\omega}\omega$. We may assume that for every $\mathbb{P}$-name $\underset{\sim}{\rho}$ we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{S}}_n} \underset{\sim}{\tau} \neq \underset{\sim}{\rho}$. If $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, then we will use the same notation $\underset{\sim}{\tau}$ for $\mathbb{S}_n$-name in $\mathbf{V}[G]$ for a member of ${}^{\omega}\omega$ that is given by the original $\underset{\sim}{\tau}$ in the extension via $\mathbb{P} * \underset{\sim}{\mathbb{S}}_n$.

Let $(p, \underset{\sim}{f}) \in \mathbb{P} * \underset{\sim}{\mathbb{S}}_n$ and let **st** be a winning strategy of Generic in $\Game^{\ominus}_n(p, \mathbb{P})$ which is nice for the set $K = \{4j + 2 : j \in \omega\}$ (see Definition 2.3(iii)).

By induction on i we are going to choose for each $i < \omega$ $s_i, \bar{\eta}^i, \bar{p}^i, \bar{q}^i, \underset{\sim}{f}_i$, and also for $m_i, \bar{\sigma}^i$ for odd $i < \omega$ such that the following conditions $(\boxtimes)_1$–$(\boxtimes)_7$ are satisfied.

$(\boxtimes)_1$ $\langle s_i, \bar{\eta}^i, \bar{p}^i, \bar{q}^i : i < \omega\rangle$ is a play of $\Game^{\ominus}_n(p, \mathbb{P})$ in which Generic uses **st**.
$(\boxtimes)_2$ $\underset{\sim}{f}_i$ is a $\mathbb{P}$-name for a condition in $\mathbb{S}_n$, and we stipulate that $\underset{\sim}{f}_{-1} = \underset{\sim}{f}$.
$(\boxtimes)_3$ $q^i_\eta \Vdash_{\mathbb{P}} \underset{\sim}{f}_{i-1} \leq^*_i \underset{\sim}{f}_i$ for each $\eta \in \max(s_i)$.

For odd $i < \omega$:

$(\boxtimes)_4$ $m_i < m_{i+2} < \omega$, $\bar{\sigma}^i = \langle \sigma^i_{\rho,\eta} : \eta \in \max(s_i) \text{ and } \rho \in {}^{\lfloor i/4\rfloor} n\rangle$, $\sigma^i_{\rho,\eta} \colon m_i \to \omega$.
$(\boxtimes)_5$ $(q^i_\eta, \underset{\sim}{f}_i * \rho) \Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{S}}_n}$ "$\underset{\sim}{\tau}{\restriction} m_i = \sigma^i_{\rho,\eta}$" for $\rho \in {}^{\lfloor i/4\rfloor} n$ and $\eta \in \max(s_i)$.
$(\boxtimes)_6$ If $\eta \in \max(s_i)$ and $\rho, \rho' \colon \lfloor i/4\rfloor \to n$ are distinct but $\sigma^i_{\rho,\eta} = \sigma^i_{\rho',\eta}$, then for every $q \geq q^i_\eta$ and a $\mathbb{P}$-name $\underset{\sim}{g}$ for an n-Silver condition and m, σ, σ' such that

$$q \Vdash_{\mathbb{P}} \underset{\sim}{f}_i \leq^*_i \underset{\sim}{g}, \quad (q, \underset{\sim}{g} * \rho) \Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{S}}_n} \underset{\sim}{\tau}{\restriction} m = \sigma, \quad (q, \underset{\sim}{g} * \rho') \Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{S}}_n} \underset{\sim}{\tau}{\restriction} m = \sigma'$$

we have $\sigma = \sigma'$.
$(\boxtimes)_7$ If $\eta, \eta' \in \max(s_i)$ are distinct, $\rho, \rho' \colon \lfloor i/4\rfloor \to n$, then $\sigma^i_{\rho,\eta} \neq \sigma^i_{\rho',\eta'}$.

So suppose that $i < \omega$ is even and we have already defined $s_{i-1}, \bar{q}^{i-1}, m_{i-1}$ and $\underset{\sim}{f}_{i-1}$ (we stipulate $s_{-1} = \{\langle\rangle\}$, $q^{-1}_{\langle\rangle} = p$, $\underset{\sim}{f}_{-1} = \underset{\sim}{f}$ and $m_{-1} = 0$). Let $j = \lfloor i/4\rfloor$ (so either $i = 4j$ or $i = 4j + 2$).

The strategy **st** and demand $(\boxtimes)_1$ determine s_i and $\bar{\eta}^i = \langle \eta^i_k : k < k_i\rangle$. To define $\bar{p}^i, \bar{q}^i$ and $\underset{\sim}{f}_i$ we consider the following run of the subgame of level i of $\Game^{\ominus}_n(p, \mathbb{P})$. Assume we are at stage $k < k_i$ of the subgame. Now, $p^i_{\eta^i_k}$ is given by the strategy **st** (and $(\boxtimes)_1$, of course). Suppose for a moment that $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, $p^i_{\eta^i_k} \in G$. Working in $\mathbf{V}[G]$ we may choose $\bar{\ell}, \bar{L}, g^*, \bar{\sigma}^*, M$ such that

$(\boxtimes)^{\alpha}_8$ $M = n^j$, $\bar{\ell} = \langle \ell_m : m \leq M\rangle$ and $j = \ell_0 < \cdots < \ell_M$, $\bar{L} = \langle L_m : m \leq M\rangle$ and $m_{i-1} < L_0 < \cdots < L_M$,
$(\boxtimes)^{\beta}_8$ $g^* \in \mathbb{S}_n$, $\underset{\sim}{f}_{i-1}[G] \leq^*_i g^*$ and $\bar{\sigma}^* = \langle \sigma^*_\rho : \rho \in {}^{\ell_M} n\rangle$, $\sigma^*_\rho \in {}^{L_M}\omega$ (for $\rho \in {}^{\ell_M} n$),
$(\boxtimes)^{\gamma}_8$ $g^* * (\rho{\restriction}\ell_m) \Vdash_{\mathbb{S}_n}$ "$\underset{\sim}{\tau}{\restriction} L_m = \sigma^*_\rho{\restriction} L_m$" for each $m \leq M$ and $\rho \in {}^{\ell_M} n$,
$(\boxtimes)^{\delta}_8$ if $\rho_0, \rho_1 \in {}^{\ell_M} n$, $\rho_0{\restriction} j \neq \rho_1{\restriction} j$ but $\sigma^*_{\rho_0}{\restriction} L_0 = \sigma^*_{\rho_1}{\restriction} L_0$, then there is no condition $g \in \mathbb{S}_n$ such that $g^* \leq^*_i g$ and for some $L < \omega$ and distinct $\sigma_0, \sigma_1 \in {}^{L}\omega$ we have that $g * \rho_0 \Vdash \underset{\sim}{\tau}{\restriction} L = \sigma_0$, $g * \rho_1 \Vdash \underset{\sim}{\tau}{\restriction} L = \sigma_1$,
$(\boxtimes)^{\varepsilon}_8$ for each $m < M$ and $\rho_0 \in {}^{\ell_m} n$ the set $\{\sigma^*_\rho{\restriction}[L_m, L_{m+1}) : \rho_0 \lhd \rho \in {}^{\ell_M} n\}$ has at least $n^j \cdot k_i + 777$ elements.

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It should be clear how the construction is done. (First we take care of clause $(\boxtimes)_8^\delta$ by going successively through all pairs of elements of ${}^j n$ and trying to force distinct values for initial segments of $\underset{\sim}{\tau}$, if this is possible. Then we ensure $(\boxtimes)_8^\varepsilon$ basically by deciding longer and longer initial segments of $\underset{\sim}{\tau}$ on fronts/levels of a fusion sequence of conditions in $\mathbb{S}_n$ and using the assumption that $\underset{\sim}{\tau}$ is forced to be "new".) Now, going back to $\mathbf{V}$, we may choose a condition $q^i_{\eta^i_k} \in \mathbb{P}$ stronger than $p^i_{\eta^i_k}$ and a $\mathbb{P}$-name $\underset{\sim}{g}^{*,k}$ for a condition in $\mathbb{S}_n$ and objects $\bar{\ell}^k, \bar{L}^k, \bar{\sigma}^{*,k}$ such that

$$q^i_{\eta^i_k} \Vdash_{\mathbb{P}} \text{“ } \bar{\ell}^k, \bar{L}^k, \underset{\sim}{g}^{*,k}, \bar{\sigma}^{*,k}, n^j \text{ satisfy clauses } (\boxtimes)_8^\alpha\text{–}(\boxtimes)_8^\varepsilon \text{ as } \bar{\ell}, \bar{L}, g^*, \bar{\sigma}^*, M \text{ there ”.}$$

The condition $q^i_{\eta^i_k}$ is treated as an inning of Antigeneric at stage k of the subgame of $\Game^\ominus_n(p, \mathbb{P})$ and the process continues.

After the subgame of level i is completed, we have defined $\bar{p}^i$ and $\bar{q}^i$. We also choose $\underset{\sim}{f}_i$ to be a $\mathbb{P}$-name for an element of $\underset{\sim}{\mathbb{S}}_n$ such that $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{f}_{i-1} \leq^*_i \underset{\sim}{f}_i$ " and $q^i_{\eta^i_k} \Vdash_{\mathbb{P}}$ " $\underset{\sim}{f}_i = \underset{\sim}{g}^{*,k}$ " for all $k < k_i$ (remember that **st** is nice, so the conditions $q^i_{\eta^i_k}$ are pairwise incompatible). This completes the description of what happens at the stage i of the construction (one easily verifies that $(\boxtimes)_1$– $(\boxtimes)_3$ are satisfied) and we proceed to the next, $i+1$, stage. Note that $\lfloor (i+1)/4 \rfloor = j$.

We let $m_{i+1} = \max(L^k_M : k < k_i) + 5$ and let $\ell = \max(\ell^k_M : k < k_i) + 5$. Similarly as at stage i, s_{i+1} and $\bar{\eta}^{i+1} = \langle \eta^{i+1}_k : k < k_{i+1} \rangle$ are determined by the strategy **st** and $(\boxtimes)_1$; note that $\max(s_{i+1}) = \{\nu^\frown\langle n \rangle : \nu \in \max(s_i)\}$ so $k_{i+1} = k_i$. To define $\bar{p}^{i+1}, \bar{q}^{i+1}$ and $\underset{\sim}{f}_{i+1}$ we consider the following round of the subgame of level $i+1$ of $\Game^\ominus_n(p, \mathbb{P})$. At a stage $k < k_{i+1}$ of the subgame, letting $\eta = \eta^{i+1}_k$, the condition p^{i+1}_η is given by the strategy **st**. Suppose for a moment that $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, $p^{i+1}_\eta \in G$. In $\mathbf{V}[G]$ we may choose a condition $h^* \in \mathbb{S}_n$ such that

$(\boxtimes)_9$ $\underset{\sim}{f}_i[G] \leq^*_{4\ell} h^*$ and for every $\rho \in {}^\ell n$ the condition $h^* * \rho$ decides the value of $\underset{\sim}{\tau} \restriction m_{i+1}$, say $h^* * \rho \Vdash_{\mathbb{S}_n}$" $\underset{\sim}{\tau} \restriction m_{i+1} = \sigma_\rho$ ".

Then, going back to $\mathbf{V}$ we choose a $\mathbb{P}$-name $\underset{\sim}{h}^{*,\eta}$ for a condition in $\mathbb{S}_n$, a sequence $\bar{\sigma}^\eta = \langle \sigma^\eta_\rho : \rho \in {}^\ell n \rangle$ and a condition $q^{i+1}_\eta \geq p^{i+1}_\eta$ such that

$$q^{i+1}_\eta \Vdash_{\mathbb{P}} \text{“ } \underset{\sim}{h}^{*,\eta}, \bar{\sigma}^\eta \text{ are as in } (\boxtimes)_9 \text{ ”.}$$

The condition q^{i+1}_η is treated as an inning of Antigeneric at stage k of the subgame of $\Game^\ominus_n(p, \mathbb{P})$ and the process continues.

After the subgame of level $i+1$ is completed, we have defined $\bar{p}^{i+1}$ and $\bar{q}^{i+1}$. Since for every $\eta \in \max(s_{i+1})$ we have that $p^{i+1}_\eta \geq q^i_{\eta \restriction (i+1)}$, we may use $(\boxtimes)_8^\varepsilon$ and choose $\rho(\eta) \colon [j, \ell) \longrightarrow n$ (for $\eta \in \max(s_{i+1})$) such that

$(\boxtimes)_{10}$ if $\eta, \eta' \in \max(s_{i+1})$ are distinct and $\theta, \theta' \in {}^j n$, and $\rho = \theta^\frown \rho(\eta)$, $\rho' = \theta'^\frown \rho(\eta')$, then $\sigma^\eta_\rho \neq \sigma^{\eta'}_{\rho'}$.

Let $\underset{\sim}{f}_{i+1}$ be a $\mathbb{P}$-name for a condition in $\mathbb{S}_n$ such that $\Vdash_{\mathbb{P}} \underset{\sim}{f}_i \leq^*_{i+1} \underset{\sim}{f}_{i+1}$ and

$$q^{i+1}_\eta \Vdash_{\mathbb{P}} \text{“ } \underset{\sim}{h}^{*,\eta} \leq^*_{i+1} \underset{\sim}{f}_{i+1} \text{ and } (\forall \theta \in {}^j n)\big(\underset{\sim}{f}_{i+1} * \theta = \underset{\sim}{h}^{*,\eta} * (\theta^\frown \rho(\eta)) \big) \text{ ”.}$$

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Also, for $\eta \in \max(s_{i+1})$ and $\rho \in {}^{j}n$, we let $\sigma^{i+1}_{\rho,\eta} = \sigma^{\eta}_{\rho^\frown\rho(\eta)}$. This completes the description of what happens at the stage $i+1$ of the construction (one easily checks that $(\boxtimes)_1$–$(\boxtimes)_7$ are satisfied). Thus we have finished the description of the inductive step of the construction of $s_i, \bar{\eta}^i, \bar{p}^i, \bar{q}^i, \underset{\sim}{f}_i$ (for $i < \omega$).

After the construction is carried out we may pick a condition $q \in \mathbb{P}$ stronger than p and such that for each $i < \omega$ the family $\{q^i_\eta : \eta \in \max(s_i)\}$ is predense above q (possible by $(\boxtimes)_1$).

Suppose that $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, $q \in G$. Then there is $\eta \in {}^\omega(n+1)$ such that $\eta\restriction(i+1) \in \max(s_i)$ and $q^i_{\eta\restriction(i+1)} \in G$ for each $i < \omega$. Therefore we may use $(\boxtimes)_3$ to conclude that there is a condition $g \in \mathbb{S}_n$ stronger than all $\underset{\sim}{f}_i[G]$. Going back to $\mathbf{V}$, we may choose a $\mathbb{P}$-name $\underset{\sim}{g}$ for a condition in $\mathbb{S}_n$ such that $q \Vdash_{\mathbb{P}} (\forall i < \omega)(\underset{\sim}{f}_i \leq \underset{\sim}{g})$.

Note that for each $i < \omega$ the family $\{(q^i_\eta, \underset{\sim}{f}_i * \rho) : \eta \in \max(s_i) \text{ and } \rho \in {}^{\lfloor i/4 \rfloor}n\}$ is predense in $\mathbb{P} * \mathbb{S}_n$ above $(q, \underset{\sim}{g})$, and hence (by $(\boxtimes)_5$)

$$(q, \underset{\sim}{g}) \Vdash_{\mathbb{P}*\mathbb{S}_n} \text{“} \underset{\sim}{\tau}\restriction m_i \in \{\sigma^i_{\rho,\eta} : \eta \in \max(s_i) \text{ and } \rho \in {}^{\lfloor i/4 \rfloor}n\} \text{ for every odd } i < \omega \text{”}.$$

Also,

$(\boxtimes)_{11}$ if $i \geq 3$ is odd, $\eta \in \max(s_i)$, $\rho \in {}^{\lfloor i/4 \rfloor}n$ and $\eta' = \eta\restriction(i-1)$ and $\rho' = \rho\restriction\lfloor (i-2)/4 \rfloor$, then $\eta' \in \max(s_{i-2})$ and $\sigma^{i-2}_{\rho',\eta'} = \sigma^i_{\rho,\eta}\restriction m_{i-2}$.

[Why? Since **st** is a nice strategy, $\eta\restriction i \in \max(s_{i-1})$ and $\eta' \in \max(s_{i-2})$. It follows from $(\boxtimes)_1$ that $q^{i-2}_{\eta'} \leq q^{i-1}_{\eta\restriction i} \leq q^i_\eta$ and by $(\boxtimes)_3$ we have $q^i_\eta \Vdash_{\mathbb{P}} \underset{\sim}{f}_{i-2} \leq^*_{i-1} \underset{\sim}{f}_i$. Therefore $q^i_\eta \Vdash_{\mathbb{P}} \underset{\sim}{f}_{i-2} * \rho' \leq \underset{\sim}{f}_i * \rho$ and $(q^{i-2}_{\eta'}, \underset{\sim}{f}_{i-2} * \rho') \leq (q^i_\eta, \underset{\sim}{f}_i * \rho)$, so using $(\boxtimes)_5$ we may conclude that $\sigma^{i-2}_{\rho',\eta'} = \sigma^i_{\rho,\eta}\restriction m_{i-2}$.]

Let $T = \{\nu \in {}^{\omega>}\omega : (\exists i < \omega \text{ odd})(\exists \eta \in \max(s_i))(\exists \rho \in {}^{\lfloor i/4 \rfloor}n)(\nu \trianglelefteq \sigma^i_{\rho,\eta})\}$. Then T is a perfect tree and $(q, \underset{\sim}{g}) \Vdash_{\mathbb{P}*\mathbb{S}_n} \underset{\sim}{\tau} \in [T]$. So the theorem will readily follow once we show that T is n-ary. To this end we are going to argue that

$(\boxtimes)_{12}$ if $i \geq 3$ is odd, $\eta \in \max(s_i)$, $\rho \in {}^{\lfloor i/4 \rfloor}n$, then

$$\left| \left\{ \sigma^i_{\pi,\nu} : \nu \in \max(s_i) \text{ and } \pi \in {}^{\lfloor i/4 \rfloor}n \ \& \ \sigma^i_{\rho,\eta}\restriction m_{i-2} = \sigma^i_{\pi,\nu}\restriction m_{i-2} \right\} \right| \leq n.$$

Case A: $i = 4j+1$ for some $j < \omega$. Suppose that $\eta, \nu \in \max(s_i)$, $\rho, \pi \in {}^{\lfloor i/4 \rfloor}n$ are such that $\sigma^i_{\rho,\eta} \neq \sigma^i_{\pi,\nu}$ but $\sigma^i_{\rho,\eta}\restriction m_{i-2} = \sigma^i_{\pi,\nu}\restriction m_{i-2}$. The latter and $(\boxtimes)_7$ (and $(\boxtimes)_{11}$) imply that $\eta\restriction(i-1) = \nu\restriction(i-1)$, and since $i-1, i \notin K$ we get that $\eta(i-1) = \nu(i-1) = n = \eta(i) = \nu(i)$ (remember that **st** is nice for K), so $\eta = \nu$. If $\rho\restriction(j-1) \neq \pi\restriction(j-1)$, then let $\rho' = \rho\restriction(j-1)^\frown\langle \pi(j-1) \rangle$, otherwise $\rho' = \pi$.

Suppose $\rho' \neq \pi$. Let $\underset{\sim}{g}$ be (a $\mathbb{P}$-name for) $\underset{\sim}{f}_i \cup \{(\mathrm{FP}_{j-1}(\underset{\sim}{f}_i), \pi(j-1))\}$ and $q = q^i_\eta$. Then $q \geq q^{i-2}_{\eta\restriction(i-1)}$, $q \Vdash \underset{\sim}{f}_{i-2} \leq^*_{i-2} \underset{\sim}{g}$, and

$$q \Vdash \text{“} \underset{\sim}{g} * \left(\rho'\restriction(j-1)\right) = \underset{\sim}{f}_i * \rho' \text{ and } \underset{\sim}{g} * \left(\pi\restriction(j-1)\right) = \underset{\sim}{f}_i * \pi \text{”}.$$

Hence

$$(q, \underset{\sim}{g} * \left(\rho'\restriction(j-1)\right) \Vdash \text{“} \underset{\sim}{\tau}\restriction m_i = \sigma^i_{\rho',\eta} \text{”} \quad \text{and} \quad (q, \underset{\sim}{g} * \left(\pi\restriction(j-1)\right) \Vdash \text{“} \underset{\sim}{\tau}\restriction m_i = \sigma^i_{\pi,\eta} \text{”}.$$

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Now we use our assumption that $\sigma^i_{\rho,\eta}\restriction m_{i-2} = \sigma^i_{\pi,\eta}\restriction m_{i-2}$ (and $(\boxtimes)_{11}$) to conclude that $\sigma^i_{\rho',\eta}\restriction m_{i-2} = \sigma^i_{\pi,\eta}\restriction m_{i-2}$. Consequently, $\sigma^i_{\rho',\eta} = \sigma^i_{\pi,\eta}$ (remember $(\boxtimes)_6$ for $i-2$, $\eta\restriction(i-1)$, $\rho'\restriction(j-1)$ and $\pi\restriction(j-1)$). Trivially the same conclusion holds if $\rho' = \pi$, so we have justified that

$$\{\sigma^i_{\pi,\nu} : \nu \in \max(s_i) \text{ and } \pi \in {}^{\lfloor i/4\rfloor}n \text{ and } \sigma^i_{\rho,\eta}\restriction m_{i-2} = \sigma^i_{\pi,\nu}\restriction m_{i-2}\}$$
$$\subseteq \{\sigma^i_{\pi,\eta} : \pi \in {}^{j}n \text{ and } \rho\restriction(j-1) = \pi\restriction(j-1)\}$$

and the latter set is of size at most n.

Case B: $i = 4j+3$ for some $j < \omega$. Again, let us assume that $\eta, \nu \in \max(s_i)$, $\rho, \pi \in {}^{\lfloor i/4\rfloor}n$ are such that $\sigma^i_{\rho,\eta} \neq \sigma^i_{\pi,\nu}$ but $\sigma^i_{\rho,\eta}\restriction m_{i-2} = \sigma^i_{\pi,\nu}\restriction m_{i-2}$. Then, as in the previous case, $(\boxtimes)_7$ implies $\eta\restriction(i-1) = \nu\restriction(i-1)$. Also $\lfloor i/4\rfloor = j = \lfloor (i-2)/4\rfloor$, so $\rho\restriction\lfloor(i-2)/4\rfloor = \rho$, $\pi\restriction\lfloor(i-2)/4\rfloor = \pi$. Now, if $\rho = \pi$, then trivially $\sigma^i_{\pi,\nu} = \sigma^i_{\rho,\nu}$. If $\rho \neq \pi$, then we use $(\boxtimes)_6$ (with $i-2, \rho, \pi, q^i_\eta, \underset{\sim}{f}_i$ here in place of $i, \rho, \rho', q, \underset{\sim}{g}$ there, respectively) to argue that $\sigma^i_{\pi,\nu} = \sigma^i_{\rho,\nu}$. Consequently

$$\{\sigma^i_{\pi,\nu} : \nu \in \max(s_i) \text{ and } \pi \in {}^{\lfloor i/4\rfloor}n \text{ and } \sigma^i_{\rho,\eta}\restriction m_{i-2} = \sigma^i_{\pi,\nu}\restriction m_{i-2}\}$$
$$\subseteq \{\sigma^i_{\rho,\nu} : \nu \in \max(s_i) \text{ and } \eta\restriction(i-2) = \nu\restriction(i-2)\}$$

and the latter set is of size at most n.

Now in both cases we easily get the assertion of $(\boxtimes)_{12}$, completing the proof of the theorem.

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