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A Singular Critical Potential for the Schrödinger Operator

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Abstract. Consider a real potential V on $\mathbf{R}^d$, $d \geq 2$, and the Schrödinger equation:

(LS) $$i\partial_t u + \Delta u - Vu = 0, \quad u_{\restriction t=0} = u_0 \in L^2.$$

In this paper, we investigate the minimal local regularity of V needed to get local in time dispersive estimates (such as local in time Strichartz estimates or local smoothing effect with gain of $1/2$ derivative) on solutions of (LS). Prior works show some dispersive properties when V (small at infinity) is in $L^{d/2}$ or in spaces just a little larger but with a smallness condition on V (or at least on its negative part).

In this work, we prove the critical character of these results by constructing a positive potential V which has compact support, bounded outside 0 and of the order $(\log|x|)^2/|x|^2$ near 0. The lack of dispersiveness comes from the existence of a sequence of quasimodes for the operator $P := -\Delta + V$.

The elementary construction of V consists in sticking together concentrated, truncated potential wells near 0. This yields a potential oscillating with infinite speed and amplitude at 0, such that the operator P admits a sequence of quasi-modes of polynomial order whose support concentrates on the pole.

1 Introduction

Consider a Schrödinger operator,

(1) $$P := -\Delta + V, \quad \Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

on $\mathbf{R}^d$, $d \geq 1$, where V is a time-independent, real potential, small at infinity, with a small negative part, and the initial value problem for the Schrödinger equation,

(2) $$\begin{cases} i\partial_t U - PU = 0, \\ U_{\restriction t=0} = U_0 \in L^2(\mathbf{R}^d). \end{cases}$$

For potentials V which are not too singular, the solutions of equation (2) enjoy many dispersive properties, such as the classical dispersion estimate, Strichartz estimates and the local smoothing effect (introduced by Constantin–Saut [7], Sjölin [14], Vega [16]):

(3) $$\|\chi U\|_{L^2(]0,T[,H^{1/2}(\mathbf{R}^d))} \leq C\|U_0\|_{L^2(\mathbf{R}^d)}, \ \chi \in C_0^\infty(\mathbf{R}^d),$$

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where T may be finite, and in some cases infinite, and C may depend on V, χ and T.

These well-known properties are essential in the description of the solutions of (2) and the study of related non-linear equations. It is natural to look for the minimal assumptions to be made on V for the dispersive properties to hold. This issue has two aspects, the boundedness of V at infinity and its local regularity. In this paper, we shall ignore the first question, taking only potentials that decay fast enough for large $|x|$, and investigate only the second one. One of our motivations is that the linearization of non-linear Schrödinger equations near a singular stationary solution may yield very harsh potentials. We now recall some important results on the subject.

First note that for potentials

$$V \in L^{d/2},\ d \geq 3 \quad \text{and} \quad V \in L^p,\ p > 1,\ d = 2$$

small at infinity, it is easy to deduce local in time Strichartz estimates from the free equation estimates using Duhamel's formula.

Using perturbative arguments, Ruiz and Vega [13] showed that for these potentials, in dimension $d \geq 3$ the smoothing effect (3) also holds. Furthermore, their results handle more singular potentials belonging to some Morrey–Campanato spaces, which are (from a local point of view), strictly larger than the Lorentz space $L^{d/2}_w$ (and thus larger than $L^{d/2}$), but strictly contained in $\bigcap_{p<d/2} L^p$. In this case a smallness condition on the potential (or at least on its most singular part) is needed. The simpler example of this kind is the inverse square potential,

$$V_a := \frac{a}{|x|^2}, \quad |a| < \varepsilon_d,$$

where ε_d is a constant depending on d.

Burq, Planchon, Stalker and Tahvildar-Zadeh [5] showed Strichartz estimates and smoothing effect on the Schrödinger equation with the potential V_a (see also [9] for the multipolar case), assuming only the following positivity property on a:

$$a + (d/2 - 1)^2 > 0.$$

When $a + (d/2 - 1)^2 < 0$, there is a sequence of eigenvalues going to $-\infty$, such that the sequence of the corresponding stationary solutions invalidate the dispersive-type estimates. Therefore, these critical potentials may not be considered as perturbations of the Laplacien, but rather as operators of the same order, which may change radically the behaviour of P.

In a recent article [12], Rodniansky and Schlag proved dispersive-type decay of solutions of Schrödinger equations in dimension 3 with potentials that are in a Kato class which is, again, locally strictly contained in $\bigcap_{p<d/2} L^p$ but larger than $L^{d/2}$. In their article, the authors make a smallness assumption on the negative, singular part of the potential, which is similar to the positivity assumption on a.

These results suggest that the critical class of potentials from the point of view of local regularity is a little larger than $L^{d/2}$, but to our knowledge little is known about the optimality of the assumptions on the positive part of V. Indeed nothing in the above works suggest that a positive, very singular potential may prevent solutions of (2) from having dispersive properties.

In this paper, we answer this question, constructing a unipolar positive potential, with compact support, whose pole is of the order $\frac{(\log|x|)^2}{|x|^2}$ near 0, and such that local in time dispersive estimates do not hold. This construction shows the quasi-optimality of the works mentioned above [5, 9, 12, 13], (see also the paper by Goldberg [10]). The lack of dispersion comes from the existence of a sequence of quasi-modes for the operator P (*i.e.*, approximate solutions of the equation $Pu - \lambda u = 0$ with λ going to infinity). Note that the existence of a finite number of eigenfunctions would show that global in time dispersive estimates cannot hold. Our result is stronger as it shows also the impossibility of local in time estimates (which are really high-frequency type estimates). Recall also that in general, there is no uniformity of the estimates with respect to the norm of the potential in the critical spaces. Thus, it is important to construct a single V invalidating the dispersive properties, rather than giving an example of a sequence of potentials (even with an uniform bound).

Theorem 1 (Existence of quasi-modes) *Let $d \geq 1$, $N \geq 0$ be integers. There exist*

- *a radial, positive potential V on $\mathbf{R}^d$, which has compact support and such that*

$$V \in C^\infty(\mathbf{R}^d \backslash \{0\}) \tag{4}$$

$$\frac{|\log r|^2}{Cr^2} \leq V(r) \leq \frac{C|\log r|^2}{r^2}, \ r \leq r_0, \tag{5}$$

- *an increasing sequence $(\lambda_n)_{n \geq n_0}$ of positive real numbers, diverging to $+\infty$,*
- *a sequence of radial C^∞ functions $(u_n)_{n \geq n_0}$, whose support is of the following form:*

$$\Big\{c_1 \frac{\log \lambda_n}{\lambda_n} \leq r \leq c_2 \frac{\log \lambda_n}{\lambda_n}\Big\}, \ 0 < c_1 < c_2;$$

such that

$$(-\Delta + V)u_n - \lambda_n^2 u_n = f_n \tag{6}$$

$$\|u_n\|_{L^1} = 1 \tag{7}$$

$$\forall j \in \mathbb{N}, \ \Big\| \frac{d^j}{dr^j} f_n \Big\|_{L^\infty} = O(\lambda_n^{j-N}), \ n \to +\infty. \tag{8}$$

Corollary 1 *Let $N \geq 2$, $P = -\Delta + V$, where V is the potential of the preceding theorem, and χ a function in $C_0^\infty(\mathbf{R}^d)$ which does not vanish in 0. Then the solutions of (2) do not enjoy any of the classical dispersive properties.*

- *local Strichartz estimates:*

$$\forall q, q_0 \in [1, +\infty], \ q > q_0, \ \forall T > 0, \ \forall K > 0, \ \exists U_0 \in C_0^\infty(\mathbf{R}^d \backslash \{0\}), \\ \|\chi U(t)\|_{L^1(]0,T[, L^q(\mathbf{R}^d))} > K \|U_0\|_{L^{q_0}(\mathbf{R}^d)}; \tag{9}$$

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- *local smoothing effect:*

$$(10)\quad \forall \sigma > 0\ \forall T > 0,\ \forall K > 0,\ \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}),$$
$$\|\chi U(t)\|_{L^1(]0,T[,H^\sigma(\mathbf{R}^d))} > K\|U_0\|_{L^2(\mathbf{R}^d)};$$

- *local dispersion:*

$$(11)\quad \forall q \in]1,+\infty],\ \forall T > 0,\ \forall K > 0,\ \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}),$$
$$\|\chi U(T)\|_{L^q(\mathbf{R}^d)} > K\|U_0\|_{L^{q'}(\mathbf{R}^d)};$$

- *Strichartz estimates with loss of derivative:*

$$(12)\quad \forall q \in [1,+\infty],\ \forall \sigma \in [0,1],\ \frac{1}{2} - \frac{1}{q} > \frac{\sigma}{d},$$
$$\forall T > 0,\ \forall K > 0,\ \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}),$$
$$\|\chi U(t)\|_{L^1(]0,T[,L^q(\mathbf{R}^d))} > K\|U_0\|_{D(P^{\sigma/2})}.$$

In the preceding statements, $U(t)$ *is the solution of the equation* (2) *with initial value* U_0*, and* q' *the conjugate exponent of* q *which is defined by* $1/q + 1/q' = 1$.

Remarks

(i) If $d > 2$, the hypotheses (4) and (5) imply:

$$V \in \bigcap_{p<d/2} L^p.$$

(ii) The sequence $(f_n)_{n \geq n_0}$ of Theorem 1 invalidates the classical non-trapping type, resolvent estimates (*cf.* [2]), which imply the local smoothing effect (3). Indeed, one may see V as a trapping potential, as it concentrates the energy of a sequence of quasi-modes on the pole 0. In similar settings, the equivalence between a non-trapping type assumption and the local smoothing effect is by now well understood (see for example [3]). As will be clear in the proof, this trapping property is a consequence of the very fast oscillations of V near 0.

(iii) It will be clear in the proof of the theorem that one may construct quasi-modes of infinite order (*i.e.*, such that(8) holds with $O(\lambda_n^{-\infty})$ instead of $O(\lambda_n^{j-N})$) by taking a singularity just a bit stronger:

$$\frac{|\log r|^{2+\varepsilon}}{Cr^2} \leq V(r) \leq \frac{C|\log r|^{2+\varepsilon}}{r^2}, \quad \varepsilon > 0.$$

With a still stronger singularity, one may force f_n to be exponentially decreasing in $-\lambda_n$.

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(iv) Classical Strichartz estimates

$$\|U\|_{L^p(]0,T[,L^q(\mathbf{R}^d))} \leq C\|U_0\|_{L^2(\mathbf{R}^d)}, \quad p > 2, \ \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

are invalidated by (9), with the exception of the case $q = 2$, $p = +\infty$ which is always true by the L^2 conservation law. Likewise, (12) shows that Strichartz estimates with loss of derivative (see [4]) of the form:

$$\|U\|_{L^p(]0,T[,L^q(\mathbf{R}^d))} \leq C\|U_0\|_{D(P^{\sigma/2})}, \quad \frac{1}{2} - \frac{1}{q} > \frac{\sigma}{d}, \ \sigma \in [0,1],$$

are false. In this inequality, the limit case $\sigma/d = 1/2 - 1/q$ is an immediate consequence of the Sobolev inclusion

$$L^q(\mathbf{R}^d) \subset H^\sigma(\mathbf{R}^d) \subset D(P^{\sigma/2}),$$

and of the conservation of the norm $D(P^{\sigma/2})$ of any solution of (2). Notice that this last inclusion implies that (12) is still true in usual Sobolev spaces, *i.e.*, when $D(P^{\sigma/2})$ is replaced by $H^\sigma(\mathbf{R}^n)$.

(v) The potential V and quasi-modes u_n being of compact support as close to 0 as desired, the preceding counter-example is still valid in any regular domain for a Laplace operator with Dirichlet or Neumann boundary conditions.

(vi) It is also possible to deduce from Theorem 1 negative results on the solutions of the wave equation (see the remark at the end of the proof of Corollary 1).

$$\partial_t^2 u + Pu = 0. \tag{13}$$

For example, the usual Strichartz and dispersive estimates, and the uniform decay of local energy do not hold. For positive results in this direction, see [13], or the recent work of D'Ancona and Pierfelice [8], in dimension 3. Their assumption on V, taken in a critical Kato class, is very close to the one of Rodniansky and Schlag [12].

(vii) Examples of linear Schrödinger equations which do not admit the classical dispersion inequality or Strichartz estimates are given in [1,6]. In these two cases, the particular behavior of the equation arises from the metric which defines the Laplace operator. Our construction is close to the one of C. Castro and E. Zuazua in [6], where the authors investigate the critical regularity of the metric defining the Laplace operator in order to get certain control properties on the wave and Schrödinger equations. Let us mention that the idea to use quasi-modes in relation with Strichartz inequalities is due to M. Zworski (see [15]).

Finally, we would like to point out that for the potential V introduced in the theorem, one may define the operator P without ambiguity. The operator $-\Delta+V$ has a natural meaning on $C_0^\infty(\mathbf{R}^d\backslash\{0\})$, and is essentially self-adjoint on this space, under a positivity assumption implied by (5). We call P its unique self-adjoint extension which is of course positive. We refer to Reed and Simon [11, Th X.30] for any precision.

Section 2 of the paper is devoted to the proof of Corollary 1, and Section 3 to the proof of Theorem 1.

2 Proof of Corollary 1

According to Hölder's inequality, for any function v on $\mathbf{R}^d$, with a finite volume support,

$$\|v\|_{L^{q_0}} \leq B^{1/q_0-1/q}\|v\|_{L^q}, \quad q > q_0, \tag{14}$$

where B is the volume of the support. Let $U_n(t) = e^{-i\lambda_n^2 t}u_n$. Then

$$i\partial_t U_n - PU_n = -e^{-i\lambda_n^2 t} f_n, \quad U_{n\restriction t=0} = u_n,$$

$$U_n(t) = e^{-itP}u_n + i\int_0^t e^{-i\lambda_n^2 s}e^{i(s-t)P} f_n ds. \tag{15}$$

Assume that (9) is false. According to (15):

$$\int_0^T \|\chi U_n(t)\|_{L^q(\mathbf{R}^d)}\, dt \leq \int_0^T \|\chi e^{-itP}u_n\|_{L^q(\mathbf{R}^d)}\, dt + \int_0^T \int_0^t \|\chi e^{i(s-t)P} f_n\|_{L^q(\mathbf{R}^d)}\, dsdt.$$

Since the support of u_n concentrates in 0, with a volume of order $(\log \lambda_n/\lambda_n)^d$, and χ does not vanish in 0, the left term of this inequality is, using (14) and taking n large enough, greater than

$$\frac{1}{C}\int_0^T \|u_n\|_{L^q(\mathbf{R}^d)}\, dt \geq \frac{1}{C}\Big(\frac{\log \lambda_n}{\lambda_n}\Big)^{d(1/q-1/q_0)} \|u_n\|_{L^{q_0}(\mathbf{R}^d)}.$$

Using the negation of (9), the right term is dominated by

$$\|u_n\|_{L^{q_0}(\mathbf{R}^d)} + \|f_n\|_{L^{q_0}(\mathbf{R}^d)}.$$

Hence, using (8) we obtain,

$$\Big(\frac{\lambda_n}{\log \lambda_n}\Big)^{d(1/q_0-1/q)} \|u_n\|_{L^{q_0}} = O\left(\|u_n\|_{L^{q_0}} + \lambda_n^{-N}\right),$$

which leads to the announced contradiction since $N \geq 1$ and $q > q_0$, the norm of u_n in L^{q_0} being greater than 1 by (7).

By Sobolev inequalities, (9) implies (10), taking in (9) $q_0 = 2$ and $q > 2$ close enough to 2. So (10) holds.

Let us assume that (11) is not true. Then by (15) we get

$$\begin{aligned}\|\chi U_n(T)\|_{L^q(\mathbf{R}^d)} &\leq C\|u_n\|_{L^{q'}(\mathbf{R}^d)} + \int_0^T \|\chi e^{-iTP}e^{isP} f_n\|_{L^1(\mathbf{R}^d)}\, ds \\ &\leq C\Big\{ \|u_n\|_{L^{q'}(\mathbf{R}^d)} + \int_0^T \|e^{isP} f_n\|_{L^{q'}(\mathbf{R}^d)}\, ds\Big\} \\ &\leq C\Big\{ \|u_n\|_{L^{q'}(\mathbf{R}^d)} + T\|f_n\|_{L^2(\mathbf{R}^d)}\Big\}.\end{aligned}$$

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To obtain the second inequality, we have bounded the $L^{q'}$ norm on the support of f_n by the L^2 norm, up to a multiplicative constant, and we have used the fact that e^{isP} is an isometry on L^2. The contradiction is again a simple consequence of (14) and the fact that $q' < q$.

We shall now prove (12). Otherwise, we would have, by (15),

$$\int_0^T \|U_n(t)\|_{L^q(\mathbf{R}^d)}\,dt \le C\Big\{ \|u_n\|_{D(P^{\sigma/2})} + \int_0^T \int_0^t \|e^{i(s-t)P} f_n\|_{D(P^{\sigma/2})}\,dsdt \Big\}. \tag{16}$$

Of course, $e^{i(s-t)P}$ maps isometrically the domain of $P^{\sigma/2}$ onto itself. We have,

$$\|f_n\|^2_{D(P^{1/2})} = \int |\nabla f_n(x)|^2\,dx + \int (1 + V(x))|f_n(x)|^2\,dx.$$

On the support of f_n, as on that of u_n, we have $|x| \ge \frac{\log \lambda_n}{C\lambda_n}$. Using the superior bound of V in (5) on the support of f_n, we get

$$|V(r)| \le C\frac{\lambda_n^2}{(\log \lambda_n)^2}(\log\log \lambda_n - \log \lambda_n)^2 \le C\lambda_n^2.$$

It follows using the bound (8) on the semi-norms of f_n and $N \ge 2$ that

$$\|f_n\|_{D(P^{1/2})} \le C\big(\|\nabla f_n\|_{L^2} + \lambda_n \|f_n\|_{L^2}\big) \le C\lambda_n^{-1}. \tag{17}$$

Furthermore, the equation $Pu_n - \lambda_n^2 u_n = f_n$ implies, using once again the bounds (7) and (8) on the norms of u_n and f_n,

$$\|u_n\|^2_{D(P^{1/2})} \le \|f_n\|_{L^2}\|u_n\|_{L^2} + \lambda_n^2\|u_n\|^2_{L^2} \le C\lambda_n^2\|u_n\|^2_{L^2}.$$

By interpolation on the norms of the left-hand side, and since $0 \le \sigma \le 1$, we get

$$\|u_n\|_{D(P^{\sigma/2})} \le C\lambda_n^{\sigma}\|u_n\|_{L^2}. \tag{18}$$

According to (17), (18) and the inequality (16),

$$\|u_n\|_{L^q} \le C\lambda_n^{\sigma}\|u_n\|_{L^2} + o(1).$$

Hence, with (14),

$$\Big(\frac{\lambda_n}{\log \lambda_n}\Big)^{d/2-d/q}\|u_n\|_{L^2} \le C\lambda_n^{\sigma}\|u_n\|_{L^2} + o(1),$$

which is absurd since $\frac{d}{2} - \frac{d}{q} > \sigma$.

Remark To get a similar result on the wave equation (13), replace $e^{-i\lambda_n^2 t}u_n$ by $\sin(\lambda_n t)u_n$, which is an approximate solution of (13).

3 Proof of the Theorem

Denote by r the Euclidean norm of x and by $'$ the radial derivative $\frac{d}{dr}$. We would like to find radial functions V, f_n, u_n such that

$$f_n(r) = -u_n''(r) - \frac{d-1}{r}u_n'(r) + V(r)u_n(r) - \lambda_n^2 u_n(r), \tag{19}$$

with f_n small and λ_n diverging to infinity. We shall first change functions to get rid of the first-order derivative in this equation. Set

$$u_n = r^{-\frac{d-1}{2}} v_n, \quad f_n = r^{-\frac{d-1}{2}} g_n, \quad W = V + \frac{d^2 - 4d + 3}{4r^2}. \tag{20}$$

Thus (19) becomes

$$g_n = -v_n'' + W v_n - \lambda_n^2 v_n. \tag{19'}$$

Let

$$y(s) = e^{-\sqrt{s^2+1}}, \quad b(s) = -\frac{1}{(s^2+1)^{3/2}} + \frac{s^2}{s^2+1},$$

which are C^∞ solutions of the equation on $\mathbf{R}$:

$$-y''(s) + b(s)y(s) = 0 \tag{21}$$

$$y(s) > 0, \forall j \in \mathbb{N}, \quad |y^{(j)}(s)| \leq C_j e^{-|s|} \tag{22}$$

$$|b(s)| \leq 1. \tag{23}$$

We shall write

$$y_{\omega,a}(r) = y(\omega(r-a)), \quad b_{\omega,a}(r) = \omega^2 b(\omega(r-a)),$$

where a and ω are two real parameters. We have

$$-y_{\omega,a}'' + b_{\omega,a} y_{\omega,a} = 0; \tag{21'}$$

Let $q(\lambda)$ be a strictly increasing positive function defined in a neighborhood of $+\infty$ such that

$$\lim_{\lambda\to+\infty} q(\lambda) = +\infty, \tag{24}$$

$$\lim_{\lambda\to+\infty} \frac{q(\lambda)}{\lambda} = 0. \tag{25}$$

Let $(\lambda_n)_{n\geq n_0}$ be the sequence defined by the equations

$$10^n = q(\lambda_n),$$

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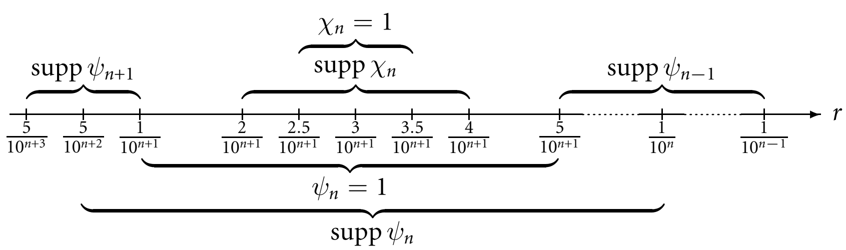

Figure 1: Cutoff functions near $3/10^{n+1}$

so that λ_n diverges faster to $+\infty$ than 10^n.

Choose a cutoff function χ: $\chi \in C_0^\infty(]-1,1[)$, $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Set

$$\psi_n(r) = \chi(10^n r) - \chi(10^{n+1} r),$$
$$\chi_n(r) = \chi\Big(10^{n+1}\Big(r - \frac{3}{10^{n+1}}\Big)\Big),$$

The ψ_n's form a partition of unity near 0. The support of each χ_n is included in that of ψ_n, away from those of the ψ_j's, $j \neq n$ (see Figure 1):

$$\operatorname{supp}\psi_n \subset \Big[\frac{5}{10^{n+2}}, \frac{1}{10^n}\Big], \quad \frac{1}{10^{n+1}} \le r \le \frac{5}{10^{n+1}} \Rightarrow \psi_n(r) = 1,$$

(26)
$$\sum_{n \ge n_0} \psi_n(r) = \chi(10^{n_0} r),$$

(27)
$$\operatorname{supp}\chi_n \subset \Big[\frac{2}{10^{n+1}}, \frac{4}{10^{n+1}}\Big] \subset \{\psi_n = 1\},$$

(28)
$$\frac{25}{10^{n+2}} \le r \le \frac{35}{10^{n+2}} \Rightarrow \chi_n(r) = 1.$$

We shall denote by y_n and b_n the following functions:

(29)
$$y_n(r) = y_{\frac{\lambda_n}{2}, \frac{3}{10^{n+1}}}(r) = y\Big(\frac{\lambda_n}{2}\Big(r - \frac{3}{10^{n+1}}\Big)\Big),$$

(30)
$$b_n(r) = b_{\frac{\lambda_n}{2}, \frac{3}{10^{n+1}}}(r) = \frac{\lambda_n^2}{4} b\Big(\frac{\lambda_n}{2}\Big(r - \frac{3}{10^{n+1}}\Big)\Big).$$

Each of the functions $-b_n$ may be seen as a potential well which (by equation $(21')$) concentrates the energy of y_n near $3/10^{n+1}$. We shall construct W so that the equation $g_n(r) = 0$ is exactly $-v_n'' + b_n v_n = 0$ on a small interval (including the support of v_n) around the point $3/10^{n+1}$. The size of this interval will be of the same order as 10^{-n}, smaller than the equation parameter $\omega = \lambda_n/2$. We shall choose v_n as a cutoff of y_n.

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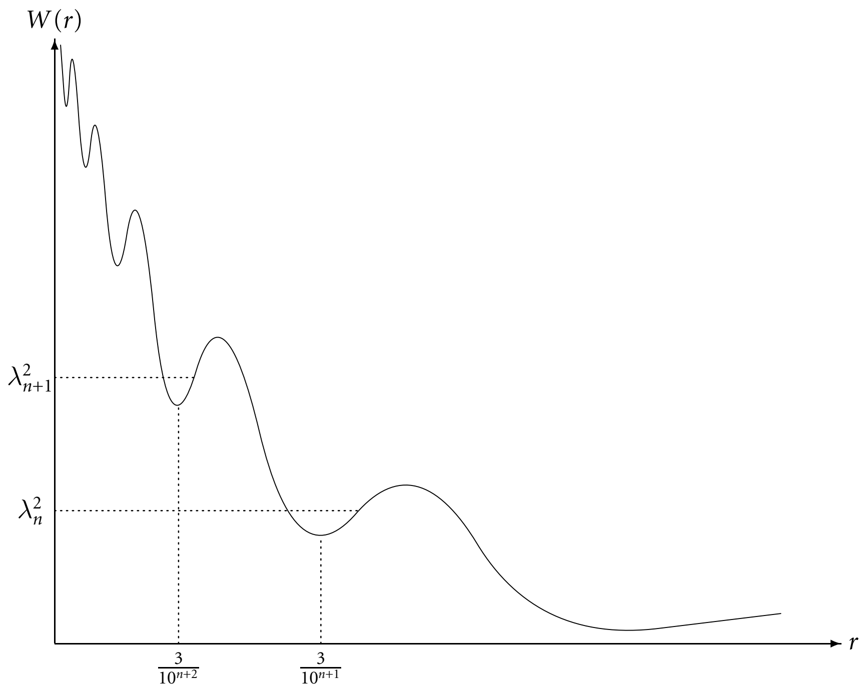

Figure 2: The potential W

The exponential decay of y, combined with the difference of order between these two scales will induce small enough error terms for (7) and (8) to hold. Set

$$W(r)=\sum_{n\geq n_0}\psi_n(r)(b_n(r)+\lambda_n^2), \tag{31}$$

$$v_n(r)=\alpha_n y_n(r)\chi_n(r), \tag{32}$$

where the α_n's are strictly positive constants yet to be determined (see Figure 2). The functions g_n, u_n, f_n and the potential V are thus defined by (19′) and (20).

Lemma 1 *The following inequalities hold:*

$$\forall j\in\mathbb{N},\ \exists C>0,\ \Big\|\frac{d^j f_n}{dr^j}\Big\|_{L^\infty(\mathbf{R}^d)}\leq C\alpha_n\lambda_n^{1+j}(q(\lambda_n))^{\frac{d+1}{2}}e^{-\lambda_n/20q(\lambda_n)} \tag{33}$$

$$\|u_n\|_{L^1(\mathbf{R}^d)}\geq\frac{\alpha_n}{C}\lambda_n^{-1}(q(\lambda_n))^{-\frac{d-1}{2}}. \tag{34}$$

Proof According to (27), W is equal to $b_n+\lambda_n^2$ on the support of χ_n. Hence,

$$g_n(r)=-v_n''(r)+b_n(r)v_n(r). \tag{35}$$

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Thus, using the equation on y_n given by (21′),

$$f_n(r) = r^{-\frac{d-1}{2}} g_n(r) = r^{-\frac{d-1}{2}} \alpha_n \left(y_n(r)\chi_n''(r) + 2y_n'(r)\chi_n'(r) \right).$$

The derivative of f_n of order j is then of the form

$$\frac{d^j f_n}{dr^j} = \sum_{j_1+j_2+j_3=j+1} \beta_{j_1,j_2,j_3} \frac{d^{j_1}}{dr^{j_1}} y_n \frac{d^{j_2}}{dr^{j_2}} \chi_n' \frac{d^{j_3}}{dr^{j_3}} r^{-\frac{d-1}{2}}. \tag{36}$$

On the support of χ_n',

$$\left| r - \frac{3}{10^{n+1}} \right| > \frac{1}{10^{n+1}} = \frac{1}{10q(\lambda_n)}.$$

So according to the bounds (22) of y and its derivatives,

$$\begin{aligned} |y_n^{(j_1)}(r)| &\leq C\lambda_n^{j_1} e^{-\frac{\lambda_n}{2}\left| r - \frac{3}{10^{n+1}} \right|} \\ &\leq C\lambda_n^{j_1} e^{-\frac{\lambda_n}{20q(\lambda_n)}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} |\chi_n^{(j_2+1)}| &\leq C\left(q(\lambda_n)\right)^{j_2+1} \leq C\lambda_n^{j_2} q(\lambda_n) \\ \left| \left(\frac{d}{dr}\right)^{j_3} \left(r^{-\frac{d-1}{2}} \right) \right| &\leq C(q(\lambda_n))^{\frac{d-1}{2}+j_3} \leq C\lambda_n^{j_3} (q(\lambda_n))^{\frac{d-1}{2}}, \end{aligned}$$

for on the support of χ_n, $r \geq \frac{1}{10^{n+1}}$. These three inequalities, together with (36), imply (33). We shall now prove (34). By the definition of y_n,

$$\frac{\lambda_n}{2}\left| r - \frac{3}{10^n} \right| \leq \frac{1}{2} \Rightarrow y_n(r) \geq m = \sup_{|s|\leq \frac{1}{2}} |y(s)|.$$

Furthermore, if r is as above, and n big enough, then $\chi_n(r) = 1$ and so

$$u_n(r) = r^{-\frac{d-1}{2}} v_n(r) = r^{-\frac{d-1}{2}} \alpha_n y_n(r).$$

Hence,

$$\|u_n\|_{L^1} \geq m\alpha_n \int_{\lambda_n\left| r - \frac{3}{10^n} \right| \leq 1} r^{-\frac{d-1}{2}} r^{d-1}\, dr \geq \frac{\alpha_n}{C} (10^{-n})^{\frac{d-1}{2}} \lambda_n^{-1}. \qquad \blacksquare$$

Choose $M > 1$ and set

$$q(\lambda) = \frac{\lambda}{M \log \lambda}. \tag{37}$$

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The positive function q is strictly increasing for big λ's and satisfies (24) and (25). In addition, bounding $q(\lambda_n)$ from above by λ_n, Lemma 1 implies

$$\Big\| \frac{d^j f_n}{dr^j} \Big\|_{L^\infty} \leq C\alpha_n \lambda_n^{\frac{d+3}{2}+j-\frac{M}{20}},$$

$$\|u_n\|_{L^1} \geq \frac{\alpha_n}{C} \lambda_n^{\frac{d+1}{2}},$$

with new constants C, which may depend on M. So the conditions (7) and (8) of the theorem are satisfied if the constants M and α_n are well chosen. The support of u_n is that of χ_n, which is of the desired form

$$\Big\{ c_1 \frac{\log \lambda_n}{\lambda_n} \leq r \leq c_2 \frac{\log \lambda_n}{\lambda_n} \Big\}$$

if the support of χ is taken to be a segment.

The assertion (5) on the potential remains to be checked. We have the following approximation of the inverse function of q:

Lemma 2 *Let q be defined by* (37). *Then*

$$q(\lambda)\log(q(\lambda)) \sim \frac{\lambda}{M}, \ \lambda \to +\infty.$$

Proof

$$q(\lambda)\log q(\lambda) = q(\lambda)\log\Big(\frac{\lambda}{M\log\lambda}\Big)$$
$$= q(\lambda)(\log\lambda - \log\log\lambda - \log M) \sim q(\lambda)\log\lambda,$$

when λ goes to infinity. ■

On the support of ψ_n,

$$\frac{1}{20r} \leq q(\lambda_n) \leq \frac{1}{r},$$

$$\frac{1}{20r}|\log r + \log 20| \leq q(\lambda_n)\log(q(\lambda_n)) \leq \frac{1}{r}|\log r|.$$

(We used that $s \mapsto s\log s$ is an increasing function for $s > e^{-1}$). Using Lemma 2 and taking r close enough to 0 we get

$$C^{-1}\frac{1}{r}|\log r| \leq \lambda_n \leq C\frac{1}{r}|\log r|.$$

Thus, by the definition (31) of W, $|b_n|$ being bounded from above by $\lambda_n^2/4$,

$$C^{-1}\frac{|\log r|^2}{r^2}\sum_{n\geq n_0}\psi_n(r) \leq W(r) \leq C\frac{|\log r|^2}{r^2}\sum_{n\geq n_0}\psi_n(r),$$

which implies, with (26), the inequality (5) on the potential W. Taking a greater C if necessary, we get that V satisfies the same inequality.

Remarks

(i) To get quasi-modes of infinite order, we would have taken $q(\lambda) = \frac{\lambda}{(\log \lambda)^{1+\varepsilon}}$ and suitably modified Lemma 2.

(ii) In the preceding construction, we could have replaced y and b by any nontrivial solutions of the equation (21) satisfying the bounds (22) and (23).

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