RESEARCH ARTICLE

Curves on K3 surfaces in divisibility 2

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Abstract

We prove a conjecture of Maulik, Pandharipande and Thomas expressing the Gromov–Witten invariants of K3 surfaces for divisibility 2 curve classes in all genera in terms of weakly holomorphic quasi-modular forms of level 2. Then we establish the holomorphic anomaly equation in divisibility 2 in all genera. Our approach involves a refined boundary induction, relying on the top tautological group of the moduli space of smooth curves, together with a degeneration formula for the reduced virtual fundamental class with imprimitive curve classes. We use double ramification relations with target variety as a new tool to prove the initial condition. The relationship between the holomorphic anomaly equation for higher divisibility and the conjectural multiple cover formula of Oberdieck and Pandharipande is discussed in detail and illustrated with several examples.

Contents

1. Introduction

Let S be a complex nonsingular projective K3 surface and $\beta \in H_2(S, \mathbb{Z})$ be an effective curve class. Gromov–Witten invariants of S are defined via intersection theory on the moduli space $\overline{M}_{g,n}(S, \beta)$ of stable maps from n-pointed genus g curves to S. This moduli space comes with a virtual fundamental class. However, the virtual class vanishes for $\beta \neq 0$, so instead we use the *reduced class*[1]

$$[\overline{M}_{g,n}(S, \beta)]^{red} \in A_{g+n}(\overline{M}_{g,n}(S, \beta), \mathbb{Q}) .$$

For integers $a_i \geq 0$ and cohomology classes $\gamma_i \in H^*(S, \mathbb{Q})$, we define

$$\left\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \right\rangle_{g, \beta}^{S} = \int_{[\overline{M}_{g,n}(S, \beta)]^{red}} \prod_{i=1}^{n} \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i) ,$$

where $\mathrm{ev}_i \colon \overline{M}_{g,n}(S, \beta) \to S$ is the evaluation at the ith marking and ψ_i is the cotangent class at the ith marking. By the deformation invariance of the reduced class, the invariant depends only on the norm $\langle \beta, \beta \rangle$ and the divisibility of the curve class β.

1.1. Quasi-modularity

Gromov–Witten invariants of K3 surfaces for primitive curve classes have been well understood since the seminal paper by Maulik, Pandharipande and Thomas [29]. The invariants are coefficients of weakly holomorphic[2] quasi-modular forms with pole of order at most 1 [29, Theorem 4]. For imprimitive curve classes, quasi-modularity is conjectured with the level structure [29, Section 7.5].

Quasi-modularity can be stated in a precise sense via elliptic K3 surfaces. Let

$$\pi \colon S \to \mathbb{P}^1$$

be an elliptic K3 surface with a section, and denote by $B, F \in H_2(S, \mathbb{Z})$ the class of the section and a fibre, respectively. For any $m \geq 1$, one defines the descendent potential

$$\mathsf{F}_{g,m}\left(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n)\right) = \sum_{h \geq 0} \left\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \right\rangle_{g, mB+hF}^{S} q^{h-m}.$$

Note that this generating series involves curve classes $mB + hF$ of different divisibilities, bounded by m.

It is convenient to use the following homogenised insertions, which will lead to quasi-modular forms of pure weight. Let $1 \in H^0(S)$ and $\mathsf{p} \in H^4(S)$ be the identity and the point class, respectively. Denote

$$W = B + F \in H^2(S),$$

let

$$U = \mathbb{Q}\langle F, W \rangle \subset H^2(S)$$

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[1]We will identify this class with its image under the cycle class map $A_* \to H_{2*}$.
[2]Weakly holomorphic means holomorphic on the upper half-plane, with possible pole at the cusp $i\infty$.

be the hyperbolic plane in $H^2(S)$ and let $U^\perp \subset H^2(S)$ be its orthogonal complement with respect to the intersection form. We consider only second cohomology classes which are pure with respect to the decomposition

$$H^2(S, \mathbb{Q}) \cong \mathbb{Q}\langle F \rangle \oplus \mathbb{Q}\langle W \rangle \oplus U^\perp.$$

Following [8, Section 4.6], define a modified degree function $\underline{\deg}$ by

$$\underline{\deg}(\gamma) = \begin{cases} 2 & \text{if } \gamma = W \text{ or } \mathsf{p}, \\ 1 & \text{if } \gamma \in U^\perp, \\ 0 & \text{if } \gamma = F \text{ or } 1. \end{cases}$$

For $m \geq 1$, consider the *Hecke congruence subgroup of level m*

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod m \right\}$$

and let $\mathsf{QMod}(m)$ be the space of quasi-modular forms for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$. Let $\Delta(q)$ be the modular discriminant

$$\Delta(q) = q \prod_{n \geq 1}(1 - q^n)^{24}.$$

Our first main result proves the level 2 quasi-modularity of $\mathsf{F}_{g,2}$, previously conjectured by Maulik, Pandharipande and Thomas [29, Section 7.5].

Theorem 1. *Let $\gamma_1, \ldots, \gamma_n \in H^*(S)$ be homogeneous on the modified degree function $\underline{\deg}$. Then $\mathsf{F}_{g,2}$ is the Fourier expansion of a quasi-modular form*

$$\mathsf{F}_{g,2}\big(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n)\big) \in \frac{1}{\Delta(q)^2}\mathsf{QMod}(2)$$

of weight $2g - 12 + \sum_i \underline{\deg}(\gamma_i)$ with pole at $q = 0$ of order at most 2.

1.2. Holomorphic anomaly equation

In the physics literature, the (conjectural) *holomorphic anomaly equation* [4, 5] predicts hidden structures of the Gromov–Witten partition function associated to Calabi–Yau varieties. In the past few years, there has been extensive work to prove the holomorphic anomaly equation in many cases: local $\mathbb{P}^2$ [26], the quintic threefold [11, 16], K3 surfaces with primitive curve classes [33], elliptic fibration [34] and $\mathbb{P}^2$ relative to a smooth cubic [6].

Every quasi-modular form for $\Gamma_0(m)$ can be written uniquely as a polynomial in C_2 with coefficients which are modular forms for $\Gamma_0(m)$ [18, Proposition 1]. Here,

$$C_2(q) = -\frac{1}{24}E_2(q)$$

is the renormalised second Eisenstein series. Assuming quasi-modularity, the holomorphic anomaly equation fixes the nonholomorphic parameter of the Gromov–Witten partition function of K3 surfaces in terms of lower weight partition functions: it computes the derivative of $\mathsf{F}_{g,m}$ with respect to the C_2 variable. See [33] for the proof of the holomorphic anomaly equation for K3 surfaces with primitive curve classes and [34] for the holomorphic anomaly equation associated to elliptic fibrations.

Define an endomorphism [33, Section 0.6]

$$\sigma : H^*(S^2) \to H^*(S^2)$$

by the following assignments:

$$\sigma(\gamma \boxtimes \gamma') = 0$$

if γ or $\gamma' \in H^0(S) \oplus \mathbb{Q}\langle F \rangle \oplus H^4(S)$, and for $\alpha, \alpha' \in U^\perp$,

$$\sigma(W \boxtimes W) = \Delta_{U^\perp}, \ \sigma(W \boxtimes \alpha) = -\alpha \boxtimes F,$$
$$\sigma(\alpha \boxtimes W) = -F \boxtimes \alpha, \ \sigma(\alpha, \alpha') = \langle \alpha, \alpha' \rangle F \boxtimes F,$$

where $\Delta_{U^\perp}$ denotes the diagonal class for the intersection pairing on $U^\perp$. We will view σ as the exterior product $\sigma_1 \boxtimes \sigma_2$ via Künneth decomposition.

Recall the virtual fundamental class for trivial curve classes, which will play a role for the holomorphic anomaly equation. For $\beta = 0$ we have an isomorphism

$$\overline{M}_{g,n}(S,0) \cong \overline{M}_{g,n} \times S,$$

and the virtual class is given by

$$[\overline{M}_{g,n}(S,0)]^{vir} = \begin{cases} [\overline{M}_{0,n} \times S] & \text{if } g = 0, \\ c_2(S) \cap [\overline{M}_{1,n} \times S] & \text{if } g = 1, \\ 0 & \text{if } g \geq 2. \end{cases}$$

Also, consider the pullback under the morphism $\pi \colon S \to \mathbb{P}^1$ of the diagonal class of $\mathbb{P}^1$:

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee.$$

Define the generating series[3]

$$
\begin{aligned}
&\mathsf{H}_{g,m}\big(\alpha; \gamma_1, \ldots, \gamma_n\big) \\
&= \mathsf{F}_{g-1,m}\big(\alpha; \gamma_1, \ldots, \gamma_n, \Delta_{\mathbb{P}^1}\big) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\ldots,n\}=I_1 \sqcup I_2 \\ i \in \{1,2\}}} \mathsf{F}_{g_1,m}\big(\alpha_{I_1}; \gamma_{I_1}, \delta_i\big) \mathsf{F}_{g_2}^{vir}\big(\alpha_{I_2}; \gamma_{I_2}, \delta_i^\vee\big) \\
&\quad - 2 \sum_{i=1}^{n} \mathsf{F}_{g,m}\big(\alpha\psi_i; \gamma_1, \ldots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \ldots, \gamma_n\big) \\
&\quad + \frac{20}{m} \sum_{i=1}^{n} \langle \gamma_i, F \rangle \mathsf{F}_{g,m}\big(\alpha; \gamma_1, \ldots, \gamma_{i-1}, F, \gamma_{i+1}, \ldots, \gamma_n\big) \\
&\quad - \frac{2}{m} \sum_{i<j} \mathsf{F}_{g,m}\big(\alpha; \gamma_1, \ldots, \underbrace{\sigma_1(\gamma_i, \gamma_j)}_{i\text{th}}, \ldots, \underbrace{\sigma_2(\gamma_i, \gamma_j)}_{j\text{th}}, \ldots, \gamma_n\big),
\end{aligned}
\tag{1}
$$

where F^{vir} denotes the generating series for virtual fundamental class. In most cases this term vanishes. The equation takes almost the same form for arbitrary m, except the last two terms acquire a factor of $\frac{1}{m}$. The appearance of these factors is explained in Section 4 (see also Example 22). We conjecture that the holomorphic anomaly equation has the following form:

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[3]Here, instead of descendent insertions we use a tautological class $\alpha \in R^*(\overline{M}_{g,n})$; see Section 3.2

Conjecture 2.

$$\frac{d}{dC_2} \mathsf{F}_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = \mathsf{H}_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n). \tag{2}$$

For primitive curve classes, the holomorphic anomaly equation is proven in [33]. In higher divisiblity, it is precisely equation (2) that would be implied by the conjectural multiple cover formula for imprimitve Gromow–Witten invariants of K3 surfaces. We explain this in the following section. We prove Conjecture 2 unconditionally when $m = 2$:

Theorem 3. *For any* $g \geq 0$,

$$\frac{d}{dC_2} \mathsf{F}_{g,2}(\alpha; \gamma_1, \ldots, \gamma_n) = \mathsf{H}_{g,2}(\alpha; \gamma_1, \ldots, \gamma_n). \tag{3}$$

1.3. Multiple cover formula

Motivated by the Katz–Klemm–Vafa (KKV) formula, Oberdieck and Pandharipande conjectured a formula which computes imprimitive invariants from the primitive invariants:

Conjecture 4 ([32], Conjecture C2). *For a primitive curve class β,*

$$\left\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \right\rangle_{g, m\beta} \tag{4}$$
$$= \sum_{d|m} d^{2g-3+\deg} \left\langle \tau_{a_1}(\varphi_{d,m}(\gamma_1)) \ldots \tau_{a_n}(\varphi_{d,m}(\gamma_n)) \right\rangle_{g, \varphi_{d,m}(\frac{m}{d}\beta)}.$$

The invariants on the right-hand side are with respect to primitive curve classes.[4] Assuming this formula, we can deduce the holomorphic anomaly equation:

Proposition 5. *Let $m \geq 1$. Assume that the multiple cover formula* (4) *holds for all curve classes of divisibility $d \mid m$ and all descendent insertions. Then the holomorphic anomaly equation* (2) *holds.*

Given this proposition, it seems a natural strategy to prove the multiple cover formula in divisibility 2 and deduce, as a consequence, the holomorphic anomaly equation. Indeed, our method does follow this logic for $m = 2$ and for low genus: we verify the multiple cover formula for $g \leq 2$ (see Example 35). For higher genus, however, our method does not seem suitable. Instead, our proof of Theorem 1 provides an algorithm, based on the degeneration to the normal cone of a smooth elliptic fibre $E \subset S$, to reduce divisibility 2 invariants to low-genus invariants for which the multiple cover formula is known.[5] The degeneration formula intertwines invariants of S with invariants of $\mathbb{P}^1 \times E$ in a nontrivial way. This phenomenon is illustrated in Example 35 for the genus 2 invariants

$$\left\langle \tau_0(\mathsf{p})^2 \right\rangle_{2, 2\beta}.$$

1.4. Hecke operator

In Section 3 we apply Conjecture 4 to an elliptic K3 surface to deduce a conjectural multiple cover formula for the descendent potentials $\mathsf{F}_{g,m}$. The multiple cover formula for any divisibility m is then simply a *Hecke operator of the wrong weight* acting on the primitive potential $\mathsf{F}_{g,1}$. Indeed, the weight of $\mathsf{F}_{g,1}$ (and conjecturally of $\mathsf{F}_{g,m}$) is $2g - 12 + \underline{\deg}$, whereas the Hecke operator has the weight of a descendent potential attached to elliptic curves, namely $2g - 2 + \underline{\deg}$. This operator can be expressed in terms of Hecke operators (of the correct weight) and translation $q \mapsto q^d$. Together with the holomorphic

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[4]Section 3 contains all relevant definitions.
[5]The cases in genus 0 and genus 1 are proved by Lee and Leung in [24, 25]. Their proof involves a degeneration formula in symplectic geometry which is not possible in algebraic geometry. We present an algebro-geometric approach using the KKV formula.

anomaly equation for primitive curve classes [33], this naturally leads to Conjecture 4 for the holomorphic anomaly equation for higher divisibility.

1.5. *Plan of the paper*

We prove quasi-modularity and the holomorphic anomaly equation by induction on the genus and the number of markings. In Section 2, we discuss Hecke theory for weakly holomorphic quasi-modular forms. This leads to a natural formulation of the multiple cover formula in Section 3 and the imprimitive holomorphic anomaly equation in Section 4. In Section 5, the compatibility of the holomorphic anomaly equation with the degeneration formula is presented. In Section 6, we derive the multiple cover formula, which implies the holomorphic anomaly equation, for genus 0, genus 1 and some genus 2 decendent invariants from the KKV formula. The genus 2 computation relies on double ramification relations with target variety. This result serves as the initial condition for our induction. In Section 7, we use previous results to prove Theorems 1 and 3. The property of the top tautological group $R^{g-1}(M_{g,n})$ reduces higher-genus cases to lower-genus invariants discussed in Section 6.

2. Quasi-modular forms and Hecke operators

We recall basic properties of quasi-modular forms and Hecke operators (see [22, 39], in particular [22, pp. 156–163] and [22, Ch. 3, Section 3]). The Hecke theory for weakly holomorphic quasi-modular forms, however, seems to be less well documented. We thus also include some proofs.

The following operators will play a central role. For any Laurent series

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n \tag{5}$$

and $d \in \mathbb{Z}_{>0}$, we define

$$\mathsf{D}_q f = q\frac{d}{dq} f, \quad \mathsf{B}_d f = \sum_{n=-\infty}^{\infty} a_n q^{dn}, \quad \mathsf{U}_d f = \sum_{n=-\infty}^{\infty} a_{dn} q^n.$$

We will apply these operators to the Laurent series associated to certain modular functions. For this we briefly review the definition of modular forms.

2.1. *Quasi-modular forms*

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the upper half-plane. The group $\mathrm{GL}_2^+(\mathbb{R})$ of real 2×2-matrices with positive determinant acts on $\mathbb{H}$ via

$$A\tau = \frac{a\tau + b}{c\tau + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

Let $f \colon \mathbb{H} \to \mathbb{C}$ be a function and let

$$q = e^{2\pi i \tau}, \quad y = \mathrm{Im}(\tau).$$

For $k \in \mathbb{Z}$ define the *kth slash operator*

$$(f|_k A)(\tau) = \det(A)^{k/2}(c\tau + d)^{-k} f(A\tau).$$

Definition 6. A *quasi-modular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ admitting a Fourier expansion

$$f(q) = \sum_{n=0}^{\infty} a_n q^n, \quad |q| < 1, \tag{6}$$

such that there exist $p \geq 0$ and holomorphic functions f_r, $r = 0, \ldots, p$ satisfying the following conditions:

(i) the (nonholomorphic) function $\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}$ satisfies the transformation law

$$\widehat{f}|_k \gamma = \widehat{f} \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

(ii) $f = f_0$ and

(iii) each f_r has an expansion of the form of equation (6).

If $p = 0$, then f is called a *modular form*. We denote the spaces of modular and quasi-modular forms by Mod and QMod, respectively.

Remark 7. If $\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}$ as with $f_p \neq 0$, then each f_r is a quasi-modular form of weight $k - 2r$ (see [39, Proposition 20]). Moreover, the last one – that is, f_p – is in fact modular (of weight $k - 2p$). The following structural results are well known [39, Proposition 4, Proposition 20]:

$$\mathsf{Mod} = \mathbb{C}[C_4, C_6], \quad \mathsf{QMod} = \mathbb{C}[C_2, C_4, C_6],$$

where

$$C_{2i}(q) = -\frac{B_{2i}}{2i \cdot (2i)!} E_{2i}(q)$$

is the renormalised $2i$th Eisenstein series. Condition (i) in Definition 6 defines the space AHM of *almost holomorphic modular forms*, and the assignment $\widehat{f} \mapsto f$ is an isomorphism

$$\mathsf{AHM} \to \mathsf{QMod}.$$

Under this map, differentiation with respect to $\frac{1}{8\pi y}$ corresponds to differentiation with respect to C_2.

The modular functions considered in this paper will usually have poles at the cusp $\tau = i\infty$ corresponding to $q = 0$. We will refer to these functions as *weakly holomorphic* with pole of specified order. We want to clarify this terminology in the context of quasi-modular forms.

Definition 8. A function f is said to be *weakly holomorphic quasi-modular with pole of order at most $m \geq 0$* if f satisfies the conditions in Definition 6, except that each f_r is allowed to have a pole at the cusp $i\infty$ of order at most m. If $p = 0$, then f is called a weakly holomorphic modular form with pole of order at most m.

By parallel arguments as in [39, Proposition 20], the assertions in Remark 7 hold analogously for weakly holomorphic quasi-modular forms. In particular, f_p is weakly holomorphic modular with pole of order at most m. The space of weakly holomorphic modular forms is generated by $\frac{1}{\Delta}$ over Mod, where

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

is the modular discriminant.[6] As a consequence,

$$f_p \in \frac{1}{\Delta^m} \mathsf{Mod},$$

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[6] See [14], where the authors examine an explicit basis of the space of weakly holomorphic modular forms.

and since f_p is of weight $k - 2p$ (and there are no nonzero modular forms of negative weight), we have $k \geq 2p - 12m$.

For quasi-modular forms we include the following observation:

Lemma 9. *The space of weakly holomorphic quasi-modular forms with pole of order at most m is given by*

$$\frac{1}{\Delta^m} \mathsf{QMod}.$$

Proof. Let f be a weakly holomorphic form with pole of order at most m and weight k, and let

$$\widehat{f} = \sum_{r=0}^{p} f_r y^{-r},$$

with $f = f_0$. Multiplying by Δ^m, we have, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$(\Delta^m \widehat{f})|_{k+12m} \gamma = (\Delta^m)|_{12m} \gamma \cdot (\widehat{f})|_k \gamma = \Delta^m \widehat{f}.$$

Since each $\Delta^m f_r$ is holomorphic at $i\infty$, this proves

$$f \in \frac{1}{\Delta^m} \mathsf{QMod}.$$

An analogous argument shows that the quotient of any quasi-modular form by Δ^m defines a weakly holomorphic quasi-modular form with pole of order at most m. □

2.2. Hecke operators

Let $m \in \mathbb{N}$ and consider the set of integral matrices of determinant m

$$H_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = m \right\}.$$

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on H_m by left multiplication. The classical *Hecke operators* T_m acting on modular forms f of weight k are defined by [39, Section 4.1]:

$$\mathsf{T}_m f = m^{k/2-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash H_m} f|_k \gamma.$$

This definition is equivalent to [22, Ch. 3, Proposition 38]:

$$\mathsf{T}_m = \sum_{ad=m} a^{k-1} \mathsf{B}_a \mathsf{U}_d. \tag{7}$$

The action of equation (7) naturally extends to the action of the q-expansion of weakly holomorphic quasi-modular forms. We prove that the action again defines a weakly holomorphic quasi-modular form. For simplicity (we will only use this case), we restrict to the case when f has a pole of order at most 1.

Lemma 10. *Let $f \in \frac{1}{\Delta} \mathsf{QMod}$ be of weight k. Then $\mathsf{T}_m f$ is a weakly holomorphic quasi-modular form of weight k with pole of order at most m – that is,*

$$\mathsf{T}_m f \in \frac{1}{\Delta^m} \mathsf{QMod}.$$

Proof. In [31] it is shown that T_m defines a map QMod $\to$ QMod preserving the weight. We briefly recall the key arguments for $f \in$ QMod. The definition of quasi-modular forms is equivalent to the condition

$$(f|_k\gamma)(\tau) = \sum_{r=0}^{p}\left(\frac{c}{c\tau+d}\right)^r f_r(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

where f_r are as in Definition 6.[7] Defining a modification of the slash operator for quasi-modular forms,[8]

$$(f||_k A)(\tau) = \sum_{r=0}^{p}(-c)^r(c\tau+d)^r(f_r|_k A)(\tau) \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

and the quasi-modularity is then equivalent to

$$f||_k \gamma = f \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

This leads to a parallel treatment of Hecke operators as in the classical context of modular forms. By [31, Proposition 2] we have

$$f||_k(\gamma A) = f||_k A, \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}), A \in \mathrm{GL}_2^+(\mathbb{R})$$

and we define

$$\mathsf{T}_m f = m^{k/2-1} \sum_{A \in \mathrm{SL}_2(\mathbb{Z}) \backslash H_m} f||_k A.$$

This definition is then independent of a choice of representatives of $\mathrm{SL}_2(\mathbb{Z}) \setminus H_m$. To conclude that $\mathsf{T}_m f$ is a quasi-modular form, we would like to argue that it is invariant under $(-)||_k\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This statement, however, is not sensible at the moment,[9] because the definition of $(-)||_k\gamma$ relies on the existence of associated functions f_r. This technicality is resolved in [31, Section 2.4, 2.5] by considering a certain period domain $\mathcal{P}$ and identifying quasi-modular forms as holomorphic functions on $\mathcal{P}$, which are left $\mathrm{SL}_2(\mathbb{Z})$-invariant and satisfy a transformation property for a right action of the subgroup of upper triangular matrices. The domain $\mathcal{P}$ is contained in $\mathrm{GL}_2(\mathbb{C})$ and contains the upper-half plane $\mathbb{H}$. The actions are given by left and right multiplication. The argument carries over to weakly holomorphic quasi-modular forms without change.

A particular set of representatives for $\mathrm{SL}_2(\mathbb{Z}) \setminus H_m$ is given by

$$\left\{\gamma_b = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{N}, ad = m, 0 \le b < d\right\}.$$

Note that $(-)||_k\gamma_b = (-)|_k\gamma_b$, because the terms for $r > 0$ vanish. Since

$$\mathsf{U}_d f(\tau) = \frac{1}{d} \sum_{0 \le b < d} f\left(\frac{\tau+b}{d}\right),$$

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[7]This notion is called 'differential modular form' in [31]. As pointed out in [39, Section 5.3], it is equivalent to a quasi-modular form.

[8]This definition differs from [31, Equation 12] by a factor m^{-p}, where p is the depth of f. Our definition of the Hecke operator differs by the same factor.

[9]We are grateful to the referee for pointing out this subtle detail.

we thus recover equation (7):

$$\mathsf{T}_m f(\tau) = m^{k/2-1} \sum_{\substack{ad=m \\ 0 \le b < d}} d^{-k} m^{k/2} f\left(\frac{a\tau + b}{d}\right)$$

$$= \sum_{ad=m} a^{k-1} \mathsf{B}_a \mathsf{U}_d f(\tau).$$

For weakly holomorphic quasi-modular forms $f \in \frac{1}{\Delta}\mathsf{QMod}$, we follow the same proof. The difference here is that the functions f_r are allowed to have simple poles at $i\infty$. The slash operator $(-)\|_k$, however, may turn a simple pole into a pole of higher order. For $(-)\|_k \gamma_b$ this order is bounded by m. As a consequence, $\mathsf{T}_m f$ is weakly holomorphic quasi-modular with pole of order at most m.   □

For our study of the multiple cover formula in Section 3, we will require a more flexible notion, where the exponent is not necessarily related to the weight. The action of this operator will preserve the weight of weakly holomorphic quasi-modular forms; it will, however, introduce poles and level structure.

Definition 11. For $\ell \in \mathbb{Z}$, we define

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} a^{\ell-1} \mathsf{B}_a \mathsf{U}_d.$$

The operator $\mathsf{T}_{m,\ell}$ is simply the mth Hecke operator of weight ℓ, which we let act on functions of weight k. By Möbius inversion we may rewrite each of them in terms of the other (see [1, Section 2.7]). For this, let μ be the Möbius function.

Lemma 12. *The action of* $\mathsf{T}_{m,\ell}$ *on weakly holomorphic quasi-modular forms of weight k is given by*

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} c_{k,\ell}(a) \mathsf{B}_a \mathsf{T}_d,$$

where

$$c_{k,\ell}(a) = \sum_{r|a} r^{\ell-1} \mu\left(\frac{a}{r}\right) \left(\frac{a}{r}\right)^{k-1}.$$

Proof. The formula for $c_{k,\ell}$ can be rewritten as

$$c_{k,\ell} = \mathrm{Id}_{\ell-1} \star (\mu \cdot \mathrm{Id}_{k-1}),$$

where $\mathrm{Id}_{\ell-1}(n) = n^{\ell-1}$ is the $(\ell-1)$th power function and $\star$ denotes Dirichlet convolution – that is, for functions g, h we have

$$(g \star h)(m) = \sum_{ad=m} g(a)h(d).$$

Note also that B is multiplicative with respect to composition – that is, for $e \mid a$ we have $\mathsf{B}_a = \mathsf{B}_e \mathsf{B}_{\frac{a}{e}}$ – and therefore

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} a^{\ell-1}\mathsf{B}_a\mathsf{U}_d$$

$$= \sum_{ad=m} \big(\mathrm{Id}_{\ell-1} \star (\mu \cdot \mathrm{Id}_{k-1}) \star \mathrm{Id}_{k-1}\big)(a)\mathsf{B}_a\mathsf{U}_d$$

$$= \sum_{ad=m} \left(\sum_{e|a} c_{k,\ell}(e)\left(\frac{a}{e}\right)^{k-1}\right)\mathsf{B}_a\mathsf{U}_d$$

$$= \sum_{uw=m} c_{k,\ell}(u)\mathsf{B}_u\left(\sum_{v|w} v^{k-1}\mathsf{B}_v\mathsf{U}_{\frac{w}{v}}\right)$$

$$= \sum_{uw=m} c_{k,\ell}(u)\mathsf{B}_u\mathsf{T}_w. \qquad \square$$

As a consequence, we obtain the following result. Here we let $\mathsf{Mod}(m)$ and $\mathsf{QMod}(m)$ be the spaces of modular and quasi-modular forms, respectively, for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$ (see the introduction).

Proposition 13. *Let $f \in \frac{1}{\Delta}\mathsf{QMod}$ be of weight k; then $\mathsf{T}_{m,\ell}f$ is a weakly holomorphic quasi-modular of weight k with pole of order at most m for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$:*

$$\mathsf{T}_{m,\ell}f \in \frac{1}{\Delta^m}\mathsf{QMod}(m).$$

Proof. We use the formula in Lemma 12 and treat each summand separately. By Lemma 9, each $\mathsf{T}_d f$ satisfies

$$\mathsf{T}_d f \in \frac{1}{\Delta^d}\mathsf{QMod}.$$

The action of B_a raises $q \mapsto q^a$, or equivalently $\tau \mapsto a\tau$, so it maps QMod to $\mathsf{QMod}(a)$ (see [22, Ch. 3, Proposition 17]). Therefore,

$$\mathsf{B}_a\mathsf{T}_d f \in \frac{1}{\Delta(q^a)^d}\mathsf{QMod}(a).$$

Finally, the weakly holomorphic modular form for $\Gamma_0(a)$ defined by

$$\frac{\Delta(q)^a}{\Delta(q^a)}$$

is in fact holomorphic at $i\infty$ – that is, contained in $\mathsf{Mod}(a)$. Hence the same is true for its dth power, and we find

$$\mathsf{B}_a\mathsf{T}_d f \in \frac{1}{\Delta^m}\mathsf{QMod}(a).$$

which concludes the proof since, $\mathsf{QMod}(a) \subset \mathsf{QMod}(m)$. $\qquad \square$

For later reference, we list the following basic commutator relations between the foregoing operators acting on weakly holomorphic quasi-modular forms f of weight k. Recall that the algebra $\mathsf{QMod}(m)$ is freely generated by the Eisenstein series C_2 over the algebra $\mathsf{Mod}(m)$ of modular forms. Formal differentiation with respect to C_2 is therefore well defined.

Lemma 14. *Let $d, e \in \mathbb{N}$ and $\ell \in \mathbb{Z}$; then*

(i) $\mathsf{B}_d\mathsf{B}_e = \mathsf{B}_{de} = \mathsf{B}_e\mathsf{B}_d,$

(ii) $\mathsf{U}_d\mathsf{U}_e = \mathsf{U}_{de} = \mathsf{U}_e\mathsf{U}_d,$

(iii) $D_q B_d = d B_d D_q, \quad U_d D_q = d D_q U_d,$

(iv) $T_{m,\ell+2} D_q = m D_q T_{m,\ell},$

(v) $\frac{d}{dC_2} T_{m,\ell+2} = m T_{m,\ell} \frac{d}{dC_2},$

(vi) $[\frac{d}{dC_2}, D_q] = -2k.$

Proof. The proof for (i)–(iv) follows directly from the definition. For (v) one may use the fact that under the isomorphism $\widehat{f} \mapsto f$, the differentiation $\frac{d}{dC_2}$ corresponds to differentiation with respect to $\frac{1}{8\pi y}$ (see Remark 7). Statement (v) is then checked as an identity of Laurent series in q with polynomial coefficients in y^{-1}. The commutator relation (vi) is well known (see, for example, [39, Section 5.3]). □

3. Multiple cover formula

This section contains a discussion of the multiple cover formula. We start by recalling the conjecture formulated in [32]. Then, we study the conjecture for the descendent potentials associated to elliptic K3 surfaces. The result is expressed in terms of Hecke operators. The discussion naturally leads to a candidate for the holomorphic anomaly equation in higher divisibility. We conclude with a proof of the multiple cover formula in the fibre direction.

3.1. Multiple cover formula

Let S be a nonsingular projective K3 surface, $\beta \in H_2(S, \mathbb{Z})$ be a *primitive* effective curve class, $m \in \mathbb{N}$ and $d \mid m$ be a divisor of m. The formula proposed by Oberdieck and Pandharipande involves a choice of a real isometry

$$\varphi_{d,m} \colon \left(H^2(S, \mathbb{R}), \langle, \rangle \right) \to \left(H^2(S_d, \mathbb{R}), \langle, \rangle \right)$$

between two K3 surfaces such that

$$\varphi_{d,m}\left(\frac{m}{d} \beta \right) \in H_2(S_d, \mathbb{Z})$$

is a primitive effective curve class.[10] The second author proved in [9] that such an isometry can always be found, and Gromov–Witten invariants are in fact independent of the choice of isometry.

Consider integers $a_i \in \mathbb{N}$ and cohomology classes $\gamma_i \in H^*(S, \mathbb{Q})$ and let deg $= \sum \deg(\gamma_i)$. Then the conjectured multiple cover formula [32, Conjecture C2], identical to our Conjecture 4, is

$$\left\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \right\rangle_{g, m\beta}$$
$$= \sum_{d \mid m} d^{2g-3+\deg} \left\langle \tau_{a_1}(\varphi_{d,m}(\gamma_1)) \dots \tau_{a_n}(\varphi_{d,m}(\gamma_n)) \right\rangle_{g, \varphi_{d,m}\left(\frac{m}{d}\beta\right)}.$$

Let S be an elliptic K3 surface with a section.[11]. The full (reduced) Gromov–Witten theory of K3 surfaces is captured by S with curve class $mB + hF$ via standard deformation arguments using the Torelli theorem. In fact, the multiple cover conjecture can be captured entirely via S as well: we may choose *the same* $S_d = S$ for any d dividing m and h. For $l \in \mathbb{Q}^*$ we define

$$\phi_l \colon H^*(S, \mathbb{Q}) \to H^*(S, \mathbb{Q})$$

---

[10]We view curve classes also as cohomology classes under the natural isomorphism $H_2(S, \mathbb{Z}) \cong H^2(S, \mathbb{Z})$.

[11]Notations here are as in Section 1 In particular, we use the modified degree function <u>deg</u>.

acting on $U = \mathbb{Q}\langle F, W \rangle$ as

$$\phi_l(F) = \frac{1}{l} F, \qquad \phi_l(W) = lW,$$

and trivially on the orthogonal complement $U^\perp$. For $d \mid m$ and $d \mid h$ we may choose $\varphi_{d,m}$ as $\phi_{\frac{d}{m}}$:

$$\phi_{\frac{d}{m}} \left(\frac{m}{d} B + \frac{h}{d} F \right) = B + \left(\frac{m(h-m)}{d^2} + 1 \right) F \quad \text{in } H_2(S, \mathbb{Z}),$$

which is a primitive curve class.

Altering the curve class via the isometry ϕ therefore results in additional factors of $\frac{d}{m}$ or $\frac{m}{d}$ while keeping the descendent insertions unchanged. This explains the change in exponents

$$2g - 3 + \deg \longleftrightarrow 2g - 3 + \underline{\deg}$$

and the factor $m^{\deg - \underline{\deg}}$ in the multiple cover formula for the descendent potential (see later). We use the operator $\mathsf{T}_{m,\ell}$ introduced in Definition 11. As pointed out in Section 1.4, this is the mth Hecke operator for functions of weight ℓ, which we let act on $\mathsf{F}_{g,1}$ (which has weight $2g - 12 + \underline{\deg}$). Before stating the conjecture, we want to discuss the role of tautological classes and compatibility with respect to restriction to boundary strata.

3.2. Compatibility I

We will find it convenient to use pullbacks of tautological classes from $\overline{M}_{g,n}$ instead of ψ-classes on $\overline{M}_{g,n}(S, \beta)$. For $2g - 2 + n > 0$, let

$$R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$$

be the tautological ring of $\overline{M}_{g,n}$. For a tautological class $\alpha \in R^*(\overline{M}_{g,n})$, we consider the invariants

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \pi^* \alpha \cup \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i),$$

where $\pi \colon \overline{M}_{g,n}(S, \beta) \to \overline{M}_{g,n}$ is the stabilisation morphism. We write

$$\mathsf{F}_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = \sum_{h \geq 0} \langle \alpha; \gamma_1, \ldots, \gamma_n \rangle_{g, mB+hF} \, q^{h-m}$$

for the generating series in divisibility m. By the usual trading of cotangent line classes, these generating series are related to the ones defined via cotangent classes on $\overline{M}_{g,n}(S, \beta)$. Any monomial in ψ- and κ-classes can be written, after adding markings, as a product of ψ-classes. This procedure leaves deg and $\underline{\deg}$ unchanged. Before stating the multiple cover formula later, we explain the compatibility with respect to restriction to boundary strata in $\overline{M}_{g,n}(S, \beta)$.

A crucial point for this compatibility is the splitting behaviour of the reduced class. Consider the pullback of the boundary divisor

$$\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$$

under the stabilisation morphism π. Let α be the push-forward of a tautological class (we will omit push-forwards in the notation later). By the restriction property of the reduced class, we obtain

$$\mathsf{F}_{g,m}(\alpha; \gamma) = \mathsf{F}_{g-1,m}(\alpha; \gamma \Delta_S).$$

Then the compatibility follows from two facts. First, for the diagonal class Δ_S we have

$$\left(\deg - \underline{\deg} \right)(\Delta_S) = 0,$$

and thus the factor $m^{\deg - \underline{\deg}}$ in Conjecture 15 remains unchanged. Second, we have $\underline{\deg}(\Delta_S) = 2$, which precisely offsets the genus reduction from g to $g - 1$ in the formula

$$\ell = 2g - 2 + \underline{\deg}.$$

Next, consider the pullback of the boundary divisor

$$\overline{M}_{g_1, n_1+1} \times \overline{M}_{g_2, n_2+1} \to \overline{M}_{g,n}$$

under the stabilisation morphism π. Let

$$\alpha = \alpha_1 \boxtimes \alpha_2, \quad \{1, \ldots, n\} = I_1 \cup I_2, \quad \gamma = \gamma_1 \boxtimes \gamma_2,$$

be the push-forward of the product of tautological classes, the splitting of markings and the splitting of the insertions, respectively. The Künneth decomposition of the class of the diagonal is denoted by

$$[\Delta_S] = \sum_j \Delta_j \boxtimes \Delta^j.$$

The splitting property implies that

$$F_{g,m}(\alpha; \gamma) = \sum_{m_1+m_2=m} \sum_j \left(F_{g_1, m_1}(\alpha_1; \gamma_{I_1} \Delta_j) \cdot F^{vir}_{g_2, m_2}(\alpha_1; \gamma_{I_1} \Delta^j) \right.$$

$$\left. + F^{vir}_{g_1, m_1}(\alpha_1; \gamma_{I_1} \Delta_j) \cdot F_{g_2, m_2}(\alpha_1; \gamma_{I_1} \Delta^j) \right).$$

The virtual class for nonzero curve classes vanishes, and thus the contribution F^{vir} is a number. As a consequence, no nontrivial products of generating series appear when we use boundary expressions. By a similar consideration, using the deg and $\underline{\deg}$ for the diagonal class, we find that the multiple cover formula is compatible with respect to this boundary divisor as well. We can now state the multiple cover formula for the generating series with tautological classes:

Conjecture 15. *For $\underline{\deg}$-homogeneous $\gamma_i \in H^*(S, \mathbb{Q})$,*

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = m^{\deg - \underline{\deg}} T_{m,\ell}\left(F_{g,1}(\alpha; \gamma_1, \ldots, \gamma_n) \right),$$

where $\deg = \sum \deg(\gamma_i)$, $\underline{\deg} = \sum \underline{\deg}(\gamma_i)$ *and* $\ell = 2g - 2 + \underline{\deg}$.

Based on the foregoing discussion, the same formula is conjectured for the potential

$$F_{g,m}\left(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \right).$$

We now show that our presentation of the multiple cover formula is equivalent to the original formula.

Lemma 16. *Conjecture 4 for all $d \mid m$ is equivalent to Conjecture 15 for m.*

Proof. By the deformation invariance of the reduced class, the Gromov–Witten invariants for arbitrary curve classes are fully captured by an elliptic K3 surface with a section. The primitive curve classes are $B + hF \in H_2(S, \mathbb{Z})$. Taking the coefficient of q^{mh-m} in Conjecture 15 gives a multiple cover formula

for the curve class $mB + mhF$ which matches the formula in Conjecture 4. It is the other implication which we have to justify.

The generating series $\mathsf{F}_{g,m}$ involves curve classes $mB + hF$ of different divisibilities bounded by m. We apply Conjecture 4 to each invariant and use the isometries ϕ. Note that each appearance of $\gamma_i = F$ introduces a factor of $\frac{m}{d}$, while each appearance of $\gamma_i = W$ gives $\frac{d}{m}$. Moreover,

$$|\{i \mid \gamma_i = F\}| - |\{i \mid \gamma_i = W\}| = \deg -\underline{\deg},$$

and therefore

$$
\begin{aligned}
\mathsf{F}_{g,m}\left(\alpha; \gamma_1, \ldots, \gamma_n\right) &= \sum_{h \geq 0} \left\langle \alpha; \gamma_1, \ldots, \gamma_n \right\rangle_{g, mB+hF} q^{h-m} \\
&= \sum_{\substack{h \geq 0 \ d \mid m \\ d \mid h}} d^{2g-3+\deg} \left(\frac{m}{d}\right)^{\deg -\underline{\deg}} \left\langle \alpha; \gamma_1, \ldots, \gamma_n \right\rangle_{g, B+\left(\frac{m(h-m)}{d^2}+1\right)F} q^{h-m} \\
&= m^{\deg -\underline{\deg}} \sum_{d \mid m} d^{2g-3+\deg} \left(\sum_{h \geq 0} \left\langle \alpha; \gamma_1, \ldots, \gamma_n \right\rangle_{g, B+\left(\frac{m}{d}(h-\frac{m}{d})+1\right)F} \left(q^d\right)^{h-\frac{m}{d}}\right) \\
&= m^{\deg -\underline{\deg}} \sum_{d \mid m} d^{2g-3+\deg} \left(\mathsf{B}_d \mathsf{U}_{\frac{m}{d}} \sum_{h \geq 0} \left\langle \alpha; \gamma_1, \ldots, \gamma_n \right\rangle_{g, B+hF} q^{h-1}\right) \\
&= m^{\deg -\underline{\deg}} \sum_{d \mid m} d^{2g-3+\deg} \mathsf{B}_d \mathsf{U}_{\frac{m}{d}} \mathsf{F}_{g,1}\left(\alpha; \gamma_1, \ldots, \gamma_n\right) \\
&= m^{\deg -\underline{\deg}} \mathsf{T}_{m,\ell}\left(\mathsf{F}_{g,1}\left(\alpha; \gamma_1, \ldots, \gamma_n\right)\right). \qquad \square
\end{aligned}
$$

As a direct consequence, the multiple cover formula implies level m quasi-modularity.

Proposition 17. *If the generating series $\mathsf{F}_{g,m}$ satisfies the multiple cover formula, it satisfies the quasi-modularity conjecture. More precisely,*

$$\mathsf{F}_{g,m} \in \frac{1}{\Delta(q)^m} \mathsf{QMod}(m).$$

Proof. The descendent potentials for primitive curve classes are weakly holomorphic quasi-modular with pole of order at most 1 and weight $2g - 12 + \underline{\deg}$ (see [29], Theorem 4] and [8, Theorem 9]). The claim thus follows from Proposition 13. $\square$

3.3. Multiple cover formula in the fibre direction

When the curve class is a multiple of the fibre class F, the multiple cover formula reduces to a property of the Gromov–Witten invariant of elliptic curves. Relevant properties are conjectured in [38].

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section and let $\beta = mF$. By Section 7, Case 1, we may assume at least one of the insertions is the point class $\gamma_1 = \mathsf{p}$ and $g \geq 1$. Let

$$\iota \colon E \hookrightarrow S$$

be the inclusion of a fibre representing the class F. Since the point class is represented by a transverse intersection of E and the section B, the Gromov–Witten theory of S localises to the Gromov–Witten theory of E with the curve class mE. Computation of the obstruction bundle shows that the invariant is of the form

$$\left\langle \tau_{a_1}(\mathsf{p}) \tau_{a_2}(\gamma_2) \ldots \tau_{a_n}(\gamma_n) \right\rangle_{g, mF}^S = \left\langle \lambda_{g-1}; \tau_{a_1}(\omega) \tau_{a_2}(\iota^* \gamma_2) \ldots \tau_{a_n}(\iota^* \gamma_n) \right\rangle_{g, mE}^E,$$

where $\lambda_{g-1} = c_{g-1}(\mathbb{E}_g)$. In particular, if $\gamma_i \in \mathbb{Q}\langle F \rangle \oplus U^{\perp} \oplus \mathbb{Q}\langle \mathsf{p} \rangle$, the invariant vanishes. Consider the generating series

$$\mathsf{F}_g^E\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big) = \sum_{m \geq 0} \big\langle \lambda_{g-1}; \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle_{g,mE}^E q^m,$$

where $\gamma_i = 1$ or ω and $\sum a_i + \sum \deg(\gamma_i) = g - 1 + n$.

The generating series F_g^E has a simple description in terms of Eisenstein series. The following formula is conjectured in [38]:

Lemma 18. *For $g \geq 1$,*

$$\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big) = \frac{g!}{2^{g-1}} C_{2g}.$$

Proof. In [38, Proposition 4.4.7], this formula is given under assumption of the Virasoro constraint for $\mathbb{P}^1 \times E$. The Virasoro constraint for any toric bundle over a nonsingular variety which satisfies the Virasoro constraint is proven in [13]. Combining this result with the Virasoro constraint for elliptic curves [35], the result follows. $\square$

When $\beta = mF$, Conjecture 4 is equivalent to the following proposition:

Proposition 19. *There exists $c \in \mathbb{Q}$ such that*

$$\mathsf{F}_g^E\big(\tau_{a_1}(\omega)\dots\tau_{a_r}(\omega)\tau_{a_{r+1}}(1)\dots\tau_{a_{r'}}(1)\big) = c\mathsf{D}_q^{r-1}\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big).$$

Proof. Boundary strata with a vertex of genus less than g do not contribute, because the invariants involving λ_h vanish on $\overline{M}_{g,n}(E, m)$ when $h \geq g$. If $r' > r$, then $\sum a_i \geq g$ and we can reduce to the case when $r' = r$ by topological recursion on the ψ-monomial in $R^{\geq g}(\overline{M}_{g,n})$ [23]. If $r' = r$, then $\sum a_i = g - 1$ and a similar argument as in Section 7, Case 3, can be applied. Therefore F_g^E is proportional to

$$\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\tau_0(\omega)^{r-1}\big) = \mathsf{D}_q^{r-1}\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big),$$

where the equality comes from the divisor equation. $\square$

Remark 20. One can find a closed formula for the constant $c \in \mathbb{Q}$ by integrating tautological classes on $\overline{M}_{g,n}$.

4. Holomorphic anomaly equation

This section contains a proof of Proposition 5. We derive the holomorphic anomaly equation for $m \geq 1$ from the conjectural multiple cover formula, such that both are compatible.[12] It turns out that the equation is almost identical to the one in the primitive case. Additional factors appear only in the last two terms, which are specific to K3 surfaces. We refer to [34, Section 7.3] for explanations of the appearance of these terms.

Proof of Proposition 5. Set $\gamma_1, \dots, \gamma_n \in H^*(S)$ with

$$\deg = \sum_i \deg(\gamma_i), \quad \underline{\deg} = \sum_i \underline{\deg}(\gamma_i).$$

We will simply write γ to denote $\gamma_1, \dots, \gamma_n$. Assume that the multiple cover formula (4) holds for all divisors $d \mid m$ and all descendent insertions. Using Lemma 16, Conjecture 15 also holds. By Proposition 17, the descendent potentials are quasi-modular forms of level m and we can consider the

---

[12]We should point out that this derivation should be lifted to the cycle-valued holomorphic anomaly equation. Tautological classes play no role here.

$\frac{d}{dC_2}$-derivative. We apply the $\frac{d}{dC_2}$-derivative to Conjecture 15 and use the commutator relations of Lemma 14 to obtain

$$\frac{d}{dC_2}\mathsf{F}_{g,m}(\alpha;\gamma) = \frac{d}{dC_2}\left(m^{\deg-\underline{\deg}}\mathsf{T}_{m,2g-2+\underline{\deg}}\mathsf{F}_{g,1}(\alpha;\gamma)\right)$$

$$= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\frac{d}{dC_2}\mathsf{F}_{g,1}(\alpha;\gamma).$$

We want to explain that the last row precisely recovers the definition of $\mathsf{H}_{g,m}$ in equation (1), after applying the holomorphic anomaly equation for the primitive series [33, Theorem 4]:

$$\frac{d}{dC_2}\mathsf{F}_{g,1}(\alpha;\gamma) = \mathsf{H}_{g,1}(\alpha;\gamma).$$

We do so by explaining how each term of $\mathsf{H}_{g,1}(\alpha;\gamma)$ is affected:

(i) The degree deg of $\mathsf{F}_{g-1,1}(\alpha;\gamma\Delta_{\mathbb{P}^1})$ has increased by one. The genus, however, has dropped by one. Thus, the first term precisely matches the multiple cover formula – that is,

$$\mathsf{F}_{g-1,m}(\alpha;\gamma\Delta_{\mathbb{P}^1}) = m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\left(\mathsf{F}_{g-1,1}(\alpha;\gamma\Delta_{\mathbb{P}^1})\right).$$

(ii) The virtual class is nonzero only for curve class $\beta = 0$ and general $0, 1$ (see Section 1). In these cases, the potential $\mathsf{F}_{g_2}^{vir}$ is simply a number, and the operator $\mathsf{T}_{m,\ell}$ acts nontrivially only on $\mathsf{F}_{g_1,m}$. We distinguish the two cases:

$g_2 = 0$. The virtual class is given by the fundamental class and the integral is given by intersection pairing on S. Nontrivial terms are obtained from $\delta_i^\vee = 1$ or F. If $\delta_i^\vee = 1$, then

$$\deg(\gamma_{I_2}) = \underline{\deg}(\gamma_{I_2}) = 2.$$

The modified degree $\underline{\deg}$ of $\mathsf{F}_{g_1,1}(\alpha_{I_1};\gamma_{I_1}\delta_i)$ has decreased by two, whereas deg has decreased by one (the insertion $\delta_i = F$ contributes $\deg = 1$). The term thus matches the multiple cover formula:

$$\mathsf{F}_{g_1,m}(\alpha_{I_1};\gamma_{I_1}\delta_i)$$
$$= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\left(\mathsf{F}_{g_1,1}(\alpha_{I_1};\gamma_{I_1}\delta_i)\right).$$

If $\delta_i^\vee = F$, then

$$\deg(\gamma_{I_2}) = 1, \quad \underline{\deg}(\gamma_{I_2}) = 2.$$

The modified degree $\underline{\deg}$ of $\mathsf{F}_{g_1,1}(\alpha_{I_1};\gamma_{I_1}\delta_i)$ has decreased by two, whereas deg has decreased by one. The term matches the multiple cover formula.

$g_2 = 1$. The virtual class is given by $c_2(S)$ and the integral is given by intersection pairing on S. Nontrivial terms are obtained only from $\delta_i^\vee = 1$, and

$$\deg(\gamma_{I_2}) = \underline{\deg}(\gamma_{I_2}) = 0.$$

Analogously to case (i), the degree deg of $\mathsf{F}_{g_1,1}(\alpha_{I_1};\gamma_{I_1}\delta_i)$ has increased by one, $\underline{\deg}$ remains unchanged and the genus has dropped by one. The term matches the multiple cover formula.

(iii) The modified degree $\underline{\deg}$ of $\mathsf{F}_{g,1}(\alpha\psi_i;\gamma_1,\ldots,\pi^*\pi_*\gamma_i,\ldots,\gamma_n)$ has decreased by two, whereas deg has decreased by one. Again we find that the term matches the multiple cover formula:

$$\mathsf{F}_{g,m}\big(\alpha\psi_i; \gamma_1, \dots, \pi^*\pi_*\gamma_i, \dots, \gamma_n\big)$$
$$= m^{\deg -\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g,1}\big(\alpha\psi_i; \gamma_1, \dots, \pi^*\pi_*\gamma_i, \dots, \gamma_n\big)\Big).$$

(iv) The degree of $\langle\gamma_i, F\rangle\mathsf{F}_{g,1}(\alpha; \gamma_1, \dots, F, \dots, \gamma_n)$ remains unchanged, whereas $\underline{\deg}$ has decreased by two. An additional factor of $\frac{1}{m}$ therefore appears:

$$\frac{1}{m}\langle\gamma_i, F\rangle\mathsf{F}_{g,m}\big(\alpha; \gamma_1, \dots, F, \dots, \gamma_n\big)$$
$$= m^{\deg -\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\langle\gamma_i, F\rangle\mathsf{F}_{g,1}\big(\alpha; \gamma_1, \dots, F, \dots, \gamma_n\big)\Big).$$

(v) The term $\mathsf{F}_{g,1}\big(\dots, \sigma_1(\gamma_i, \gamma_j), \dots, \sigma_2(\gamma_i, \gamma_j), \dots\big)$ is similar to the previous case: deg remains unchanged, whereas $\underline{\deg}$ has decreased by two, giving rise to an additional factor of $\frac{1}{m}$:

$$\frac{1}{m}\mathsf{F}_{g,m}\big(\gamma_1, \dots, \sigma_1(\gamma_i, \gamma_j), \dots, \sigma_2(\gamma_i, \gamma_j), \dots, \gamma_n\big)$$
$$= m^{\deg -\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g,1}\big(\gamma_1, \dots, \sigma_1(\gamma_i, \gamma_j), \dots, \sigma_2(\gamma_i, \gamma_j), \dots, \gamma_n\big)\Big).$$

We arrive at the level m holomorphic anomaly equation (1) from Section 1.   □

4.1. Divisor equation

For primitive curve classes, it was pointed out in [33, Section 3.6, Case (i)] that the holomorphic anomaly equation in genus 0 is compatible with the divisor equation. For divisibility m, set

$$\frac{d}{d\gamma} = \langle\gamma, F\rangle\mathsf{D}_q + m\langle\gamma, W\rangle, \quad \gamma \in H^2(S).$$

The divisor equation implies that

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\tau_0(\gamma_n)\big)$$
$$= \frac{d}{d\gamma_n}\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)$$
$$+ \sum_{i=1}^{n-1}\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_i-1}(\gamma_i \cup \gamma_n)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big).$$

The compatibility with the divisor equation corresponds to

$$\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\tau_0(\gamma_n)\big)$$
$$= \frac{d}{d\gamma_n}\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)$$
$$- 2k\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big) \tag{8}$$
$$+ \sum_{i=1}^{n-1}\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_i-1}(\gamma_i \cup \gamma_n)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big),$$

where k is the weight of $\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)$ and we have used the commutator relation

$$\left[\frac{d}{dC_2}, \mathsf{D}_q\right] = -2k.$$

The same check as in the primitive case works for arbitrary divisibility. This relies on the fact that the divisor equation for W is the same as applying the differential operator

$$\mathsf{D}_q = q\frac{d}{dq}$$

to the generating series. Indeed, for the curve class $\beta = mB + hF$,

$$\langle \beta, W \rangle = -2m + h + m = h - m,$$

which matches the exponent of q^{h-m} in the generating series $\mathsf{F}_{g,m}$. The divisor equation for F acts as multiplication by m on the generating series.

In Section 7, the refined induction reduces any generating series ultimately to genera 0 and 1. We thus have to justify compatibility of the holomorphic anomaly equation for generating series of the form

$$\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\tau_0(\gamma_1)\ldots\tau_0(\gamma_n)\big), \quad \gamma_i \in H^2(S).$$

This compatibility is true. By Proposition 28, the multiple cover formula, which is compatible with the divisor equation, holds in genus ≤ 1. Thus we also find compatibility for the holomorphic anomaly equation.

Example 21. We consider $\mathsf{F}_{0,m}\big(\tau_0(W)^2\big)$ to illustrate compatibility. To compute $\mathsf{H}_{0,m}$, we use the fact that $\sigma(W \boxtimes W) = U^\perp$, where the endomorphism σ is as defined in Section 1. Since the curve classes are contained in U, application of the divisor equation to a basis of $U^\perp$ implies

$$\mathsf{F}_{0,m}\big(\tau_0(U^\perp)\big) = 0.$$

We find that

$$\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) = -4\mathsf{F}_{0,m}\big(\tau_1(1)\tau_0(W)\big) + \frac{40}{m}\mathsf{F}_{0,m}\big(\tau_0(F)\tau_0(W)\big).$$

In this notation, $\gamma_n = W$ is the second W and $k = -10$ is the weight of $\mathsf{F}_{0,m}\big(\tau_0(W)\big)$. We have to check that

$$\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) = \mathsf{D}_q\mathsf{H}_{0,m}\big(\tau_0(W)\big) + 20\mathsf{F}_{0,m}\big(\tau_0(W)\big).$$

By the dilaton equation, we can verify

$$
\begin{aligned}
&\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) - \mathsf{D}_q\mathsf{H}_{0,m}\big(\tau_0(W)\big)\\
&= -2\mathsf{D}_q\mathsf{F}_{0,m}\big(\tau_1(1)\big) - 4\mathsf{F}_{0,m}\big(\tau_0(W)\big) + \frac{20}{m}\mathsf{F}_{0,m}\big(\tau_0(F)\tau_0(W)\big)\\
&= 4\mathsf{D}_q\mathsf{F}_{0,m}\big(\emptyset\big) - 4\mathsf{D}_q\mathsf{F}_{0,m}\big(\emptyset\big) + 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\\
&= 20\mathsf{F}_{0,m}\big(\tau_0(W)\big).
\end{aligned}
$$

Example 22. The previous example in genus 0 illustrates how the second-to-last term in the holomorphic anomaly equation (2) plays a role. We consider

$$\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(W)\big)$$

to show how the last term – that is, the term involving σ – interacts nontrivially with the other terms.

The corresponding series $H_{1,m}$ are

$$
\begin{aligned}
H_{1,m}\big(\tau_1(W)\tau_0(W)\big) = {} & 2F_{0,m}\big(\tau_1(W)\tau_0(W)\tau_0(1)\tau_0(F)\big) \\
& - 2\Big(F_{1,m}\big(\tau_2(1)\tau_0(W)\big) + F_{1,m}\big(\tau_1(W)\tau_1(1)\big)\Big) \\
& + \frac{20}{m}\Big(F_{1,m}\big(\tau_1(F)\tau_0(W)\big) + F_{1,m}\big(\tau_1(W)\tau_0(F)\big)\Big) \\
& - \frac{2}{m}F_{1,m}\big(\psi_1; \Delta_{U^\perp}\big),
\end{aligned}
$$

$$
\begin{aligned}
H_{1,m}\big(\tau_1(W)\big) = {} & 2F_{0,m}\big(\tau_1(W)\tau_0(1)\tau_0(F)\big) \\
& - 2F_{1,m}\big(\tau_2(1)\big) \\
& + \frac{20}{m}F_{1,m}\big(\tau_1(F)\big).
\end{aligned}
$$

Let $k = -8$ be the weight of $F_{1,m}\big(\tau_1(W)\big)$. Then equation (8) is equivalent to

$$
H_{1,m}\big(\tau_1(W)\tau_0(W)\big) = D_q H_{1,m}\big(\tau_1(W)\big) - 2kF_{1,m}\big(\tau_1(W)\big).
$$

The term $F_{1,m}\big(\psi_1; \Delta_{U^\perp}\big)$ can be computed using

$$
\psi_1 = [\delta_1] + \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,2}),
$$

where $[\delta_0] \in A^1(\overline{M}_{1,2})$ is the class of the push-forward of the fundamental class under the map

$$
\overline{M}_{0,4} \to \overline{M}_{1,2}
$$

gluing the third and fourth markings, and $[\delta_1]$ is the class of the boundary divisor of curves with a rational component carrying both markings. The genus 0 contribution vanishes by the divisor equation. Since the rank of $U^\perp$ is 20, we obtain the genus 1 contribution

$$
F_{1,m}\big(\psi_1; \Delta_{U^\perp}\big) = 20F_{1,m}\big(\tau_0(\mathsf{p})\big).
$$

The divisor equation for F implies that

$$
\frac{20}{m}F_{1,m}\big(\tau_1(W)\tau_0(F)\big) = 20F_{1,m}\big(\tau_1(W)\big) + \frac{20}{m}F_{1,m}\big(\tau_0(\mathsf{p})\big).
$$

We can now verify compatibility by a direct computation using the divisor and dilaton equation:

$$
\begin{aligned}
H_{1,m}\big(\tau_1(W)\tau_0(W)\big) = {} & D_q H_{1,m}\big(\tau_1(W)\big) - 2F_{1,m}\big(\tau_1(W)\big) - 2F_{1,m}\big(\tau_1(W)\tau_1(1)\big) \\
& + \frac{20}{m}F_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{20}{m}F_{1,m}\big(\tau_1(W)\tau_0(F)\big) \\
& - \frac{2}{m}F_{1,m}\big(\psi_1; \Delta_{U^\perp}\big) \\
= {} & D_q H_{1,m}\big(\tau_1(W)\big) - 4F_{1,m}\big(\tau_1(W)\big) \\
& + \frac{20}{m}F_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{20}{m}F_{1,m}\big(\tau_1(W)\tau_0(F)\big) \\
& - \frac{40}{m}F_{1,m}\big(\tau_0(\mathsf{p})\big) \\
= {} & D_q H_{1,m}\big(\tau_1(W)\big) + 16F_{1,m}\big(\tau_1(W)\big).
\end{aligned}
$$

5. Relative holomorphic anomaly equation

In this section, we first state the degeneration formula for the reduced virtual class under the degeneration to the normal cone. For the primitive curve class, the formula is proven in [29]. For the sake of completeness, we summarise a proof for arbitrary divisibility in Appendix A. Then we state the relative holomorphic anomaly equation and prove the compatibility with the degeneration formula.

5.1. Degeneration formula

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. For $m \geq 1$, let $\beta = mB + hF$ be a curve class. Choose a smooth fibre E of $S \to \mathbb{P}^1$. Let $\epsilon \colon \mathcal{S} \to \mathbb{A}^1$ be the total space of the degeneration to the normal cone of E in S. This space corresponds to the degeneration

$$S \rightsquigarrow S \cup_E \mathbb{P}^1 \times E. \tag{9}$$

Over the centre $\iota \colon 0 \hookrightarrow \mathbb{A}^1$, the fibre is $S \cup_E \mathbb{P}^1 \times E$, and over $t \neq 0$, the fibre is isomorphic to S. Let $\overline{M}_{g,n}(\epsilon, \beta)$ be the moduli space of stable maps to the degeneration $\mathcal{S}$. Over $t \neq 0$, this moduli space is isomorphic to $\overline{M}_{g,n}(S, \beta)$, and over $t = 0$, it parametrises stable maps to the expanded target

$$\widetilde{\mathcal{S}}_0 = S \cup_E \mathbb{P}^1 \times E \cup_E \cdots \cup_E \mathbb{P}^1 \times E.$$

Let

$$\nu = (g_1, g_2, n_1, n_2, h_1, h_2)$$

be a splitting of the discrete data g, n, h and let $\beta_i = mB + h_i F$ be the splitting of the curve class. An ordered partition of m

$$\mu = (\mu_1, \ldots, \mu_l)$$

specifies the contact order along the relative divisor E.

Let $l = \text{length}(\mu)$ and let $\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu$ be the fibre product

$$\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu = \overline{M}_{g_1,n_1}(S/E, \beta_1)_\mu \times_{E^l} \overline{M}^\bullet_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_\mu \tag{10}$$

of the boundary evaluations at relative markings,[13] and let

$$\iota_{\nu\mu} \colon \overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu \to \overline{M}_{g,n}(\mathcal{S}_0, \beta)$$

be the finite morphism. Let $\Delta_{E^l} \colon E^l \to E^l \times E^l$ be the diagonal embedding.

Theorem 23. *The reduced virtual class of maps to the degeneration* (9) *satisfies the following properties:*

(i) *For $\iota_t \colon \{t\} \hookrightarrow \mathbb{A}^1$, the Gysin pullback of reduced class is given by*

$$\iota_t^! [\overline{M}_{g,n}(\epsilon, \beta)]^{red} = [\overline{M}_{g,n}(\mathcal{S}_t, \beta)]^{red}.$$

(ii) *For the special fibre,*

$$[\overline{M}_{g,n}(\mathcal{S}_0, \beta)]^{red} = \sum_{\nu,\mu} \frac{\prod_i \mu_i}{l!} \iota_{\nu\mu*} [\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu]^{red}.$$

---

[13]We use $\bullet$ to indicate (possibly) disconnected theory – namely, for each connected component C of the domain curve, the intersection of C with the relative divisor E is nontrivial.

(iii) *On the special fibre, we have the factorisation*

$$[\overline{M}_{g,n}(\mathcal{S}_0, \nu)_{\underline{\mu}}]^{red} = \Delta^!_{E^l}\Big([\overline{M}_{g_1,n_1}(S/E, \beta_1)_{\underline{\mu}}]^{red}$$
$$\times [\overline{M}^{\bullet}_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_{\underline{\mu}}]^{vir}\Big).$$

Proof. When $m \geq 1$, the reduced class of the disconnected moduli space $\overline{M}^{\bullet}_{g,n}(S/E, \beta)$ vanishes on all components parametrising maps with at least two connected components. Therefore, disconnected theory can only appear on the bubble $\mathbb{P}^1 \times E$. The proof is given in Appendix A. $\square$

Denote an ordered cohomology weighted partition by

$$\underline{\mu} = \big((\mu_1, \delta_1), \ldots, (\mu_l, \delta_l)\big), \quad \delta_i \in H^*(E),$$

and let $\omega \in H^2(E)$ be the point class. The descendent potential for the pair (S, E) is defined analogously to the absolute case:

$$\mathsf{F}^{\mathrm{rel}}_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu}) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu} \big\rangle^{S/E}_{g, mB+hF} q^{h-m}.$$

The descendent potential for the pair $(\mathbb{P}^1 \times E, E)$ is defined by

$$\mathsf{G}^{\mathrm{rel},\bullet}_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu}) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu} \big\rangle^{\mathbb{P}^1 \times E/E, \bullet}_{g, mB+hF} q^h.$$

As a corollary, we get the degeneration formula of reduced Gromov–Witten invariants:

Corollary 24. Set $\gamma_1, \ldots, \gamma_n \in H^*(S)$ and choose a lift of these cohomology classes to the total space $\mathcal{S}$. Then

$$\mathsf{F}_{g,m}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n)) = \sum_{\nu} \sum_{\underline{\mu} \neq \underline{\mu}_{\omega}} \frac{\prod_i \mu_i}{l!} \mathsf{F}^{\mathrm{rel}}_{g_1,m}(\ldots \mid \underline{\mu}) \cdot \mathsf{G}^{\mathrm{rel},\bullet}_{g_2,m}(\ldots \mid \underline{\mu}^{\vee}), \qquad (11)$$

where

$$\underline{\mu}^{\vee} = \big((\mu_1, \delta_1^{\vee}), \ldots, (\mu_l, \delta_l^{\vee})\big) \text{ and } \underline{\mu}_{\omega} = \big((\mu_1, \omega), \ldots, (\mu_l, \omega)\big).$$

Proof. By Theorem 23, we are left to prove that the relative profile $\underline{\mu}_{\omega}$ on S/E has vanishing contribution. Let x be the intersection of the section of the elliptic fibration and the fibre E. We consider (E, x) as an abelian variety. Let K be the kernel of the following morphism between abelian varieties:

$$E^l \to \mathrm{Pic}^0(E), \quad (x_i)_i \mapsto \mathcal{O}_E\Big(\sum_i \mu_i(x_i - x)\Big).$$

Consider a stable map f from a curve C to an expanded degeneration of S/E. The equality $f_*[C] = \beta_1$ (after push-forward to S) in $H_2(S, \mathbb{Z})$ lifts to a rational equivalence of line bundles on S because the cycle-class map

$$c_1 \colon \mathrm{Pic}(S) \to H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$$

is injective. Intersecting with the relative divisor, the two line bundles are, respectively, $\mathcal{O}_E(\sum \mu_i x_i)$ and $\mathcal{O}_E(mx)$. Thus, we see that the evaluation map $\overline{M}_{g_1,n_1}(S/E, \beta_1) \to E^l$ factors through K. Since $K \subset E^l$ has codimension 1, a generic point on E^l does not lie on K, and thus the contribution from the relative profile $\underline{\mu}_{\omega}$ vanishes. $\square$

5.2. *Relative holomorphic anomaly equations*

Assuming quasi-modularity, we have two ways to compute the derivative of $\mathsf{F}_{g,m}$ with respect to C_2:

(i) Apply the degeneration formula from Corollary 24, together with the holomorphic anomaly equations for (S, E) and $(\mathbb{P}^1 \times E, E)$.
(ii) Apply the holomorphic anomaly equation (3) for S, followed by the degeneration formula for each term.

We argue that both methods yield the same result. This compatibility is parallel to the compatibility proved in [34, Section 4.6]. We first state the holomorphic anomaly equations for the relevant relative geometries.

Relative $(\mathbb{P}^1 \times \mathbf{E}, \mathbf{E})$

Consider $\pi\colon \mathbb{P}^1 \times E \to \mathbb{P}^1$ as a trivial elliptic fibration over $\mathbb{P}^1$. For the pair $(\mathbb{P}^1 \times E, E)$, the holomorphic anomaly equation holds for cycle-valued generating series [34]. The equation for descendent potentials can thus be obtained by integrating against tautological classes $\alpha \in R^*(\overline{M}_{g,n})$. For insertions $\gamma_i \in H^*(\mathbb{P}^1 \times E, \mathbb{Q})$, we will simply write γ. Let $\underline{\mu} = \big((\mu_1, \delta_1), \dots, (\mu_l, \delta_l)\big)$ and $\underline{\mu}'$ be ordered cohomology weighted partitions. We denote by

$$\mathsf{G}^{\sim,\bullet}_{g,m}\big(\underline{\mu} \mid \alpha; \gamma \mid \underline{\mu}'\big) = \sum_{h \geq 0} \big\langle \underline{\mu} \mid \alpha; \gamma \mid \underline{\mu}' \big\rangle^{\mathbb{P}^1 \times E, \sim, \bullet}_{g, m\mathbb{P}^1 + hE} q^h$$

the disconnected rubber generating series for $\mathbb{P}^1 \times E$ relative to divisors at 0 and ∞. Let $\Delta_E \subset E \times E$ be the class of the diagonal. Define the generating series

$$
\begin{aligned}
&\mathsf{P}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma \mid \underline{\mu}\big) \\
&= \mathsf{G}^{\mathrm{rel},\bullet}_{g-1,m}\big(\alpha; \gamma, \Delta_{\mathbb{P}^1} \mid \underline{\mu}\big) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,n\}=I_1 \sqcup I_2 \\ \forall i \in I_2 : \gamma_i \in H^2(E) \\ h \geq 0}} \sum_{\substack{b;b_1,\dots,b_h \\ l_1,\dots,l_h}} \frac{\prod_{i=1}^{h} b_i}{h!} \mathsf{G}^{\mathrm{rel},\bullet}_{g_1,m}\big(\alpha_{I_1}; \gamma_{I_1} \mid ((b,1), (b_i, \Delta_{E,l_i})_{i=1}^{h})\big) \\
&\quad \times \mathsf{G}^{\sim,\bullet}_{g_2,m}\big(((b,1), (b_i, \Delta^{\vee}_{E,\ell_i})_{i=1}^{h}) \mid \alpha_{I_2}; \gamma_{I_2} \mid \underline{\mu}\big) \\
&\quad - 2 \sum_{i=1}^{n} \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha\psi_i; \gamma_1, \dots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \dots, \gamma_n \mid \underline{\mu}\big) \\
&\quad - 2 \sum_{i=1}^{l} \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma \mid (\mu_1, \delta_1), \dots, (\mu_i, \psi^{\mathrm{rel}}_i \pi^*\pi_*\delta_i), \dots, (\mu_l, \delta_l)\big),
\end{aligned}
$$

where ψ^{rel}_i is the cotangent line class at the ith relative marking and $\Delta_E = \sum \Delta_{E,l_i} \otimes \Delta^{\vee}_{E,l_i}$ is the pullback of the Künneth decomposition of Δ_E at the corresponding relative marking. The holomorphic anomaly equation takes the following form:

Proposition 25 ([34], Proposition 20). $\mathsf{G}^{\mathrm{rel},\bullet}_{g,m}(\alpha; \gamma \mid \underline{\mu})$ *is a quasi-modular form and*

$$\frac{d}{dC_2} \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}(\alpha; \gamma \mid \underline{\mu}) = \mathsf{P}^{\mathrm{rel},\bullet}_{g,m}(\alpha; \gamma \mid \underline{\mu}).$$

Relative $(\mathbf{S}, \mathbf{E})$

Since the log canonical bundle of (S, E) is nontrivial, relative moduli spaces in fibre direction have nontrivial virtual fundamental class. Define

$$\mathsf{F}_{g,0}^{vir-\text{rel}}(\alpha; \gamma \mid \emptyset) = \sum_{h \geq 0} \big\langle \alpha; \gamma \mid \emptyset \big\rangle_{g,hF}^{S/E,vir} q^h.$$

Recall that we denote the pullback of the diagonal of $\mathbb{P}^1$ as

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee.$$

Define a generating series

$$
\begin{aligned}
&\mathsf{H}_{g,m}^{\text{rel}}(\alpha; \gamma \mid \underline{\mu}) \\
&= \mathsf{F}_{g-1,m}^{\text{rel}}(\alpha; \gamma, \Delta_{\mathbb{P}^1} \mid \underline{\mu}) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,n\}=I_1 \sqcup I_2 \\ i \in \{1,2\}}} \mathsf{F}_{g_1,m}^{\text{rel}}(\alpha_{I_1}; \gamma_{I_1}, \delta_i \mid \underline{\mu}) \, \mathsf{F}_{g_2,0}^{vir-\text{rel}}(\alpha_{I_2}; \gamma_{I_2}, \delta_i^\vee \mid \emptyset) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,n\}=I_1 \sqcup I_2 \\ \forall i \in I_2 : \gamma_i \in H^2(E) \\ h \geq 0}} \sum_{\substack{b;b_1,\dots,b_h \\ l_1,\dots,l_h}} \frac{\prod_{i=1}^{h} b_i}{h!} \mathsf{F}_{g_1,m}^{\text{rel}}\big(\alpha_{I_1}; \gamma_{I_1} \mid ((b,1),(b_i, \Delta_{E,l_i})_{i=1}^h)\big) \\
&\quad \times \mathsf{G}_{g_2,m}^{\sim;\bullet}\big(((b,1),(b_i, \Delta_{E,\ell_i}^\vee)_{i=1}^h) \mid \alpha_{I_2}; \gamma_{I_2} \mid \underline{\mu}\big) \\
&\quad - 2 \sum_{i=1}^{n} \mathsf{F}_{g,m}^{\text{rel}}(\alpha \psi_i; \gamma_1, \dots, \gamma_{i-1}, \pi^* \pi_* \gamma_i, \gamma_{i+1}, \dots, \gamma_n \mid \underline{\mu}) \\
&\quad - 2 \sum_{i=1}^{l} \mathsf{F}_{g,m}^{\text{rel}}\big(\alpha; \gamma \mid ((\mu_1, \delta_1), \dots, (\mu_i, \psi_i^{\text{rel}} \pi^* \pi_* \delta_i), \dots, (\mu_l, \delta_l))\big) \\
&\quad + \frac{20}{m} \sum_{i=1}^{n} \langle \gamma_i, F \rangle \mathsf{F}_{g,m}^{\text{rel}}(\alpha; \gamma_1, \dots, \gamma_{i-1}, F, \gamma_{i+1}, \dots, \gamma_n \mid \underline{\mu}) \\
&\quad - \frac{2}{m} \sum_{i<j} \mathsf{F}_{g,m}^{\text{rel}}(\alpha; \gamma_1, \dots, \underbrace{\sigma_1(\gamma_i, \gamma_j)}_{i\text{th}}, \dots, \underbrace{\sigma_2(\gamma_i, \gamma_j)}_{j\text{th}}, \dots, \gamma_n \mid \underline{\mu}).
\end{aligned}
$$

The conjectural holomorphic anomaly equation for (S, E) has the form

$$\mathsf{F}_{g,m}^{\text{rel}}(\alpha; \gamma \mid \underline{\mu}) \in \frac{1}{\Delta(q)^m} \mathsf{QMod}(m),$$

and

$$\frac{d}{dC_2} \mathsf{F}_{g,m}^{\text{rel}}(\alpha; \gamma \mid \underline{\mu}) = \mathsf{H}_{g,m}^{\text{rel}}(\alpha; \gamma \mid \underline{\mu}). \tag{12}$$

Proposition 26. *Let $m \geq 1$. Assuming quasi-modularity for $\mathsf{F}_{g,m}$ and $\mathsf{F}_{g,m}^{\text{rel}}$, the holomorphic anomaly equations are compatible with the degeneration formula in the previous sense.*

Proof. The proof given in [34, Proposition 21] treats virtual fundamental classes, not reduced classes. The splitting behaviour of the reduced class with respect to restriction to boundary divisors [29, Section 7.3] calls for a slight adaptation of the proof. For this, we introduce a formal variable ε with $\varepsilon^2 = 0$. We can then interpret reduced Gromov–Witten invariants of the K3 surface as integrals against the class

$$[\overline{M}_{g,n}(S,\beta)]^{vir} + \varepsilon \, [\overline{M}_{g,n}(S,\beta)]^{red}$$

followed by taking the $[\varepsilon]$-coefficient.[14] We consider a similar class for S/E. This class has the advantage of satisfying the usual splitting behaviour of virtual fundamental classes. Thus, for this class one can follow the proof of compatibility given in [34, Proposition 21]. All the terms appearing in computation (ii) also appear in computation (i). We are left with proving the cancellation of the remaining terms in (i). This follows from comparing the ψ_i^{rel}-class and the ψ-class pulled back from the stack of target degeneration [34, Lemma 22]. In particular, we match the third term of $\mathsf{H}^{\mathrm{rel}}$ times $\mathsf{G}^{\mathrm{rel},\bullet}$ with the fourth term of $\mathsf{F}^{\mathrm{rel}}$ times $\mathsf{P}^{\mathrm{rel},\bullet}$, and analogously the fifth term of $\mathsf{H}^{\mathrm{rel}}$ times $\mathsf{G}^{\mathrm{rel},\bullet}$ with the second term of $\mathsf{F}^{\mathrm{rel}}$ times $\mathsf{P}^{\mathrm{rel},\bullet}$. □

The main advantage of the holomorphic anomaly equation is that it is compatible with the degeneration formula. Thus, the genus reduction from the degeneration formula connects the low-genus results with arbitrary-genus predictions. On the other hand, it is not even clear what the compatibility of the multiple cover formula and the degeneration formula should be.

6. Tautological relations and initial condition

This section contains a proof of the multiple cover formula in genera 0 and 1 for any divisibility m. It is a direct consequence of the KKV formula. However, as an initial condition for our induction we also require a special case in genus 2, which cannot be easily deduced from the KKV formula. We treat this descendent potential separately, using double ramification relations [3] for K3 surfaces. This approach is likely to give *relations in any genus* and will be pursued in the future.

6.1. Double ramification relations

In this section we recall *double ramification relations with target variety* developed in [2, 3].

Let $\mathfrak{Pic}_{g,n}$ be the Picard stack for the universal curve over the stack of prestable curves $\mathfrak{M}_{g,n}$ of genus g with n markings. Let

$$\pi \colon \mathfrak{C} \to \mathfrak{Pic}_{g,n}, \quad s_i \colon \mathfrak{Pic}_{g,n} \to \mathfrak{C}, \quad \mathfrak{L} \to \mathfrak{C}, \quad \omega_\pi \to \mathfrak{C} \tag{13}$$

be the universal curve, the ith section, the universal line bundle and the relative dualising sheaf of π, respectively. The following operational Chow classes on $\mathfrak{Pic}_{g,n}$ are obtained from the universal structure (13):

○ $\psi_i = c_1(s_i^*\omega_\pi) \in A^1_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$,
○ $\xi_i = c_1(s_i^*\mathfrak{L}) \in A^1_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$,
○ $\eta = \pi_*\big(c_1(\mathfrak{L})^2\big) \in A^1_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$.

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector of integers satisfying

$$\sum_i a_i = d, \tag{14}$$

where d is the degree of the line bundle. We denote by $P^{c,r}_{g,A,d}$ the codimension c component of the class

---

[14]We thank G. Oberdieck for pointing this out.

$$\sum_{\substack{\Gamma \in \mathsf{G}_{g,n,d} \\ w \in \mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\operatorname{Aut}(\Gamma_\delta)|} j_{\Gamma*} \left[\prod_{i=1}^n \exp\left(\frac{1}{2} a_i^2 \psi_i + a_i \xi_i\right) \prod_{v \in V(\Gamma_\delta)} \exp\left(-\frac{1}{2}\eta(v)\right)\right.$$

$$\left. \prod_{e=(h,h') \in E(\Gamma)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}\right].$$

We refer to [3] for details about the notations. This expression is polynomial in r when r is sufficiently large. Let $P_{g,A,d}^c$ be the constant part of $P_{g,A,d}^{c,r}$.

Theorem 27 ([3], Theorem 8). *$P_{g,A,d}^c = 0$ for all $c > g$ in $A_{\mathrm{op}}^c(\mathfrak{Pic}_{g,n})$.*

After restricting $P_{g,A,d}^c$ to equation (14), this expression is a polynomial in $a_1, \ldots, a_{n-1}$. The polynomiality will be used to get refined relations.

Let L be a line bundle on S with degree

$$\int_\beta c_1(L) = d.$$

The choice of a line bundle L induces a morphism

$$\varphi_L \colon \overline{M}_{g,n}(S,\beta) \to \mathfrak{Pic}_{g,n}, \quad [f \colon C \to S] \mapsto (C, f^*L).$$

Then Theorem 27 gives relations

$$P_{g,A,d}^c(L) = \varphi_L^* P_{g,A,d}^c \cap [\overline{M}_{g,n}(S,\beta)]^{red} = 0 \text{ for all } c > g \tag{15}$$

in $A_{g+n-c}(\overline{M}_{g,n}(S,\beta))$.

6.2. Compatibility II

The relations among descendent potentials coming from tautological relations on $\overline{M}_{g,n}(S,\beta)$ are compatible with the multiple cover formula. This follows from two observations. First, the splitting behaviour of the reduced class, discussed in Section 3.2, is crucial. It is already crucial to justify compatibility with respect to boundary restriction for tautological classes pulled back from $\overline{M}_{g,n}$. For tautological relations on $\overline{M}_{g,n}(S,\beta)$, a second fact is essential for the compatibility.

For $c > g > 0$, $A \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, consider the series of relations

$$P_{g,bA,db}^c(L^{\otimes b}) = 0$$

obtained by tensoring the line bundle L by b times. For each coefficient of a monomial in a_i-variables, this expression is polynomial in b, and hence each b-variable is a relation. As a consequence, each term of a relation $P_{g,A,m}^c(F)$ gives the same value of

$$m^{\deg - \underline{\deg}},$$

where $\deg(\xi) = 1$ and $\underline{\deg}(\xi) = 0$, as in Definition 1.1. The same holds true with the roles of F and W interchanged. Thus, the relations are compatible with the operator

$$m^{\deg - \underline{\deg}} \mathsf{T}_{m, 2g-2+\underline{\deg}},$$

which gives the multiple cover formula in Conjecture 15.

6.3. Initial condition

The KKV formula implies that the generating series of λ_g-integrals

$$\mathsf{F}_{g,m}(\lambda_g; \emptyset)$$

satisfies the multiple cover formula [36]. Here, $\lambda_g = c_g(\mathbb{E}_g)$ is the top Chern class of the rank g Hodge bundle $\mathbb{E}_g$ on $\overline{M}_g(S, \beta)$. The KKV formula will be the starting point of our genus induction.

The class λ_g is a tautological class by the Grothendieck–Riemann–Roch computation [15], but the formula is rather complicated. Instead we use an alternative expression of λ_g in terms of double ramification cycle, proven in [17]. We recall that the class $(-1)^g \lambda_g$ is equal to the double ramification cycle $\mathsf{DR}_g(\emptyset)$ with the empty condition. By [17, Theorem 1], the class $\mathsf{DR}_g(\emptyset)$ can be written as a graph sum of tautological classes *without* κ-classes.

Proposition 28. *The multiple cover formula holds in genera 0 and 1 for all $m \geq 1$.*

Proof. When $g = 0, 1$, the tautological ring $R^*(\overline{M}_{g,n})$ is additively generated by boundary strata [19, 37]. Thus, one can replace descendents $\alpha \in R^*(\overline{M}_{g,n})$ by classes in $H^*(S)$. By the divisor equation and the dimension constraint, we can reduce to the case $\mathsf{F}_{0,m}(\emptyset)$ and $\mathsf{F}_{1,m}(\tau_0(\mathsf{p}))$. The genus 0 case is covered by the full Yau–Zaslow formula [21, 36]. The genus 1 case follows from the genus 2 KKV formula. Using the boundary expression of λ_2 on $\overline{M}_2$, we have

$$\mathsf{F}_{2,m}(\lambda_2; \emptyset) = \frac{1}{240} \mathsf{F}_{1,m}(\psi_1; \Delta_S) + \frac{1}{1152} \mathsf{F}_{0,m}(; \Delta_S, \Delta_S)$$
$$= \frac{1}{10} \mathsf{F}_{1,m}(\tau_0(\mathsf{p})) + \frac{1}{60} \mathsf{D}_q^2 \mathsf{F}_{0,m}(\emptyset),$$

where $\Delta_S \subset S \times S$ is the diagonal class. Therefore, $\mathsf{F}_{1,m}(\tau_0(\mathsf{p}))$ satisfies Conjecture 15.   □

In the argument to follow, we will use tautological relations on $\overline{M}_{g,n}$ which have recently been obtained by r-spin relations. For convenience, we summarise the result.

Proposition 29 ([23]). *Set $g \geq 2$ and $n \geq 1$. Consider tautological classes on $\overline{M}_{g,n}$.*

(i) *(Topological recursion relations) Any monomial of ψ-classes of degree at least g can be represented by a tautological class supported on boundary strata without κ-classes.*

(ii) *Any tautological class of degree $g - 1$ can represented by a sum of a linear combination of $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$ and a tautological class supported on boundary strata.*

Proof. The proof of (i) follows from the proof of [23, Lemma 5.2] (see also [12, page 3]). By [23, Proposition 3.1] (or [10, Theorem 1.1]), the degree $g-1$ part $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$. Since relations used in the proof are all tautological, the boundary expression is tautological, and thus we obtain (ii).   □

Together with the boundary expression for λ_{g+1}, we obtain the following more general consequence of the KKV formula:

Proposition 30. *Set $m \geq 1$ and $g \geq 1$. Assume that the multiple cover formula from Conjecture 15 holds for m and all descendents of genus $< g$. Then Conjecture 15 holds for*

$$\mathsf{F}_{g,m}(\tau_{g-1}(\mathsf{p})).$$

Proof. Let $\delta \in R^1(\overline{M}_g)$ be the boundary divisor corresponding to a curve with nonseparating node. Denote two half-edges as h and h'. Recall that $(-1)^g \lambda_g$ is equal to the double ramification cycle $\mathsf{DR}_g(\emptyset)$ with the empty condition. We use this formula for genus $g + 1$. By [17, Theorem 1],

$$(-1)^{g+1}\lambda_{g+1} = \mathrm{DR}_{g+1}(\emptyset)$$

$$= \frac{1}{2}\Big[-\frac{1}{(g+1)!}\sum_{w=0}^{r-1}\big(\frac{w^2}{2}(\psi_h+\psi_{h'})\big)^g\Big]_{r^1}\delta + \text{ lower genus},$$

where $[\cdots]_{r^1}$ is the coefficient of the linear part of a polynomial in r. The leading term is nonzero by Faulhaber's formula.

By Proposition 29(i), any ψ-monomial in $R^{\geq g}(\overline{M}_{g,n})$ can be represented by a sum of tautological classes supported on boundary strata without κ classes. There is only one graph with a genus g vertex (with a rational component carrying both markings). The graph is decorated with a polynomial of degree $g-1$ in ψ- and κ-classes. By Proposition 29(ii), this tautological class can be represented by a sum of a multiple of ψ^{g-1} and tautological classes supported on boundary strata. We find that

in $R^g(\overline{M}_{g,2})$ for some $c \in \mathbb{Q}$.[15] Therefore, it suffices to prove that c is nonzero. Recall that $\lambda_g\lambda_{g-1}$ vanishes on $\overline{M}_{g,n}\setminus M_{g,n}^{rt}$, so

$$\int_{\overline{M}_{g,2}}(\psi_1+\psi_2)^g\lambda_g\lambda_{g-1} = c\int_{\overline{M}_{g,1}}\psi_1^{g-1}\lambda_g\lambda_{g-1}.$$

The left-hand side of the equation is nonzero by [17, Lemma 8], which concludes the proof. □

We now consider the case of genus 2. By the Getzler–Ionel vanishing on $\overline{M}_{2,n}$, the dimension constraint and the divisor equation, any descendent insertion reduces to the following three cases:

$$\mathsf{F}_{2,m}(\tau_1(\mathsf{p})), \quad \mathsf{F}_{2,m}(\tau_0(\mathsf{p})^2), \quad \mathsf{F}_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p})), \quad \text{with } \gamma \in H^2(S).$$

The first case is treated in Proposition 30 and follows from the KKV formula in genus ≤ 3. The second case, for $m = 2$, is treated as part of the proof of Theorem 1 in Section 7. We use the double ramification relation (15) to prove the multiple cover formula for the third case. The point class p will be obtained as the product of F and W.

Proposition 31. *For $\gamma \in H^2(S)$, the generating series $\mathsf{F}_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p}))$ satisfies Conjecture 15.*

Proof. We will use relations in $A_{2+n-3}\big(\overline{M}_{2,n}(S,\beta)\big)$,

$$P_{2,A,m}^3(F) = 0,$$

associated to the line bundle $\mathcal{O}_S(F)$ on S. More precisely, we will distinguish two cases $\gamma \in U$ and $\gamma \in U^\perp$ and set respectively

$$A = (a_1, m - a_1), \quad A = (a_1, a_2, m - a_1 - a_2).$$

Refined relations are then obtained by considering particular monomials in the a_i, as outlined in the previous section. The η-class vanishes in this case, because $\langle F, F\rangle = 0$, and ξ_i^2 vanishes for the same reason. Define

$$X = \mathsf{F}_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p})).$$

---

[15]The number under each vertex is the genus, and legs correspond to markings.

The case $\gamma = F$ is treated first. As explained in Section 6.1, the tautological relations are polynomial in a_i, and we may obtain a refined relation by considering the $[a_1^4]$-coefficient of

$$P_{2,A,m}^3(F)|_{a_2=m-a_1}.$$

We will only need to consider boundary strata which both

○ contribute to X and
○ contribute to the $[a_1^4]$-coefficient.

These two properties simplify the calculation significantly. By the splitting property of the reduced class, a relevant boundary stratum is a tree with one genus 2 vertex and contracted genus 0 components. The integrals are given by the intersection product of the corresponding insertions. In the case with only two markings, the only relevant stratum[16] is
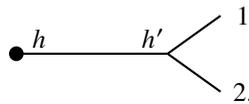
The weight factor for this stratum is

$$\frac{w(h)w(h')}{2} = -\frac{m^2}{2}.$$

This stratum, therefore, cannot contribute to the $[a_1^4]$-coefficient, since ψ-classes on the genus 0 component vanish. It remains to determine the contributions from the trivial graph
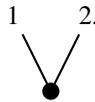
We will order the terms by the total degree $\deg(\psi)$ in the ψ-classes:

0. $\deg(\psi) = 0$. The relation we consider is of codimension 3. This case is therefore impossible, by virtue of $\xi_i^2 = 0$.
1. $\deg(\psi) = 1$. This case results in nontrivial terms, discussed later.
2. $\deg(\psi) \geq 2$. We may apply Proposition 29(i) to reduce to the descendent $\mathsf{F}_{2,m}(\tau_1(\mathsf{p}))$. This descendent is covered by Proposition 30.

Therefore, up to lower-genus data, the $[a_1^4]$-coefficient is

$$-\frac{1}{2}\psi_1\xi_1\xi_2 - \frac{1}{2}\psi_2\xi_1\xi_2.$$

Integrating

$$\mathrm{ev}_2^*(W)P_{2,A,m}^3(F)|_{a_2=m-a_1}$$

against the reduced class, we find (up to lower-genus data)

$$-\frac{1}{2}X - \frac{m}{2}\mathsf{F}_{2,m}(\tau_1(\mathsf{p})),$$

---

[16] The genus 2 vertex is represented by a filled node, and other nodes represent genus 0 vertices. Labeled half-edges correspond to markings.

where the second term is obtained by application of the divisor equation. We thus find that X is a linear combination of terms which satisfy Conjecture 15. Switching the role of F and W, we obtain the same result for $\gamma = W$.

Next we consider $\gamma \in U^{\perp}$. The following vanishing of intersection products will be used frequently:

$$\langle \gamma, F \rangle = 0, \quad \langle \gamma, W \rangle = 0, \quad \langle \gamma, \beta \rangle = 0.$$

We use a similar argument as before;[17] this time, however, we use three markings and consider the $[a_1^3 a_2]$-coefficient of

$$\mathrm{ev}_1^*(\gamma)\, \mathrm{ev}_2^*(W) P_{2,A,m}^3(F)|_{a_3 = m - a_1 - a_2}. \tag{16}$$

By the given vanishing of intersection products, the only possible trees with nontrivial contribution are
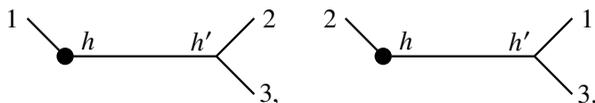
The weight factor for the right stratum is

$$\frac{w(h)w(h')}{2} = -\frac{(m - a_2)^2}{2}.$$

Since ψ-classes on the genus 0 component vanish, the power of a_1 in any monomial obtained from this stratum is bounded by 2. The contribution to the $[a_1^3 a_2]$-coefficient is, therefore, zero.

Next we explain the contributions from the left stratum. Note that the left vertex is of genus 2 with two markings, and we can apply the same reasoning as in the previous discussion for $\gamma = F$. Here the $\deg(\psi) = 0$ term $\xi_1 \xi_2$ has trivial contribution due to $\langle \gamma, F \rangle = 0$. The $\deg(\psi) = 1$ terms $\psi_h \xi_2$, $\psi_h \xi_3$ have vanishing contribution by application of the divisor equation for γ. Nontrivial contributions are obtained only from

$$\psi_1 \xi_2, \quad \psi_1 \xi_3.$$

These two terms have contributions

$$-\frac{(m - a_1)^2}{4} a_1^2 a_2 X, \quad -\frac{(m - a_1)^2}{4} a_1^2 a_3 X.$$

The $[a_1^3 a_2]$-coefficients, however, cancel due to $a_3 = m - a_1 - a_2$. It remains to determine the contributions from the trivial graph
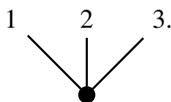
As before, we order the terms by the total degree $\deg(\psi)$ in the ψ-classes:

0. $\deg(\psi) = 0$. The relation we consider is of codimension 3. Since $\xi_i^2 = 0$, the class ξ_1 must appear. This term, however, vanishes because $\langle \gamma, F \rangle = 0$.
1. $\deg(\psi) = 1$. This case results in nontrivial terms corresponding to ψ_1 or ψ_3, discussed later. The choice of the monomial $[a_1^3 a_2]$ excludes the appearance of ψ_2.
2. $\deg(\psi) = 2$. This case results in nontrivial terms corresponding to $\psi_1 \psi_3$ or ψ_3^2, discussed later. The choice of the monomial $[a_1^3 a_2]$ excludes the appearance of ψ_1^2.

---

[17]We are grateful to the referee for pointing out a mistake in an earlier version of the text. It has become clear that the choice of monomial leading to nontrivial relations is a very subtle one. Symmetry in the a_i and the insertions causes cancellation in many cases. We plan to come back to this in future work.

3. $\deg(\psi) = 3$. As before, this case reduces to the descendent $F_{2,m}(\tau_1(\mathsf{p}))$, which is covered already.

The contributions from $\deg(\psi) \in \{1, 2\}$ are

$$
\begin{aligned}
\psi_1 \xi_2 \xi_3 \quad &\to \quad \frac{1}{2} a_1^2 a_2 a_3 F_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p})\tau_0(F)) \\
&= \frac{1}{2} a_1^2 a_2 (m - a_1 - a_2) mX, \\
\psi_3 \xi_2 \xi_3 \quad &\to \quad \frac{1}{2} a_3^2 a_2 a_3 F_{2,m}(\tau_0(\gamma)\tau_0(\mathsf{p})\tau_1(F)) \\
&= 0, \\
\psi_1 \psi_3 \xi_2 \quad &\to \quad \frac{1}{2} a_1^2 \frac{1}{2} a_3^2 a_2 F_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p})\tau_1(1)) \\
&= a_1^2 a_2 (m - a_1 - a_2)^2 X, \\
\psi_1 \psi_3 \xi_3 \quad &\to \quad \frac{1}{2} a_1^2 \frac{1}{2} a_3^3 F_{2,m}(\tau_1(\gamma)\tau_0(W)\tau_1(F)) \\
&= \frac{1}{4} a_1^2 (m - a_1 - a_2)^3 X + \text{(lower genus)}, \\
\psi_3^2 \xi_2 \quad &\to \quad \frac{1}{8} a_3^4 a_2 F_{2,m}(\tau_0(\gamma)\tau_0(\mathsf{p})\tau_2(1)) \\
&= \frac{1}{8} a_2 (m - a_1 - a_2)^4 X, \\
\psi_3^2 \xi_3 \quad &\to \quad \frac{1}{8} a_3^4 a_3 F_{2,m}(\tau_0(\gamma)\tau_0(W)\tau_2(F)) \\
&= 0.
\end{aligned}
$$

The third calculation uses the dilaton equation. All of the other calculations are obtained by application of the divisor equation. Additionally, the fourth calculation involves Proposition 29. The only stratum with a genus 2 vertex (that is, with both markings on a contracted genus 0 component) has vanishing contribution because $\langle \gamma, F \rangle = 0$, and therefore the relation reduces to lower-genus descendents. The total contribution to $[a_1^3 a_2]$ is

$$
-\frac{1}{2} mX - 2mX + \frac{3}{2} mX - \frac{1}{2} mX = -\frac{3}{2} mX.
$$

We find that X is a linear combination of terms which satisfy Conjecture 15.   □

Remark 32. In fact, for $\gamma \in U^\perp$ the previous generating series vanishes (and thus trivially satisfies the multiple cover formula). A proof in the primitive case is given in [9, Lemma 4].

7. Proof of Theorems 1 and 3

7.1. Proof of Theorem 1

The proof proceeds via induction on the pair (g, n) ordered by lexicographic order: $(g', n') < (g, n)$ if

○ $g' < g$ or
○ $g' = g$ and $n' < n$.

Recall the dimension constraint of insertions:

$$
g + n = \deg(\alpha) + \sum_i \deg(\gamma_i).
$$

We separate the proof into several steps.

Case 0. The genus 0 case is covered by Proposition 28. This serves as the start for our induction.

Case 1. If all cohomology classes γ_i satisfy $\deg(\gamma_i) \leq 1$, then $\deg(\alpha) \geq g$, and by the strong form of Getzler–Ionel vanishing [15, Proposition 2] we have $\alpha = \iota_* \alpha'$ with $\alpha' \in R^*(\partial \overline{M}_{g,n})$ and $\iota \colon \partial \overline{M}_{g,n} \to \overline{M}_{g,n}$. We are thus reduced to lower (g, n).

Case 2. Assume $\deg(\alpha) \leq g - 2$ or, equivalently, that there exist at least two descendents of the point class. We use the degeneration to the normal cone of a smooth elliptic fibre:

$$S \rightsquigarrow S \cup_E (\mathbb{P}^1 \times E).$$

We specialise the point class to the bubble $\mathbb{P}^1 \times E$. Let $C = C' \cup C''$ be the splitting of a domain curve appearing in the degeneration formula in Theorem 23. Namely, C' is the component on S and C'' is the component on $\mathbb{P}^1 \times E$. We argue that this splitting has nontrivial contribution only for $g(C') < g$. If $g(C') = g$, this forces C'' to be a disconnected union of two rational curves. Since the degree of the curve class along the divisor is $\langle 2B + hF, F \rangle = 2$, the two descendents of the point class then force the cohomology weighted partition to be $(1, 1)^2$ on the bubble or, equivalently, $(1, \omega)^2$ for (S, E). This contribution vanishes because there are no curves which can satisfy this condition (if $(1, \omega)^2$ is represented by a generic point in E^2, see Corollary 24).

Case 3. Assume $\deg(\alpha) = g - 1$ or, equivalently, that there exists only one descendent of the point class. We may thus assume $\gamma_1 = \mathsf{p}$. If $n = 1, g \geq 2$, we can move $\tau_{g-1}(\mathsf{p})$ to the bubble and the genus on S drops.

When $n \geq 2$, moving the point class to the bubble as in case 2 may not reduce the genus. In particular, moving $\tau_0(\mathsf{p})$ to the bubble has nontrivial contribution from rational curves on the bubble. On the other hand, if $a \geq 1$, moving $\tau_a(\mathsf{p})$ to the bubble reduces the genus on S because of the dimension constraint.

We use Buryak, Shadrin and Zvonkine's description of the top tautological group $R^{g-1}(M_{g,n})$ [10]. For any $\alpha \in R^{g-1}(\overline{M}_{g,n})$, the restriction of α to $M_{g,n}$ is a linear combination of

$$R^{g-1}(M_{g,n}) = \mathbb{Q}\langle \psi_1^{g-1}, \psi_2^{g-1}, \dots, \psi_n^{g-1} \rangle \tag{17}$$

and the boundary term is also a tautological class in $R^{g-1}(\partial \overline{M}_{g,n})$. By the divisor equation and subsequent use of equation (17), we can reduce to cases for $\leq (g, 2)$. When $g \geq 3$, equation (17) has a different basis,

$$R^{g-1}(M_{g,2}) = \mathbb{Q}\langle \psi_1^{g-1}, \psi_1 \psi_2^{g-2} \rangle,$$

which is an easy consequence of the generalised top intersection formula. Therefore, we may assume the descendent of the point class is of the form $\tau_a(\mathsf{p})$ with $a \geq 1$. Specialising this insertion to the bubble $\mathbb{P}^1 \times E$ reduces the genus, and hence the same argument as in case 2 applies. The genus 2 case is covered by Proposition 31.

Relative vs. absolute. We reduced to invariants for (S, E) with genus $g' < g$. As explained in the proof of [29, Lemma 31] (see also [28]), the degeneration formula provides an upper triangular relation between absolute and relative invariants for all pairs $\leq (g', n')$. Thus our induction applies. □

7.2. Proof of Theorem 3

We argue by showing that each induction step in the proof of Theorem 1 is compatible with the holomorphic anomaly equation. A nontrivial step appears when the degeneration formula is used. From the compatibility result in Proposition 26, we are reduced to proving the relative holomorphic anomaly equation for lower-genus relative generating series $\mathsf{F}_{g',2}^{\mathrm{rel}}$ for (S, E) and relative generating series for

$(\mathbb{P}^1 \times E, E)$. The holomorphic anomaly equation for $(\mathbb{P}^1 \times E, E)$ is established in [33]. Because of the relative versus absolute correspondence [28], we are reduced to proving the holomorphic anomaly equation for $\mathsf{F}_{g',2}$ in genera 0, 1 and some genus 2 descendents. We proved the multiple cover formula for these cases in Section 6, which implies the holomorphic anomaly equation by Proposition 5.   □

Remark 33. Parallel argument shows that we can always reduce the proof for arbitrary descendent insertions to the case where the number of point insertions is less than or equal to $m - 1$.

8. Examples

Example 34. We compute $\mathsf{F}_{1,2}\big(\tau_1(F)\big)$ via topological recursion in genus 1 and illustrate Conjecture 15. Let $[\delta_0] \in A^1(\overline{M}_{1,1})$ be the push-forward of the fundamental class under the gluing map

$$\overline{M}_{0,3} \to \overline{M}_{1,1}.$$

Since

$$\psi_1 = \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,1}),$$

we obtain

$$\mathsf{F}_{1,1}\big(\tau_1(F)\big) = \frac{1}{24}\mathsf{F}_{0,1}\big(\tau_0(F)\tau_0(\Delta_S)\big) = \frac{1}{12}\mathsf{F}_{0,1}\big(\tau_0(F)\tau_0(F \times W)\big)$$

$$= \frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1},$$

where $\Delta_S \subset S \times S$ is the diagonal class. Analogously,

$$\mathsf{F}_{1,2}\big(\tau_1(F)\big) = \frac{1}{24}\mathsf{F}_{0,2}\big(\tau_0(F)\tau_0(\Delta_S)\big) = \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2}.$$

Using the multiple cover formula in genus 0,

$$\mathsf{F}_{0,2} = \mathsf{T}_2\mathsf{F}_{0,1} + \frac{1023}{8192}\mathsf{F}_{0,1}(q^2),$$

we obtain

$$\mathsf{F}_{1,2}\big(\tau_1(F)\big) = \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} = 2\mathsf{T}_2\frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1} + \frac{1023}{1024}\mathsf{B}_2\frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1}$$

$$= 2\mathsf{T}_2\mathsf{F}_{1,1}\big(\tau_1(F)\big) + (2^0 - 2^{-10})\mathsf{B}_2\mathsf{F}_{1,1}\big(\tau_1(F)\big),$$

in perfect agreement with Conjecture 15 using the formula for $\mathsf{T}_{2,0}$ from Lemma 12.

Example 35. We compute $\mathsf{F}_{2,2}(\tau_0(\mathsf{p})^2)$ via the degeneration formula and verify the multiple cover formula. The first two terms are computed by the classical geometry of K3 surfaces in [32]. For simplicity, we write $\mathsf{F}_{1,2} = \mathsf{F}_{1,2}(\tau_0(\mathsf{p}))$. The relative invariants for (S, E) can be written in terms of absolute invariants:

Lemma 36.

  (i) $\mathsf{F}^{\mathrm{rel}}_{0,2}\big(\emptyset \mid (1,1)^2\big) = 2\mathsf{F}_{0,2}$,
  (ii) $\mathsf{F}^{\mathrm{rel}}_{1,2}\big(\emptyset \mid (1,1), (1,\omega)\big) = \mathsf{F}_{1,2} - 2\mathsf{F}_{0,2}\mathsf{D}_q C_2$,
  (iii) $\mathsf{F}^{\mathrm{rel}}_{1,2}\big(\emptyset \mid (2,1)\big) = \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} - 4C_2\mathsf{F}_{0,2}$.

Proof. This is a standard computation of the relative versus absolute correspondence [28].   □

The relative invariants for $(\mathbb{P}^1 \times E, E)$ can be computed by the Gromov–Witten invariants of E:

Lemma 37.

(i) $G^{\mathrm{rel}}_{0,1}\big(\tau_0(\mathsf{p}) \mid (1,1)\big) = 1,$ $G^{\mathrm{rel}}_{0,1}\big(\emptyset \mid (1,\omega)\big) = 1,$

(ii) $G^{\mathrm{rel}}_{1,1}\big(\tau_0(\mathsf{p}) \mid (1,\omega)\big) = D_q C_2,$ $G^{\mathrm{rel}}_{1,1}\big(\tau_0(\mathsf{p})^2 \mid (1,1)\big) = 2D_q C_2,$

(iii) $G^{\mathrm{rel}}_{2,1}\big(\tau_0(\mathsf{p})^2 \mid (1,\omega)\big) = (D_q C_2)^2,$

(iv) $G^{\mathrm{rel}}_{1,2}\big(\tau_0(\mathsf{p})^2 \mid (2,\omega)\big) = D_q^2 C_2,$ $G^{\mathrm{rel}}_{1,2}\big(\tau_0(\mathsf{p})^2 \mid (1,\omega)^2\big) = D_q^3 C_2.$

Consider the degeneration where two point insertions move to the bubble $\mathbb{P}^1 \times E$. By Theorem 23,

$$F_{2,2}(\tau_0(\mathsf{p})^2) = \big(F_{1,2} - 2F_{0,2}D_q C_2\big)4D_q C_2 + \big(\tfrac{1}{3}D_q F_{0,2} - 4C_2 F_{0,2}\big)2D_q^2 C_2$$

$$+ (2F_{0,2})\tfrac{1}{2}\big(D_q^3 C_2 + 4(D_q C_2)^2\big)$$

$$= 36q + 8760q^2 + 754992q^3 + 36694512q^4 + \cdots.$$

On the other hand, the primitive generating series

$$F_{2,1}(\tau_0(\mathsf{p})^2) = \frac{(D_q C_2)^2}{\Delta(q)}$$

is computed in [7], and one can apply the multiple cover formula to obtain a candidate for $F_{2,2}(\tau_0(\mathsf{p})^2)$. The first few terms of the two generating series match. It is enough to conclude that the two generating series are indeed equal because the space of quasi-modular forms with given weight is finite-dimensional. However, it seems nontrivial to match the formula from the degeneration with the formula provided by Conjecture 15.

Appendix A. A proof of the degeneration formula

For a self-contained exposition, we present a proof of the degeneration formula which is parallel to the proof in [29, 30]. When $m = 1, 2$, a proof using symplectic geometry was presented in [24].

Perfect obstruction theory

For simplicity, assume $n = 0$. General cases easily follow from this case. Let $\epsilon \colon \mathcal{S} \to \mathbb{A}^1$ be the total family of the degeneration and

$$\overline{M}_g(\epsilon, \beta) \to \mathbb{A}^1$$

be the moduli space of stable maps to the expanded target $\widetilde{\mathcal{S}}$. For the relative profile μ, the embedding

$$\iota_\mu \colon \overline{M}_g(\mathcal{S}_0, \mu) \hookrightarrow \overline{M}_g(\epsilon, \beta)$$

can be realised as a Cartier pseudo-divisor (L_μ, s_μ).

Let $\mathrm{E}_\epsilon \to \mathbb{L}_{\overline{M}_g(\epsilon,\beta)}$ be the perfect obstruction theory constructed in [27]. Then the perfect obstruction theories E_0 and E_μ of $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \mu)$ sit in exact triangles

$$L_0^\vee \to \iota_0^* \mathrm{E}_\epsilon \to \mathrm{E}_0 \xrightarrow{[1]}$$

$$L_\mu^\vee \to \iota_\mu^* \mathrm{E}_\epsilon \to \mathrm{E}_\mu \xrightarrow{[1]}.$$

On $\overline{M}_g(\mathcal{S}_0, \mu)$, the perfect obstruction theory splits as follows. Let E_1 and E_2 be the perfect obstruction theory of relative stable map spaces $\overline{M}_g(S/E, \beta_1)_\mu$ and $\overline{M}_g(\mathbb{P}^1 \times E/E, \beta_2)_\mu$, respectively. There exists an exact triangle

$$\bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E\times E})_i \to E_1 \boxplus E_2 \to E_\mu \xrightarrow{[1]} \tag{A.1}$$

where $(N^\vee_{\Delta_E/E\times E})_i$ is the pullback of the conormal bundle of the diagonal $\Delta_E \subset E \times E$ along the ith relative marking.

Reduced class

Let $\rho \colon \widetilde{\mathcal{S}} \to S \times \mathbb{A}^1 \to S$ be the projection. By pulling back the holomorphic symplectic form on S via ρ, one can define a cosection of the obstruction sheaf of E_ϵ

$$Ob_{\overline{M}_g(\epsilon,\beta)} \to \mathcal{O}$$

(see [20, Section 5]). Dualising the cosection gives a morphism

$$\gamma \colon \mathcal{O}[1] \to E_\epsilon.$$

Let E_ϵ^{red} be the cone of γ which gives the reduced class on $\overline{M}_g(\epsilon, \beta)$. Similarly, we can construct

$$\gamma_{\mathrm{rel}} \colon \mathcal{O}[1] \to E_1$$

for the moduli space of relative stable maps $\overline{M}_g(S/E, \beta)$.

Degeneration formula for reduced class

Restricting γ to $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \mu)$, we get

$$\gamma_0 \colon \mathcal{O}[1] \to \iota_0^* E_\epsilon \to E_0$$
$$\gamma_\mu \colon \mathcal{O}[1] \to \iota_\mu^* E_\epsilon \to E_\mu,$$

where the compositions induce reduced classes. The exact triangles

$$L_0^\vee \to \iota_0^* E_\epsilon^{red} \to E_0^{red} \xrightarrow{[1]}$$
$$L_\mu^\vee \to \iota_\mu E_\epsilon^{red} \to E_\mu^{red} \xrightarrow{[1]}$$

still hold.

Lemma 38. *We have an exact triangle*

$$N^\vee_{\Delta_{E^l}/E^l\times E^l} \to E_1^{red} \boxplus E_2 \to E_\mu^{red} \xrightarrow{[1]}$$

on $\overline{M}_g(\mathcal{S}_0, \mu)$ compatible with the structure maps to the cotangent complex.

Proof. Consider the diagram of complexes

$$
\begin{array}{ccc}
\mathcal{O}[1] \boxplus 0 & =\!=\!= & \mathcal{O}[1] \\
\big\downarrow{\scriptstyle \gamma_{\mathrm{rel}} \boxplus 0} & & \big\downarrow{\scriptstyle \gamma_\mu} \\
\bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E \times E})_i \longrightarrow E_1 \boxplus E_2 \longrightarrow E_\mu \\
\big\| & & \big\downarrow \qquad\qquad \big\downarrow \\
\bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E \times E})_i \longrightarrow E_1^{red} \boxplus E_2 \longrightarrow E_\mu^{red},
\end{array}
$$

where the middle horizontal morphisms are the exact triangle from formula (A.1). The square on the top commutes because the cosections for $\overline{\mathcal{S}}$ and (S, E) are both coming from the holomorphic symplectic form on S. The vertical morphisms are exact triangles and hence induce a map between cones. □

Now Theorem 23 is a direct consequence of Lemma 38.

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