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The construction of diagrams for abelian groups

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Christine W. Ayoub has defined diagrams for abelian groups and has shown that the structure of an abelian group A is determined by certain subgroups T_i of A which arise naturally from a given diagram for A. In this paper it is shown that a given diagram for A is formed from certain elementary diagrams for the T_i.

Ayoub, [1], considered certain series of finite length for abelian groups which arose in connection with the study of the group of units of primary homogeneous rings. She has used these diagrams to determine the structure of these groups, [2]. Her definition is essentially the following.

DEFINITION 1. Let $S_n = \{0, 1, \ldots, n\}$. A function $j : S_n \to S_n$ is called admissible if $j(i) = j(i') \neq n$ implies $i = i'$ and for $0 \leq i < n$, $j(i) > i$ with $j(n) = n$. The series $A = A_0 > A_1 > \ldots > A_n = 0$ for the abelian group A is a j-diagram (with respect to the prime p) if j is an admissible function,

(1) $j(i) = n$ implies $pA_i = 0$, and

(2) $j(i) < n$ implies the mapping,

$$a_i + A_{i+1} \to pa_i + A_{j(i)+1}$$

for $a_i \in A_i$, is an isomorphism of A_i/A_{i+1} onto

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$$A_{j(i)}/A_{j(i)+1} .$$

We shall denote the range of j by $R(j)$, use $\operatorname{diag}(A)$ to represent the j-diagram for A, and when (2) holds write $A_i/A_{i+1} \underset{p}{\approx} A_{j(i)}/A_{j(i)+1}$. All groups considered are additive abelian groups.

In this paper we shall show that all j-diagrams for a group A may be constructed in a natural way from certain elementary diagrams for direct summands of A. Results from [1] will be used, some of which are stated in the following theorem.

THEOREM 1 (Ayoub). *Let* A *have the* j*-diagram* $A = A_0 > A_1 > \dots > A_n = 0$. *Then for* $0 \le i < n$,

(1) A_i/A_{i+1} *is an elementary* p*-group,*

(2) *if* $a \in A_i/A_{i+1}$ *then* $o(a) = p^\nu$ *where* $j^\nu(i) = n$, $j^{\nu-1}(i) < n$ *(and therefore* A *is a bounded* p*-group),*

(3) *there are subgroups* $T_i \le A_i$ *satisfying* $A_i = A_{i+1} + T_i$, $A_{i+1} \cap T_i = pT_i = T_{j(i)}$,

(4) *if the* T_i *are chosen as in* (3), *then* $A = \bigoplus_{i \notin R(j)} T_i$,

and

(5) *let* T_i *be as in* (4); *then* T_i *equals the direct sum of cyclic groups of order* p^{ν_i}, *where* $j^{\nu_i}(i) = n$, $j^{\nu_i - 1}(i) < n$, *and* $\operatorname{rank}(T_i) = \dim(A_i/A_{i+1})$ *(as a vector space over* Z_p*).*

EXAMPLE 1. Let B be the direct sum of cyclic groups of order p^ν. Then B has a j-diagram, $B = B_0 > B_1 > \dots > B_\nu = 0$, where $B_i = pB_{i-1}$ for $1 < i \le \nu$ and $j(i) = i + 1$ for $0 \le i < \nu$. Thus if $B = \bigoplus_m \langle b_m \rangle$ then $B_i = \bigoplus_m \langle p^i b_m \rangle$, $0 \le i < \nu$.

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Miller [3] considered the problem of constructing a diagram for A from diagrams for direct summands of A.

DEFINITION 2. Let $B^{(i)} = B_0^{(i)} > B_1^{(i)} > \ldots > B_{n_i}^{(i)} = 0$ be a j_i-diagram for the group $B^{(i)}$, $1 \le i \le k$. A j-diagram for A is said to be woven from the j_i-diagrams for the $B^{(i)}$, $1 \le i \le k$, if

$A = \bigoplus_{i=1}^{k} B^{(i)}$, for every $r \ge 0$, $A_r = \bigoplus_{i=1}^{k} B_{n_{r,i}}^{(i)}$, and given

$A_r, A_{r+1} = \bigoplus_{i=1}^{k} B_{n_{r+1,i}}^{(i)}$ where for some u, $1 \le u \le k$, $n_{r+1,u} = n_{r,u} + 1$ and for all $t \ne u$, $n_{r+1,t} = n_{r,t}$.

EXAMPLE 2. Let $B^{(1)}$ be a cyclic group of order p^2 with diagram $B^{(1)} > pB^{(1)} > p^2B^{(1)} = 0$ and $B^{(2)}$ be a group of order p with diagram $B^{(2)} > pB^{(2)} = 0$. Then a diagram for $A = B^{(1)} \oplus B^{(2)}$ which is woven from the diagrams for $B^{(1)}$ and $B^{(2)}$ is

$$A_0 = B^{(1)} \oplus B^{(2)} > A_1 = pB^{(1)} \oplus B^{(2)} > A_2 = pB^{(1)} \oplus pB^{(2)} > A_3 = = p^2B^{(1)} \oplus pB^{(2)} = 0$$

where $A_0/A_1 \underset{p}{\approx} A_2/A_3$.

We will now proceed to show that if $\operatorname{diag}(A)$ is a j-diagram for A and the T_i are as in Theorem 1 each with a diagram as that given in Example 1 then $\operatorname{diag}(A)$ is woven from the diagrams for the T_i. Thus for each u, $0 \le u < n$, $A_u = \bigoplus_{i \notin R(j)} \left[\bigoplus_{(i,m)} \left\langle p^{k_i} t_{(i,m)} \right\rangle\right]$ where for each $i \notin R(j)$, $T_i = \bigoplus_{(i,m)} \langle t_{(i,m)} \rangle$ and k_i is chosen appropriately.

LEMMA 1. *Let A have a j-diagram, $A = A_0 > A_1 > \ldots > A_n = 0$, and $\bar{j}_u$ be the restriction of j to the set $\{u, \ldots, n\}$. Then*

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$A = \bigoplus_{i \notin R(j)} T_i$ *implies* $A_u = \left(\bigoplus_{\substack{i \ge u \\ i \notin R(j)}} T_i \right) \oplus \left(\bigoplus_{\substack{i < u \\ i \notin R(j)}} p^{k_i} T_i \right)$, *for*

$0 \le u < n$, *where if* $v \notin R(\bar{j}_u)$ *and* $v \in R(j)$ *then* $j^{k_i}(i) = v$ *with* $i \notin R(j)$ *and* $k_i \ge 1$.

Proof. $A_u > \ldots > A_n = 0$ with the given $\bar{j}_u$ is clearly a $\bar{j}_u$-diagram for A_u. Thus by (4) of Theorem 1, there exists $S_v \le A_u$ such that $A_u = \bigoplus_{v \notin R(\bar{j}_u)} S_v$. Now $v \notin R(\bar{j}_u)$ implies either $v = u$, $v > u$ with $v \notin R(j)$, or $v > u$ with $v \in R(j)$ and $v = j^{k_i}(i)$ where $i \notin R(j)$ and $k_i \ge 1$. Thus from (3) of Theorem 1 we will be done if for $v \ge u$ and $v \notin R(j)$ we let $S_v = T_v$, and if $v \ge u$ with $v \in R(j)$ we let $S_v = p^{k_i} T_i$ where $j^{k_i}(i) = v$, $i \notin R(j)$.

THEOREM 2. *Let* A *have a* j*-diagram and* $A = \bigoplus_{i \in R(j)} T_i$ *with the* T_i *as in Theorem* 1. *Then* diag(A) *is woven from the following diagrams for the* T_i : $T_i = T_{i,0} > T_{i,1} > \ldots > T_{i,\nu_i} = 0$ *with* $T_{i,k} = pT_{i,k-1}$ *and* $j_i(k) = k + 1$ *for* $0 \le k < \nu_i$.

Proof. From Lemma 1 we have $A_u = \left(\bigoplus_{\substack{i \ge u \\ i \notin R(j)}} T_i \right) \oplus \left(\bigoplus_{\substack{i < u \\ i \notin R(j)}} p^{k_i} T_i \right)$ and

$A_{u+1} = \left(\bigoplus_{\substack{i \ge u+1 \\ i \notin R(j)}} T_i \right) \oplus \left(\bigoplus_{\substack{i < u+1 \\ i \notin R(j)}} p^{k_i} T_i \right)$. Now if $u \notin R(j)$ then

$A_u = T_u \oplus \left(\bigoplus_{\substack{i > u \\ i \notin R(j)}} T_i \right) \oplus \left(\bigoplus_{\substack{i < u \\ i \notin R(j)}} p^{k_i} T_i \right)$. Since $j(u) \notin R(\bar{j}_{u+1})$ or $j(u) = n$ we have in the former case, from the proof of Lemma 1, that

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$$A_{u+1} = pT_u \oplus \Big(\bigoplus_{\substack{i>u \\ i \notin R(j)}} T_i \Big) \oplus \Big(\bigoplus_{\substack{i<u \\ i \notin R(j)}} p^{k_i} T_i \Big)$$ or in the latter case

$$A_{u+1} = \Big(\bigoplus_{\substack{i>u \\ i \notin R(j)}} T_i \Big) \oplus \Big(\bigoplus_{\substack{i<u \\ i \notin R(j)}} p^{k_i} T_i \Big)$$ since $pT_u = 0$. If $u \in R(j)$ then

$$A_u = \Big(\bigoplus_{\substack{i>u \\ i \notin R(j)}} T_i \Big) \oplus p^{k_h} T_h \oplus \Big(\bigoplus_{\substack{i<u \\ i \notin R(j) \\ i \neq h}} p^{k_i} T_i \Big)$$ where $j^{k_h}(h) = u$ and

$h \notin R(j)$. Again considering whether $j(u) \notin R(\bar{j}_{u+1})$ or $j(u) = n$ we

have that either $A_{u+1} = \Big(\bigoplus_{\substack{i>u \\ i \notin R(j)}} T_i \Big) \oplus p^{k_h+1} T_h \oplus \Big(\bigoplus_{\substack{i<u \\ i \notin R(j) \\ i \neq h}} p^{k_i} T_i \Big)$ or

$$A_{u+1} = \Big(\bigoplus_{\substack{i>u \\ i \notin R(j)}} T_i \Big) \oplus \Big(\bigoplus_{\substack{i<u \\ i \notin R(j) \\ i \neq h}} p^{k_i} T_i \Big)$$. Thus we see that diag(A) is woven

from the j_i-diagrams for the T_i , $i \notin R(j)$.

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