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THE UNIQUE CONTINUATION PROPERTY OF p-HARMONIC FUNCTIONS ON THE HEISENBERG GROUP

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Abstract

We introduce an Almgren frequency function of the sub-p-Laplace equation on the Heisenberg group to establish a doubling estimate under the assumption that the frequency function is locally bounded. From this, we obtain some partial results on unique continuation for the sub-p-Laplace equation.

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1. Introduction

We investigate the unique continuation property for a class of quasilinear subelliptic equations on the Heisenberg group. We recall that the Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group of step two whose underlying manifold is $\mathbb{R}^{2n} \times \mathbb{R}$ with coordinates $(z, t) = (x, y, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ and whose group action $\circ$ is given by

$$(x_0, y_0, t_0) \circ (x, y, t) = \Big(x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i})\Big). \tag{1.1}$$

The left invariant vector fields corresponding to the canonical basis of the Lie algebra are

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad X_{n+i} = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

and the only nontrivial commutators are

$$[X_i, X_{n+i}] \equiv X_i X_{n+i} - X_{n+i} X_i = -4\partial_t \equiv -4T$$

for $1 \le i \le n$. The horizontal gradient of a function f is defined by

$$\nabla_H f = Xf = (X_1 f, \ldots, X_n f, X_{n+1} f, \ldots, X_{2n} f).$$

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For $(z,t) \in \mathbb{H}^n$, the gauge norm is defined by

$$\rho(z,t) = \bigg(\Big(\sum_{i=1}^{n}(x_i^2 + y_i^2)\Big)^2 + t^2\bigg)^{1/4} \equiv (|z|^4 + t^2)^{1/4}. \tag{1.2}$$

(See [4] for more on the Heisenberg group; relevant facts are collected in Section 2.)

The sub-p-Laplace equation on $\mathbb{H}^n$ is

$$\Delta_H^p u = \sum_{i=1}^{2n} X_i(|Xu|^{p-2}X_i u) = 0, \quad 1 < p < \infty. \tag{1.3}$$

For $p = 2$, this is the Kohn–Laplace equation

$$\Delta_H u = \sum_{i=1}^{2n} X_i^2 u = 0.$$

The operator Δ_H fails to be elliptic at every point. However, thanks to Hörmander's celebrated result in [14], Δ_H is hypoelliptic. Moreover, Δ_H shares many properties with the Laplace operator on $\mathbb{R}^n$, including the mean value formula and the strong maximum principle. For the Heisenberg group, Mukherjee and Zhong [20, 24] recently proved the optimal result that weak solutions of $\Delta_H^p u = 0$ ($p \neq 2$) are locally in the class $C^{1,\alpha}$ for some $\alpha \in (0,1)$; the first published proof valid for $p > 4$ is due to Ricciotti [22]. The $C^{1,\alpha}$ regularity will play an important role in our paper.

A differential operator L is said to have the unique continuation property in Ω if every solution u of $Lu = 0$ which vanishes on an open subset of Ω vanishes throughout Ω. There are many results on (strong) unique continuation for second order elliptic operators on $\mathbb{R}^n$ with linear primary parts (see, for example, [9]). In addition, there are some results about the unique continuation property for subelliptic equations [8, 10, 17]. However, little is known about the unique continuation problem for nonlinear elliptic equations (such as the p-Laplace equation), except for the planar case using the theory of quasiregular mappings [13, 19]. Recently, Granlund and Marola [11] studied the unique continuation problem of the p-Laplace equation by introducing a generalisation of Almgren's frequency function and obtained the unique continuation principle for the p-Laplace equation under the assumption that the frequency function is locally bounded.

The goal of this paper is to study the unique continuation problem of the sub-p-Laplace equation (1.3) on the Heisenberg group. In order to describe our results, we first introduce some definitions. For $u \in HW^{1,p}(\Omega) \cap C^1(\Omega)$ (see Section 2 for the definition of $HW^{1,p}$) and $\overline{B}_r \subset \Omega$, we define the height

$$H_p(r) = \int_{\partial B_r} \frac{|u|^p}{|\nabla \rho|} \psi^{p/2}\, dH^{2n},$$

where $\psi = |z|^2/\rho^2$ (this function will be explained in Section 2), and the sub-p-Dirichlet integral

$$D_p(r) = \int_{B_r} |Xu|^p\, dz\, dt.$$

The frequency function is

$$N_p(r) = \frac{r^{p-1}D_p(r)}{H_p(r)} \quad \text{if } H_p(r) \neq 0.$$

The frequency function was first introduced by Almgren [2] for harmonic functions. Garofalo and Lin [9] showed the applications of the frequency function in the strong unique continuation problem. Here $N_p(r)$ is a generalisation of the frequency function

$$N_2(r) = \frac{r\int_{B_r} |Xu|^2}{\int_{\partial B_r} u^2\psi/|\nabla\rho|}$$

for the sub-Laplace equation $\Delta_H u = 0$ on $\mathbb{H}^n$ introduced by Garofalo and Lanconelli [8]. On the other hand, $N_p(r)$ is also a generalisation of the frequency function for the p-harmonic functions on $\mathbb{R}^n$ defined by Granlund and Marola [11]. To the best of our knowledge, $N_p(r)$ $(p \neq 2)$ in the subelliptic setting has not been previously studied.

Our main results are the following two theorems.

THEOREM 1.1. *Let u be an arbitrary function in $C^1(\Omega)$. Assume that there exist two concentric balls $B_{r_0} \subset \overline{B}_{R_0} \subset \Omega$ such that the frequency function $N_p(r)$ is well defined, that is, $H_p(r) > 0$ for every $r \in (r_0, R_0]$ and $\|N_p\|_{L^\infty(r_0,R_0)} < \infty$. Then, for any $r_2 \in (r_0, R_0)$, there exists some $r^* \in (r_0, R_0)$ such that for any $r_1 \in [r^*, r_2]$, the following doubling property holds:*

$$\int_{\partial B_{r_2}} \frac{|u|^p}{|\nabla\rho|}\psi^{p/2} \le 2\int_{\partial B_{r_1}} \frac{|u|^p}{|\nabla\rho|}\psi^{p/2}. \tag{1.4}$$

Based on the doubling estimate in Theorem 1.1, we are able to establish the unique continuation property for the sub-p-Laplace equation.

THEOREM 1.2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of the sub-p-Laplace equation (1.3). For arbitrary balls $B_{r_0} \subset \overline{B}_{R_0} \subset \Omega$ such that $H_p(r) > 0$ for $r \in (r_0, R_0]$, assume that $\|N_p\|_{L^\infty(r_0,R_0)} < \infty$. Then, if u vanishes on some open ball in Ω, u is identically zero in Ω.*

The rest of the paper is organised as follows. In the next section, we collect some facts about the Heisenberg group and the sub-p-Laplace equation. In Section 3, we first study the behaviour of $H_p(r)$ and $D_p(r)$ for weak solutions of the sub-p-Laplace equation and then prove the main theorems following the argument in [11].

2. Preliminaries

In this section, we gather some notation about the Heisenberg group and well-known results about the sub-p-Laplace equation.

The Heisenberg group $\mathbb{H}^n$ has a family of dilations that are group homomorphisms, parameterised by $\lambda > 0$ and given by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t),$$

which leads to a homogeneous dimension $Q = 2n + 2$.

The gauge norm ρ defined in (1.2) satisfies

$$\rho(\delta_\lambda(z,t)) = \lambda\rho(z,t),$$

that is, ρ is homogeneous of degree one with respect to the dilation δ_λ. The associated distance between (z,t) and (z_0,t_0) is defined by

$$\rho(z,t;z_0,t_0) = \rho((z_0,t_0)^{-1}\circ(z,t)),$$

where $(z_0,t_0)^{-1}$ denotes the inverse of (z_0,t_0) with respect to the group action (1.1), that is, $(z_0,t_0)^{-1} = (-z_0,-t_0)$.

For vector fields $X = \{X_1,\dots,X_{2n}\}$, one can usually define the Carnot–Carathéodory distance d_{CC} in the following way. A Lipschitz path $\gamma : [0,T] \to \mathbb{H}^n$ is said to be a subunit with respect to the fields X if there exist measurable coefficients $c_j(s)$ such that

$$\dot\gamma(s) = \sum_{j=1}^{2n} c_j(s)X_j(\gamma(s)) \quad\text{and}\quad \sum_{j=1}^{2n} c_j^2(s)\le 1 \quad\text{for a.e. } s\in[0,T].$$

Then the Carnot–Carathéodory distance d_{CC} is defined by

$$d_{CC}(\xi,\xi') = \inf\{T\ge 0 \mid \text{there exists a subunit path } \gamma : [0,T]\to\mathbb{H}^n \text{ joining } \xi \text{ to } \xi'\}.$$

By a simple version of the 'ball–box' theorem (see, for example, [3]), d_{CC} is equivalent to the gauge distance.

In the following, we let

$$B_r = \{(z,t)\in\mathbb{H}^n \mid \rho(z,t) < r\},\qquad \partial B_r = \{(z,t)\in\mathbb{H}^n \mid \rho(z,t) = r\}$$

and call these sets respectively the Heisenberg ball and the Heisenberg sphere centred at the origin with radius r. Since $\rho\in C^\infty(\mathbb{H}^n\backslash\{(0,0)\})$, the outer unit normal on ∂B_r is given by $\vec{n} = |\nabla\rho|^{-1}\nabla\rho$, where $\nabla\rho$ means the Euclidean gradient of ρ. Balls and spheres centred at (z_0,t_0) are defined by left translation, that is,

$$B_r(z_0,t_0) = \{(z,t)\in\mathbb{H}^n \mid \rho(z,t;z_0,t_0) < r\}$$

and

$$\partial B_r(z_0,t_0) = \{(z,t)\in\mathbb{H}^n \mid \rho(z,t;z_0,t_0) = r\}.$$

Introducing the function

$$\psi(z,t) = |\nabla_H\rho|^2 = \frac{|z|^2}{\rho(z,t)^2} \tag{2.1}$$

allows us to write

$$|B_r| = \int_{B_r}\psi\,dz\,dt.$$

Using polar coordinates that match the gauge norm (1.2) for $\mathbb{H}^n$ (introduced by Greiner [12]), it is not hard to see that there exists a constant $\omega_Q > 0$ depending only on Q such that

$$|B_r| = \omega_Q r^Q.$$

We now recall the co-area formula (see [16, Theorem 4.2.1]): if $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz and $m \geq n$, then

$$\int_{\mathbb{R}^m} g(x) J_n f(x)\, dH^m_x = \int_{\mathbb{R}^n} \int_{f^{-1}(y)} g(x)\, dH^{m-n}_x\, dH^n_y \tag{2.2}$$

for every integrable function g.

Applying (2.2) with $f = \rho(z,t)$ and $g = \psi(z,t)/|\nabla\rho(z,t)|$ gives

$$|B_r| = \int_{B_r} \psi(z,t)\, dH^{2n+1} = \int_0^r ds \int_{\partial B_s} \frac{\psi(z,t)}{|\nabla\rho(z,t)|}\, dH^{2n}. \tag{2.3}$$

Next, we collect some basic facts about the horizontal gradient ∇_H on $\mathbb{H}^n$. For a multi-index $J = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{N}^{2n}$, let

$$X^J f = X_{i_1}^{\alpha_{i_1}} X_{i_2}^{\alpha_{i_2}} \cdots X_{i_{2n}}^{\alpha_{i_{2n}}} f$$

denote a horizontal derivative of f of order $|J| = \sum_{j=1}^{2n} \alpha_j$. The natural volume in $\mathbb{H}^n$ is the Haar measure, which coincides with Lebesgue measure L^{2n+1} in $\mathbb{R}^{2n+1}$. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. If $k \in \mathbb{N}$ and $1 \leq p < \infty$, the horizontal Sobolev spaces $HW^{k,p}(\Omega)$ can be defined in the natural way (see, for example, [23]):

$$HW^{k,p}(\Omega) = \{f : \Omega \to \mathbb{R} \mid |X^J f| \in L^p(\Omega) \text{ for } 0 \leq |J| \leq k\}.$$

This is a Banach space with the norm

$$\|f\|_{HW^{1,p}(\Omega)} = \bigg(\int_\Omega \bigg(\sum_{i=1}^{2n} |X_i f|^p + |f|^p \bigg) \bigg)^{1/p}.$$

The closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ is denoted by $HW_0^{1,p}(\Omega)$.

Now we recall some properties of the sub-p-Laplace equation (1.3). We say that a function $u \in HW^{1,p}(\Omega)$ is a weak solution of (1.3) if

$$\int_\Omega |Xu|^{p-2} \langle Xu, X\phi \rangle = 0 \quad \text{for all } \phi \in HW_0^{1,p}(\Omega).$$

It is easy to show that a function $u \in HW^{1,p}(\Omega)$ is a local minimiser of the functional

$$I(v) = \int_\Omega |Xv|^p, \quad 1 < p < \infty,$$

if and only if u is a weak solution of (1.3).

For $p = 2$, it is now classical that the solutions of the equation $\Delta_H u = 0$ are C^∞ [14]. For $p \neq 2$, it is well known that weak solutions of the p-Laplace equation in Euclidean space are of the class $C^{1,\alpha}$ (see [7]). The $C^{1,\alpha}$ regularity is optimal when $p \geq 2$, as shown by examples in [15]. The corresponding optimal regularity of the sub-p-Laplace equation on the Heisenberg group was resolved recently by Zhong [24] for $p > 2$ and Mukherjee and Zhong [20] for $1 < p < 2$, following earlier work of Ricciotti [21, 22].

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Next, we recall some basic identities on $\mathbb{H}^n$. We denote by S the $2n \times (2n+1)$ matrix relating the horizontal gradient ∇_H in $\mathbb{H}^n$ and the standard gradient ∇ in $\mathbb{R}^{2n+1}$, that is, $\nabla_H = S \cdot \nabla$, where

$$S = \begin{pmatrix} I_{n\times n} & 0_{n\times n} & (2y)^T \\ 0_{n\times n} & I_{n\times n} & (-2x)^T \end{pmatrix}.$$

Hence,

$$\Delta_H u = \sum_{i=1}^{2n} X_i(X_i u) = \operatorname{div}(S^T S \nabla u).$$

It is easy to check that

$$\Delta_H f(\rho) = \psi\Big(f''(\rho) + \frac{Q-1}{\rho} f'(\rho)\Big),$$

so that

$$\Delta_H \rho = \frac{Q-1}{\rho}\psi \quad \text{in } \mathbb{H}^n \backslash \{0\}. \tag{2.4}$$

To end this section, we give a simple and basic identity that will be used later.

Lemma 2.1. *Let ρ and ψ be the gauge norm and the function defined above. Then*

$$\langle X\rho, X\psi\rangle \equiv \sum_{i=1}^{2n} X_i\rho X_i\psi = 0 \quad \textit{in } \mathbb{H}^n \backslash \{0\}. \tag{2.5}$$

Proof. The horizontal derivatives of $\rho = (|z|^4 + t^2)^{1/4}$ and $\psi = |z|^2/\rho^2$ are

$$X_i\rho = \frac{1}{\rho^3}(|z|^2 x_i + y_i t), \quad X_{n+i}\rho = \frac{1}{\rho^3}(|z|^2 y_i - x_i t)$$

and

$$X_i\psi = \frac{2x_i}{\rho^2} - \frac{2|z|^2}{\rho^3} X_i\rho, \quad X_{n+i}\psi = \frac{2y_i}{\rho^2} - \frac{2|z|^2}{\rho^3} X_{n+i}\rho.$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{2n} X_i\rho X_i\psi &= \sum_{i=1}^{2n} \Big(\frac{2x_i}{\rho^2} X_i\rho + \frac{2y_i}{\rho^2} X_{n+i}\rho\Big) - \frac{2|z|^2}{\rho^3}|X\rho|^2 \\ &= \frac{2}{\rho^5}\sum_{i=1}^{n} (x_i(|z|^2 x_i + y_i t) + y_i(|z|^2 y_i - x_i t)) - \frac{2|z|^4}{\rho^5} \\ &= 0. \end{aligned}$$

□

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3. Proofs of the main results

We first prove some properties of $D(r)$ and $H(r)$ and then prove the main theorems.

LEMMA 3.1. *Let u be a weak solution of the sub-p-Laplace equation (1.3) in B_R. Then, for any $r \in (0, R)$,*

$$D_p(r) = \int_{\partial B_r} |Xu|^{p-2} u \frac{\langle Xu, X\rho\rangle}{|\nabla\rho|}\, dH^{2n}. \tag{3.1}$$

PROOF. As in the case $p = 2$, the proof is based on the divergence theorem. However, for the general case $1 < p < \infty$, u is not C^2. Hence, we need to use an approximation argument. Let $0 < \varepsilon < 1$. For $\overline{B}_r \subset \overline{D} \subset \Omega$, we construct a sequence of functions $u_\varepsilon \in HW^{1,p}(D)$ which minimise the variational integral

$$I_\varepsilon(v) = \int_D (\varepsilon + |Xv|^2)^{p/2}$$

over the admissible functions in $\mathcal{K}_u(D) = \{v \in HW^{1,p}(D) \mid v - u \in HW_0^{1,p}(D)\}$. It is well known that the minimising function u_ε is unique and u_ε is a weak solution to

$$\sum_{i=1}^{2n} X_i((\varepsilon + |Xu_\varepsilon|^2)^{(p-2)/2} X_i u_\varepsilon) = 0. \tag{3.2}$$

Recall that, for $\varepsilon > 0$, weak solutions u_ε to the above nondegenerate sub-p-Laplace equation are smooth. This was proved by Capogna in [5] for $p \geq 2$ and extended to the full range $1 < p < \infty$ in [21] by adapting techniques of Domokos [6].

By integration by parts and (3.2),

$$\begin{aligned}
&\int_{B_r} (\varepsilon + |Xu_\varepsilon|^2)^{(p-2)/2} |Xu_\varepsilon|^2 \\
&= -\int_{B_r} u_\varepsilon \sum_{i=1}^{2n} X_i((\varepsilon + |Xu_\varepsilon|^2)^{(p-2)/2} X_i u_\varepsilon) + \int_{\partial B_r} (\varepsilon + |Xu_\varepsilon|^2)^{(p-2)/2} u_\varepsilon \frac{\langle Xu_\varepsilon, X\rho\rangle}{|\nabla\rho|} \\
&= \int_{\partial B_r} (\varepsilon + |Xu_\varepsilon|^2)^{(p-2)/2} u_\varepsilon \frac{\langle Xu_\varepsilon, X\rho\rangle}{|\nabla\rho|}.
\end{aligned} \tag{3.3}$$

By the recent results on the Hölder continuity of the horizontal gradient of the solution to the sub-p-Laplace equation on $\mathbb{H}^n$ (see [20, 22, 24]), there exists $\alpha > 0$, depending only on p, Q and a positive constant $M < \infty$, depending on p, Q and D, such that

$$\max_{(z,t)\in\overline{D}} |Xu_\varepsilon(z, t)| \leq M \tag{3.4}$$

and, for each $(z_1, t_1), (z_2, t_2) \in D$,

$$|Xu_\varepsilon(z_1, t_1) - Xu_\varepsilon(z_2, t_2)| \leq M\rho\,((z_1, t_1); (z_2, t_2))^\alpha\,. \tag{3.5}$$

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We note that α and M are independent of ε. By (3.4) and the Poincaré inequality for the horizontal vector fields X (see, for example, [18, Theorem C]),

$$\|u_\varepsilon\|_{HW^{1,p}(D)} \le C,$$

where the constant C is independent of ε. Then, from the weak compactness of $HW^{1,p}$, there exist a subsequence of $\{u_\varepsilon\}$ (still denoted by u_ε) and a function $w \in \mathcal{K}_u(D)$ such that

$$u_\varepsilon \rightharpoonup w \quad \text{weakly in } HW^{1,p}.$$

It is not hard to prove that w minimises the sub-p-Dirichlet integral $\int_D |Xv|^p$ over $\mathcal{K}_u(D)$ and so $w = u$.

Furthermore, (3.4) and (3.5) imply that the sequences $\{Xu_\varepsilon\}$ are uniformly bounded and equicontinuous. On the other hand, applying the maximum principle of the sub-p-Laplace equation (see, for example, [1, Lemma 3]) and noting that $u_\varepsilon - u \in HW_0^{1,p}(D)$, we see that the sequences $\{u_\varepsilon\}$ are uniformly bounded. The equicontinuity of $\{u_\varepsilon\}$ follows from (3.4) and Morrey's lemma on the Heisenberg group (see, for example, [5, Lemma 4.5]). Therefore, by the Ascoli–Arzelà theorem, there is a subsequence of $\{u_\varepsilon\}$ and of $\{Xu_\varepsilon\}$ (both still denoted by $\{u_\varepsilon\}$ and $\{Xu_\varepsilon\}$) such that

$$u_\varepsilon \to u \quad \text{and} \quad Xu_\varepsilon \to Xu \quad \text{uniformly in } \overline{D}.$$

We get the desired identity (3.1) by taking $\varepsilon \to 0$ in (3.3). This completes the proof of the lemma. □

Lemma 3.2. *Let u be a weak solution of (1.3) in B_R. Then there exists r_0, depending only on Q, such that either $u \equiv 0$ in B_{r_0} or $H_p(r) \neq 0$ for every $r \in (0, r_0)$.*

Proof. Suppose that $H_p(r_0) = 0$ for some $r_0 \le R$. Then $u = 0$ on ∂B_{r_0}. Therefore, from (3.1), $D_p(r_0) = 0$, which implies that $Xu = 0$ in B_{r_0}. Thus, $u \equiv 0$ in B_{r_0}. □

Lemma 3.3. *Let u be an arbitrary function in $C^1(B_R)$. Then, for any $r \in (0, R)$,*

$$H_p'(r) = \frac{Q-1}{r} H_p(r) + p \int_{\partial B_r} |u|^{p-2} u \psi^{(p/2)-1} \frac{\langle Xu, X\rho\rangle}{|\nabla\rho|}\, dH^{2n}$$

and

$$H_p'(r) \le \frac{Q-1}{r} H_p(r) + p \int_{\partial B_r} |u|^{p-1} |Xu| \psi^{(p-1)/2} \frac{1}{|\nabla\rho|}\, dH^{2n}. \tag{3.6}$$

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PROOF. From (2.1) and the divergence theorem,

$$\begin{aligned}
H_p(r) &= \int_{\partial B_r} |u|^p \psi^{(p/2)-1} \frac{\langle X\rho, X\rho\rangle}{|\nabla\rho|} = \int_{\partial B_r} |u|^p \psi^{(p/2)-1} \langle S^T X\rho, \vec{n}\rangle \\
&= \int_{B_r} \operatorname{div}(S^T X\rho |u|^p \psi^{(p/2)-1}) \\
&= \int_{B_r} |u|^p \psi^{(p/2)-1} \Delta_H \rho + p \int_{B_r} |u|^{p-2} u \, \langle Xu, X\rho\rangle \, \psi^{(p/2)-1} \\
&\quad + ((p/2)-1) \int_{B_r} \psi^{(p/2)-2} |u|^p \, \langle X\rho, X\psi\rangle \\
&= \int_{B_r} |u|^p \psi^{(p/2)-1} \Delta_H \rho + p \int_{B_r} |u|^{p-2} u \, \langle Xu, X\rho\rangle \, \psi^{(p/2)-1},
\end{aligned}$$

where we used (2.5) in the last equality. Then, by the co-area formula (2.3) and (2.4),

$$\begin{aligned}
H_p'(r) &= \int_{\partial B_r} \frac{|u|^p \Delta_H \rho}{|\nabla\rho|} \psi^{(p/2)-1} + p \int_{\partial B_r} \frac{|u|^{p-2} u \langle Xu, X\rho\rangle}{|\nabla\rho|} \psi^{(p/2)-1} \\
&= \frac{Q-1}{r} \int_{\partial B_r} \frac{|u|^p}{|\nabla\rho|} \psi^{p/2} + p \int_{\partial B_r} \frac{|u|^{p-2} u \langle Xu, X\rho\rangle}{|\nabla\rho|} \psi^{(p/2)-1} \\
&= \frac{Q-1}{r} H_p(r) + p \int_{\partial B_r} \frac{|u|^{p-2} u \langle Xu, X\rho\rangle}{|\nabla\rho|} \psi^{(p/2)-1}
\end{aligned}$$

and (3.6) follows from this and (2.1). This completes the proof of the lemma. □

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. First fix $r_2 \in (r_0, R_0]$. We can assume that the function $H_p(r)$ is not decreasing. Otherwise, the desired doubling property (1.4) is obviously true. Since $H_p(r)$ is continuous and $H_p(r)$ is not decreasing, there exist some $r_1 \in (r_0, r_2]$ such that $H_p(r) \le H_p(r_2)$ for any $r \in [r_1, r_2]$.

Integrating both sides of (3.6) over (r_1, r_2),

$$\begin{aligned}
H_p(r_2) - H_p(r_1) &\le (Q-1) \int_{r_1}^{r_2} \frac{H_p(r)}{r} \, dr + p \int_{r_1}^{r_2} \Big(\int_{\partial B_r} |u|^{p-1} |Xu| \psi^{(p-1)/2} \frac{1}{|\nabla\rho|} dH^{2n} \Big) dr \\
&\le (Q-1) H_p(r_2) \log \frac{r_2}{r_1} + \varepsilon \int_{r_1}^{r_2} r^{p-1} \Big(\int_{\partial B_r} \frac{|Xu|^p}{|\nabla\rho|} dH^{2n} \Big) dr \\
&\quad + C(p,\varepsilon) \int_{r_1}^{r_2} \frac{1}{r} \Big(\int_{\partial B_r} \frac{|u|^p}{|\nabla\rho|} \psi^{p/2} dH^{2n} \Big) dr \\
&\le (Q-1) H_p(r_2) \log \frac{r_2}{r_1} + \varepsilon r_2^{p-1} \int_{B_{r_2}} |Xu|^p + C(p,\varepsilon) H_p(r_2) \log \frac{r_2}{r_1},
\end{aligned}$$

where we have applied Young's inequality in the second inequality. We shall fix ε later. Dividing the above inequality by $H_p(r_2)$ gives

$$\frac{H_p(r_2) - H_p(r_1)}{H_p(r_2)} \le (Q - 1 + C(p,\varepsilon)) \log \frac{r_2}{r_1} + \varepsilon N_p(r_2), \tag{3.7}$$

where r_2 is fixed and $H_p(r_2) = \max_{r \in [r_1, r_2]} H_p(r)$.

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Now, we shall estimate the right-hand side in (3.7). The frequency function $N_p(r)$ is locally bounded by hypothesis, say $\|N_p(r)\|_{L^\infty(r_0,R_0)} = M$. We first set $\varepsilon = 1/(4M)$ and then choose $r^* \in (r_0, r_2]$ sufficiently close to r_2 so that, for any $r_1 \in [r^*, r_2]$,

$$(Q - 1 + C(p,\varepsilon)) \log \frac{r_2}{r_1} \le \frac{1}{4}.$$

Therefore, for $r_1 \in [r^*, r_2]$,

$$\frac{H_p(r_2) - H_p(r_1)}{H_p(r_2)} \le \frac{1}{2},$$

which implies that

$$H_p(r_2) \le 2H_p(r_1).$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Suppose that u is a nontrivial solution to the sub-p-Laplace equation (1.3) and vanishes in $\overline{B}_{r_1}$, but that u is not identically zero in B_{r_2}, where $\overline{B}_{r_1} \subset B_{r_2} \subset \Omega$. For $s > 0$, define

$$s^* = \sup\{s > 0 : u|_{\partial B_s} \equiv 0\},$$

so that $s^* \in [r_1, r_2)$. By Lemma 3.1, for any radius $\lambda \in (s^*, r_2]$, we have $H_p(\lambda) \neq 0$. Theorem 1.1 implies that there exists $r^* \in (s^*, r_2]$ such that

$$H_p(r*) \le 2H_p(r)$$

for every $r \in (s^*, r^*]$. This is a contradiction, because $H_p(r) \to 0$ as $r \to s^*$. □

Finally, we give a sufficient condition for the boundedness of $N_p(r)$.

Lemma 3.4. *Suppose that u is a nontrivial solution to the sub-p-Laplace equation (1.3). Assume that there exists a positive constant $A < \infty$ such that*

$$\int_{\partial B_r} \frac{|Xu|^p}{|\nabla\rho|}\, dH^{2n} \le A \int_{\partial B_r} \frac{|u|^p}{|\nabla\rho|} \psi^{p/2}\, dH^{2n}. \tag{3.8}$$

Then

$$N_p(r) < \infty.$$

Proof. By using (3.1), (3.8), Young's inequality and the fact that $|X\rho|^2 = \psi$,

$$\begin{aligned} D_p(r) &= \int_{\partial B_r} |Xu|^{p-2} u \frac{\langle Xu, X\rho\rangle}{|\nabla\rho|} \\ &\le \int_{\partial B_r} \frac{|Xu|^{p-1}}{|\nabla\rho|^{1-1/p}} \cdot \frac{|u|\psi^{1/2}}{|\nabla\rho|^{1/p}} \\ &\le \varepsilon \int_{\partial B_r} \frac{|Xu|^p}{|\nabla\rho|} + C(\varepsilon) \int_{\partial B_r} \frac{|u|^p}{|\nabla\rho|} \psi^{p/2} \\ &\le (A\varepsilon + C(\varepsilon)) H_p(r). \end{aligned}$$

This gives $N_p(r) \le (A\varepsilon + C(\varepsilon))$. □

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References

[1] T. Adamowicz and B. Warhurst, 'Three-spheres theorems for subelliptic quasilinear equations in Carnot groups of Heisenberg-type', *Proc. Amer. Math. Soc.* **144** (2016), 4291–4302.

[2] F. J. Almgren Jr, 'Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents', in: *Minimal Submanifolds and Geodesics* (ed. M. Obata) (North-Holland, Amsterdam, 1979), 1–6.

[3] A. Bellaïche, 'The tangent space in sub-Riemannian geometry', *Progr. Math.* **144** (1996), 1–78.

[4] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer Monographs in Mathematics (Springer, Berlin–Heidelberg–New York, 2007).

[5] L. Capogna, 'Regularity of quasi-linear equations in the Heisenberg group', *Comm. Pure Appl. Math.* **50** (1997), 867–889.

[6] A. Domokos, 'Differentiability of solutions for the non-degenerate p-Laplacian in the Heisenberg group', *J. Differential Equations* **204** (2004), 439–470.

[7] L. Evans, 'A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e.', *J. Differential Equations* **45** (1982), 356–373.

[8] N. Garofalo and E. Lanconelli, 'Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation', *Ann. Inst. Fourier* **44** (1990), 313–356.

[9] N. Garofalo and F. Lin, 'Unique continuation for elliptic operators: a geometric-variational approach', *Comm. Pure Appl. Math.* **40** (1987), 347–366.

[10] N. Garofalo and D. Vassilev, 'Strong unique continuation properties of generalized Baouendi–Grushin operators', *Comm. Partial Differential Equations* **32** (2007), 643–663.

[11] S. Granlund and N. Marola, 'On the problem of unique continuation for the p-Laplace equation', *Nonlinear Anal. Theory Methods Appl.* **101** (2014), 89–97.

[12] P. C. Greiner, 'Spherical harmonics on the Heisenberg group', *Canad. Math. Bull.* **23** (1980), 383–396.

[13] C. Guo and M. Kar, 'Quantitative uniqueness estimates for p-Laplace type equations in the plane', *Nonlinear Anal. Theory Methods Appl.* **143** (2016), 19–44.

[14] L. Hörmander, 'Hypoelliptic second-order differential equations', *Acta Math.* **119** (1967), 147–171.

[15] J. L. Lewis, 'Smoothness of certain degenerate elliptic equations', *Proc. Amer. Math. Soc.* **80** (1980), 259–265.

[16] F. Lin and X. Yang, *Geometric Measure Theory: An Introduction* (International Press, Boston, MA, 2002).

[17] H. Liu and X. Yang, 'Strong unique continuation of sub-elliptic operator on the Heisenberg group', *Chin. Ann. Math. Ser. B* **34** (2013), 461–478.

[18] G. Lu, 'Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications', *Rev. Mat. Iberoam.* **8** (1992), 367–439.

[19] J. J. Manfredi, 'p-harmonic functions in the plane', *Proc. Amer. Math. Soc.* **103** (1988), 473–479.

[20] S. Mukherjee and X. Zhong, '$C^{1,\alpha}$-regularity for variational problems in the Heisenberg group', Preprint, 2017, arXiv:1711.04671v2.

[21] D. Ricciotti, *p-Laplace Equation in the Heisenberg Group, Regularity of Solutions*, Springer Briefs in Mathematics (Springer, Cham, Switzerland, 2015).

[22] D. Ricciotti, 'On the $C^{1,\alpha}$ regularity of p-harmonic functions in the Heisenberg group', *Proc. Amer. Math. Soc.* **146** (2018), 2937–2952.

[23] L. P. Rothschild and E. M. Stein, 'Hypoelliptic differential operators and nilpotent groups', *Acta Math.* **137** (1976), 247–320.

[24] X. Zhong, 'Regularity for variational problems in the Heisenberg group', Preprint, 2017, arXiv:1711.03284.

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