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DISTRIBUTION OF MINIMAL PATH LENGTHS WHEN EDGE LENGTHS ARE INDEPENDENT HETEROGENEOUS EXPONENTIAL RANDOM VARIABLES

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Abstract

We find the joint distribution of the lengths of the shortest paths from a specified node to all other nodes in a network in which the edge lengths are assumed to be independent heterogeneous exponential random variables. We also give an efficient way to simulate these lengths that requires only one generated exponential per node, as well as efficient procedures to use the simulated data to estimate quantities of the joint distribution.

Keywords: Shortest path; exponential edge length; joint distribution; simulation

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1. Introduction

Consider the complete directed graph on nodes $0, 1, \dots, n$, and suppose that, for $i \neq j$, $X_{i,j}$ is the time taken to traverse the edge from i to j. With L_i equal to the minimal length, measured in units of traversal time, of a path from node 0 to node i, $i = 1, \dots, n$, we are interested in studying the random variables $L_1, \dots, L_n$ when $X_{i,j}$, $i \neq j$, are independent exponential random variables, with $\mathrm{E}[X_{i,j}] = 1/\lambda_{i,j}$. In Section 2 we show that the joint distribution of $L_1, \dots, L_n$ is the same as the joint distribution of component lifetimes of a certain n component system. Utilizing this equivalence, we derive the joint distribution of $L_1, \dots, L_n$ in Section 2.1. In Section 3 we give an effective way of simulating $L_1, \dots, L_n$ that involves only n random numbers, each one being used to generate an exponential random variable. Also, in Section 3 we show how to efficiently use simulation to estimate such quantities as the mean and the probability distribution of L_i. Finally, in Section 4 we consider the exchangeable case, which results when $\lambda_{0,i} = \lambda$ and $\lambda_{i,j} = \mu$, $i, j = 1, \dots, n$.

The problem of finding the distribution of minimal cost paths when the edge costs are independent exponential random variables has previously been solved in [4] for the symmetric case where all the $\lambda_{i,j}$ are equal. Other papers dealing with the symmetric case are [1], [2], and [3]. The heterogeneous case was previously considered in [1] and [5] where approaches for finding $\mathrm{E}[L_i]$ were presented. The computational requirements of these approaches, however, require solving a set of recursive equations whose cardinality grows exponentially in n.

2. The component system model

Consider n components, where component j initially has the failure rate $\lambda_{0,j}$. Suppose, however, that a component failure increases the failure rates of the still working components,

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in that the failure of component i increases the failure rate of the still working component j by the amount $\lambda_{i,j}$. That is, if L_j^* is the lifetime of component j then, with $F(t)$ denoting the set of failed components at time t, we suppose that

$$\mathrm{P}\Big(t < L_j^* < t+h \;\Big|\; F(t)\Big) = \Big(\lambda_{0,j} + \sum_{i \in F(t)} \lambda_{i,j}\Big)h + o(h), \qquad j \notin F(t).$$

Moreover, we assume that

$$\mathrm{P}(t < L_i^* < t+h,\; t < L_j^* < t+h \mid F(t)) = o(h), \qquad i \neq j.$$

As a consequence, $\{F(t),\ t \geq 0\}$ is a continuous-time Markov chain with instantaneous transition rates

$$q_{S,S\cup\{j\}} = \lambda_{0,j} + \sum_{i \in N-S} \lambda_{i,j}, \qquad j \notin S,$$

where $S \subset N = \{1, \dots, n\}$.

The following lemmas will be used to show that $L_1, \dots, L_n$ and $L_1^*, \dots, L_n^*$ have the same distribution.

Lemma 1. *Assuming that* $c_{i,j} > 0,\ i, j = 0, 1, \dots, n$, *there is a unique solution* $(y_1, \dots, y_n)$ *to the equation*

$$y_j = \min\Big(c_{0,j}, \min_{i \neq j}(y_i + c_{i,j})\Big), \qquad j = 1, \dots, n. \tag{1}$$

Proof. To argue that there is always a solution to the above equation, consider the complete directed graph on nodes $0, 1, \dots, n$ in which $c(i, j)$ is the length of the edge (i, j). Then, if m_j represents the minimal distance from node 0 to node j, it follows that $(m_1, \dots, m_n)$ satisfies (1). It remains to prove uniqueness, which we do by induction on n. As uniqueness is immediate when $n = 1$, assume uniqueness for $n - 1$. Now, let $y_1, \dots, y_n$ be a solution of (1), and suppose that $y_1 = \min_j y_j$. Then

$$y_1 = \min\Big(c_{0,1}, \min_{i \neq 1}(y_i + c_{i,1})\Big) \geq \min\Big(c_{0,1}, \min_{i \neq 1}(y_1 + c_{i,1})\Big),$$

which, since $c_{i,1} > 0$, implies that $y_1 \geq c_{0,1}$ and, thus, by (1), that $y_1 = c_{0,1}$. Consequently, $y_2, \dots, y_n$ satisfy

$$\begin{aligned} y_j &= \min\Big(c_{0,j},\ c_{0,1} + c_{1,j}, \min_{1<i\neq j}(y_i + c_{i,j})\Big) \\ &= \min\Big(c_{0,j}^*, \min_{1<i\neq j}(y_i + c_{i,j})\Big), \qquad j = 2, \dots, n, \end{aligned}$$

where $c_{0,j}^* = \min(c_{0,j}, c_{0,1} + c_{1,j})$. The result now follows by the induction hypothesis.

Lemma 2. *For* $i \neq j,\ i, j = 0, 1, \dots, n$, *let* $X_{i,j}$ *be exponential with rate* $\lambda_{i,j}$. *Moreover, assume that these random variables are independent. Let* $T_j,\ j = 1, \dots, n$, *be the unique solution of*

$$T_j = \min\Big(X_{0,j}, \min_{i \neq j}(T_i + X_{i,j})\Big), \qquad j = 1, \dots, n. \tag{2}$$

Then $(T_1, \dots, T_n)$ *and* $(L_1^*, \dots, L_n^*)$ *have the same distribution.*

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Proof. Using the fact that the failure rate function of the minimum of independent random variables is equal to the sum of their failure rate functions, we can interpret our model as follows. Each component j, $j = 1, \dots, n$, has a primary event associated with it, with the primary event associated with j occurring at time $X_{0,j}$. The occurrence of the primary event associated with i causes i, if it is still working at that time, to immediately fail. In addition, the failure of component i results in the origin of events $A_{i,j}$, $j \neq i$, with the event $A_{i,j}$ occurring after an exponentially distributed time $X_{i,j}$ having rate $\lambda_{i,j}$, $j \neq i$. If $A_{i,j}$ occurs while j is still working then it immediately causes j to fail. Moreover, the $X_{i,j}$, $i \neq j$, $i, j = 0, 1, \dots, n$, are independent random variables. With this equivalent description, if T_j is the time at which component j fails then the T_j, $j = 1, \dots, n$, satisfy (2), proving the lemma.

Theorem 1. *The random variables $L_1, \dots, L_n$ and $L_1^*, \dots, L_n^*$ have the same distribution.*

Proof. It follows from the proof of Lemma 1, and from Lemma 2, that both $L_1, \dots, L_n$ and $L_1^*, \dots, L_n^*$ satisfy the equation

$$T_j = \min\Big(X_{0,j}, \min_{i \neq j}(T_i + X_{i,j})\Big), \qquad j = 1, \dots, n.$$

The result now follows from Lemma 1.

Throughout the rest of the paper, although we will use the notation $L_1, \dots, L_n$, in our analyses we will assume the component model interpretation.

2.1. The distribution of $(L_1, \dots, L_n)$

For a vector $\boldsymbol{t} = (t_1, \dots, t_n)$, let $t_0 = 0$ and set

$$r_j(\boldsymbol{t}) = \sum_{0 \le i \neq j} \lambda_{i,j}(t_j - t_i)^+, \qquad j = 1, \dots, n,$$

and

$$\alpha_j(\boldsymbol{t}) = \sum_{0 \le i \neq j} \lambda_{i,j}\, \mathbf{1}\{t_i < t_j\}, \qquad j = 1, \dots, n.$$

Proposition 1. *The joint density of $L_1, \dots, L_n$ is*

$$f(t_1, \dots, t_n) = \prod_{j=1}^{n} \alpha_j(\boldsymbol{t}) \mathrm{e}^{-r_j(\boldsymbol{t})}.$$

Proof. Since we can always renumber components $1, \dots, n$, assume without loss of generality that $t_1 < t_2 < \cdots < t_n$. Note that in this case

$$r_j(\boldsymbol{t}) = \sum_{i=0}^{j-1} \lambda_{i,j}(t_j - t_i), \qquad j = 1, \dots, n,$$

and

$$\alpha_j(\boldsymbol{t}) = \sum_{i=0}^{j-1} \lambda_{i,j}, \qquad j = 1, \dots, n.$$

In order for L_j to equal t_j for all $j = 1, \dots, n$, there must be no failures before time t_1, component 1 must fail at time t_1, no failures must occur for the next $t_2 - t_1$ time units,

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component 2 must fail at time t_2, and so on. Consequently, for $t_1 < t_2 < t_3 < \cdots < t_n$, the joint density of $L_1, \dots, L_n$ is

$$\begin{aligned} f(t_1, \dots, t_n) = \alpha_1(\boldsymbol{t}) \exp\Bigg[-t_1 \sum_{i=1}^n \lambda_{0,i}\Bigg] \alpha_2(\boldsymbol{t}) \exp\Bigg[-(t_2 - t_1) \sum_{i=2}^n (\lambda_{0,i} + \lambda_{1,i})\Bigg] \\ \times \alpha_3(\boldsymbol{t}) \exp\Bigg[-(t_3 - t_2) \sum_{i=3}^n (\lambda_{0,i} + \lambda_{1,i} + \lambda_{2,i})\Bigg] \cdots \\ \times \alpha_n(\boldsymbol{t}) \exp\Big[-(t_n - t_{n-1})(\lambda_{0,n} + \lambda_{1,n} + \cdots + \lambda_{n-1,n})\Big]. \end{aligned} \tag{3}$$

To show that the right-hand side is equal to $\prod_{j=1}^n \alpha_j(\boldsymbol{t}) \mathrm{e}^{-r_j(\boldsymbol{t})}$, we use induction on n. As the proof is immediate for $n = 1$, we assume that the equation holds for $n - 1$. Noting that

$$\begin{aligned} r_j(t_1, \dots, t_n) &= r_j(t_1, \dots, t_{n-1}), \qquad j = 1, \dots, n-1, \\ \alpha_j(t_1, \dots, t_n) &= \alpha_j(t_1, \dots, t_{n-1}), \qquad j = 1, \dots, n-1, \end{aligned}$$

it follows from (3) and the induction hypothesis that, for $t_1 < \cdots < t_n$,

$$f(t_1, \dots, t_n) = \alpha_n(\boldsymbol{t}) \mathrm{e}^{-A(\boldsymbol{t})} \prod_{j=1}^{n-1} \alpha_j(\boldsymbol{t}) \mathrm{e}^{-r_j(\boldsymbol{t})},$$

where

$$\begin{aligned} A(\boldsymbol{t}) &= t_1 \lambda_{0,n} + (t_2 - t_1)(\lambda_{0,n} + \lambda_{1,n}) + \cdots + (t_n - t_{n-1})(\lambda_{0,n} + \cdots + \lambda_{n-1,n}) \\ &= t_n \lambda_{0,n} + (t_n - t_1)\lambda_{1,n} + (t_n - t_2)\lambda_{2,n} + \cdots + (t_n - t_{n-1})\lambda_{n-1,n} \\ &= r_n(\boldsymbol{t}). \end{aligned}$$

This completes the proof.

3. Efficient simulation procedures

Using the component model interpretation, we now present an efficient way to simulate the vector $L_1, \dots, L_n$. The following algorithm requires only the generation of n exponentials.

1. Let $\theta_i = \lambda_{0,i}$, $i = 1, \dots, n$. Let F be the null set.
2. Generate $X_1, \dots, X_n$ independent exponentials with respective rates $\theta_1, \dots, \theta_n$.
3. Let $w = \operatorname{argmin}_{i \notin F} X_i$.
4. $L_w = X_w$, $F = F \cup \{w\}$.
5. If $F = \{1, \dots, n\}$ stop.
6. For $j \notin F$, reset $X_j = X_w + \theta_j(X_j - X_w)/(\theta_j + \lambda(w, j))$.
7. For $j \notin F$, reset $\theta_j = \theta_j + \lambda(w, j)$.
8. Go to step 3.

The key step to understanding the preceding is step 6, which uses the fact that if the X_j, $j \notin F$, are independent exponentials with respective rates θ_j then, conditional on X_w being

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the smallest among them, $X_j - X_w$ is, for $j \neq w$, independent of X_w and is exponential with rate θ_j. Consequently, $\theta_j(X_j - X_w)/(\theta_j + \lambda(w, j))$ is exponential with rate $\theta_j + \lambda(w, j)$, and is independent of X_w.

Suppose now that we want to use simulation to estimate quantities related to the L_j. For instance, we might be interested in such quantities as $\mathrm{E}[L_j]$, $\mathrm{P}(L_j > t)$, $\mathrm{E}[L_i L_j]$, or $\mathrm{P}(L_i > s, L_j > t)$. Rather than directly using the simulated values of the L_j, we can improve the estimates by letting $I_1, \dots, I_n$ be the order in which the components fail, and then using a conditional expectation estimator by conditioning on $I_1, \dots, I_n$. To obtain these conditional expectation estimators, suppose that we are given $I_1, \dots, I_n$. Let X_{I_k} be exponential with rate $\sum_{j \notin \{I_1,\dots,I_{k-1}\}}(\lambda_{0,j} + \sum_{r=1}^{k-1} \lambda_{I_r,j})$, $k = 1, \dots, n$, and take these exponentials to be independent. Now use the fact that, given $I_1, \dots, I_n$, the vector $L_{I_1}, L_{I_2}, \dots, L_{I_n}$ has the same distribution as $X_{I_1}, X_{I_1} + X_{I_2}, \dots, X_{I_1} + X_{I_2} + \cdots + X_{I_n}$. Consequently, given $I_1, \dots, I_n$, the lifetimes L_j are all distributed as hypoexponential random variables. (Hypoexponential random variables are defined as sums of independent heterogeneous exponentials. The density and distribution functions of a hypoexponential random variable can be found in Section 5.2.4 of [6].)

For instance, suppose that $n = 4$ and that the simulated failure order was $\mathbb{I} = 4, 1, 3, 2$. Then we could estimate $\mathrm{P}(L_1 > s)$ by $\mathrm{P}(X_4 + X_1 > s)$, where X_4 is exponential with rate $\sum_{j=1}^{4} \lambda_{0,j}$ and X_1 is an independent exponential with rate $\sum_{j=1}^{3}(\lambda_{0,j} + \lambda_{4,j})$. If we wanted to estimate $\mathrm{P}(L_1 > s, L_2 > t)$ then we would use the fact that, conditional on $\mathbb{I} = 4, 1, 3, 2$, the random variable L_1 is distributed as in the preceding, and L_2 given $(\mathbb{I}, L_1)$ is distributed as L_1 plus a hypoexponential distributed as the sum of an exponential with rate $\sum_{i \in \{0,4,1\}}(\lambda_{i,3} + \lambda_{i,2})$ and an exponential with rate $\lambda_{0,2} + \lambda_{1,2} + \lambda_{3,2} + \lambda_{4,2}$.

We can generate $I_1, \dots, I_n$ either directly, first generating I_1, then I_2 given the value of I_1, and so on; or we could use the algorithm given at the beginning of this section, keeping track of the failure order.

Additional variance reduction can be obtained by stratifying on I_1. That is, to estimate an expectation, call it $\mathrm{E}[R]$, we would use the fact that

$$\mathrm{E}[R] = \sum_{j=1}^{n} \mathrm{E}[R \mid I_1 = j] \frac{\lambda_{0,j}}{\sum_{j=1}^{n} \lambda_{0,j}}.$$

If we are simulating $I_1, \dots, I_n$ via the algorithm, then to simulate conditional on $I_1 = j$ just let $X_j = 0$.

4. The exchangeable case

The random vector $L_1, \dots, L_n$ is exchangeable when, for some μ and λ, we have $\lambda_{0,j} = \lambda$ and $\lambda_{i,j} = \mu$ for all $i, j = 1, \dots, n$. For this case, letting $L_{(i)}$ be the ith smallest of $L_1, \dots, L_n$, the joint density of $L_{(1)}, \dots, L_{(n)}$ is

$$f_{L_{(1)},\dots,L_{(n)}}(t_1, \dots, t_n) = \prod_{i=1}^{n} a_i \mathrm{e}^{-a_i(t_i - t_{i-1})}, \qquad 0 = t_0 < t_1 < \cdots < t_n,$$

where $a_i = (n - i + 1)(\lambda + (i-1)\mu)$. By symmetry, the joint density of the unordered variables is

$$f_{L_1,\dots,L_n}(t_1, \dots, t_n) = \frac{1}{n!} \prod_{i=1}^{n} a_i \mathrm{e}^{-a_i(t_i - t_{i-1})}, \qquad 0 = t_0 < t_1 < \cdots < t_n.$$

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If we let $W_1, \ldots, W_n$ be independent exponential random variables, with W_i having rate a_i, and let $S_j = \sum_{i=1}^{j} W_i$, then $L_{(1)}, \ldots, L_{(n)}$ has the same joint distribution as $S_1, \ldots, S_n$. The density of L_j is

$$f_{L_j}(t) = \frac{1}{n} \sum_{j=1}^{n} f_{S_j}(t).$$

This yields, for instance,

$$\mathrm{E}[L_j] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{j} \frac{1}{(n-i+1)(\lambda+(i-1)\mu)} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda+(i-1)\mu}.$$

All the exchangeable results are in agreement with the results of [4] and [7] when $\lambda = \mu$.

Remark. It is not immediately clear that our results in the exchangeable case should agree with those presented in [4] and [7] when $\lambda = \mu$. The reason is that in our model we consider a directed graph whereas in the model of [4] and [7] an undirected graph is considered. Thus, although both models assume that the random variables $X_{i,j}$ are exponential with parameter μ, our model assumes that $X_{i,j}$ and $X_{j,i}$ are independent whereas the model of [4] and [7] assumes that $X_{i,j} = X_{j,i}$. That the joint distribution of the shortest paths from node 0 is, nevertheless, the same in the two models can be seen by the Dijkstra algorithm for finding the shortest paths in order of their distance from 0. Let us employ this algorithm to generate the shortest paths in the two models. In both models, we start by generating all edge distances $X_{0,j},\ j = 1, \ldots, n$. We then determine i_1 such that $X_{0,i_1} = \min_j X_{0,j}$, and note that i_1 would be the nearest edge from node 0. Now suppose that the rth nearest edge from 0, call it i_r, has just been discovered by the algorithm, and that the nearest edges so far discovered are, in order of discovery, $i_1, \ldots, i_r$. At this point, to find the next nearest edge, the Dijkstra algorithm would generate the values $X_{i_r,j},\ j \neq 0,\ i_1, \ldots, i_{r-1}$. Because the edge value $X_{i,j}$ will be generated only if the shortest path to i is discovered before the shortest path to j (and so $L_j > L_i$), the algorithm would never generate both of the values $X_{i,j}$ and $X_{j,i}$. (To understand why, note that if $L_j > L_i$ then (j, i) would never be an edge of a shortest path.) Thus, it is irrelevant whether $X_{i,j}$ and $X_{j,i}$ are independent or equal because only one of them will be used in finding all shortest paths.

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