Canad. Math. Bull. Vol. 62 (4), 2019 pp. 703–714
http://dx.doi.org/10.4153/S0008439518000656

On Differential Torsion Theories and Rings with Several Objects

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Abstract. Let $\mathcal{R}$ be a small preadditive category, viewed as a "ring with several objects." A right $\mathcal{R}$-module is an additive functor from $\mathcal{R}^{op}$ to the category *Ab* of abelian groups. We show that every hereditary torsion theory on the category $(\mathcal{R}^{op}, Ab)$ of right $\mathcal{R}$-modules must be differential.

1 Introduction

Let R be a ring equipped with a derivation $\delta : R \to R$ and let $\text{Mod}-R$ be the category of right R-modules. A δ-derivation on a right R-module M is an additive map $d : M \to M$ satisfying $d(mr) = d(m)r + m\delta(r)$ for any $m \in M$ and any $r \in R$. Following Bland [3], a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on $\text{Mod}-R$ is said to be differential if the torsion submodule $M^\tau \subseteq M$ satisfies $d(M^\tau) \subseteq M^\tau$ for every $M \in \text{Mod}-R$ and every δ-derivation $d : M \to M$ on M.

The significance of differential torsion theories in the literature is the fact that they allow a δ-derivation on a module M to be extended to the "module of quotients" $Q^\tau(M)$ of M with respect to the torsion theory τ. For a hereditary torsion theory τ, we recall (see, for instance, [3, §2]) that the module of quotients $Q^\tau(M)$ is the "τ-injective envelope" $E^\tau(M/M^\tau)$ of the torsion free quotient M/M^τ of M. It was shown by Golan [8] that if $d : M \to M$ is a δ-derivation satisfying $d(M^\tau) \subseteq M^\tau$, then there is a δ-derivation $\overline{d} : Q^\tau(M) \to Q^\tau(M)$ on $Q^\tau(M)$ extending d. However, the question of uniqueness of the extension $\overline{d}$ was left open in [8], and the uniqueness was finally established by Bland [3, Proposition 2.1]. The differentiability of torsion theories and related questions on extending derivations were also studied extensively in [15, 17, 18]. A striking result of Lomp and van den Berg [10] showed that, in fact, every hereditary torsion theory on $\text{Mod}-R$ must be differential.

In this paper, we prove that all hereditary torsion theories are differential for modules over a preadditive category $\mathcal{R}$, which we treat as a "ring with several objects," following the philosophy of Mitchell [14]. Indeed, if $\mathcal{R}$ is a preadditive category with a single object $*$, then $\mathcal{R}$ is described completely by means of the Hom-object $\mathcal{R}(*, *)$, which is an ordinary ring. As such, an arbitrary preadditive category $\mathcal{R}$ becomes a more general kind of ring, *i.e.,* a "ring with several objects." Then a right module $\mathcal{M}$ over $\mathcal{R}$ is an additive functor $\mathcal{M} : \mathcal{R}^{op} \to Ab$, where Ab is the category of abelian groups. In fact, the idea of replacing rings with preadditive categories has proved to be very influential in several fields of mathematics, such as commutative algebra

Received by the editors September 14, 2018.
Published online on Cambridge Core May 2, 2019.
AMS subject classification: **16S90,** 18E05, 18E40.
Keywords: hereditary torsion theory, differential torsion theory, ring with several objects.

(see, for example, [19, 20]), algebraic geometry (see, for example, [6]), and cohomology theories (see, for example, [11–13]). In [1], we have previously worked with torsion theories on $(\mathcal{R}^{op}, Ab)$, where $\mathcal{R}$ is a small abelian category.

Accordingly, we consider a small preadditive category $\mathcal{R}$ and the category $(\mathcal{R}^{op}, Ab)$ of right $\mathcal{R}$-modules, which is a locally finitely presentable Grothendieck category. A derivation δ on $\mathcal{R}$ consists of additive maps $\delta(a,b) : \mathcal{R}(a,b) \to \mathcal{R}(a,b)$ on each of the Hom-objects of $\mathcal{R}$ satisfying $\delta(f \circ g) = \delta(f) \circ g + f \circ \delta(g)$ with respect to composition of morphisms in the category $\mathcal{R}$ (see Definition 2.1). Then our first result is Theorem 2.5, which shows that every Gabriel filter on the Grothendieck category $(\mathcal{R}^{op}, Ab)$ is δ-invariant.

By a δ-derivation d on an $\mathcal{R}$-module $\mathcal{M} \in (\mathcal{R}^{op}, Ab)$, we will mean a family of homomorphisms $d = \{d(r) : \mathcal{M}(r) \to \mathcal{M}(r)\}_{r \in Ob(\mathcal{R})}$ satisfying

$$d(r) \circ \mathcal{M}(h) = \mathcal{M}(h) \circ d(a) + \mathcal{M}(\delta(h))$$

for any $h \in \mathcal{R}(r,a)$, r, $a \in Ob(\mathcal{R})$ (see Definition 2.7). We consider a hereditary torsion theory $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ on $(\mathcal{R}^{op}, Ab)$ and let $\mathcal{M}^\tau \subseteq \mathcal{M}$ be the torsion subobject of $\mathcal{M}$. For the category $(\mathcal{R}^{op}, Ab)$, we know that hereditary torsion classes correspond to localizing subcategories as well as to Gabriel filters in $(\mathcal{R}^{op}, Ab)$. This enables us to prove Theorem 2.9, which shows that any δ-derivation d on any $\mathcal{M} \in (\mathcal{R}^{op}, Ab)$ satisfies $d(a)(\mathcal{M}^\tau(a)) \subseteq \mathcal{M}^\tau(a)$ for every $a \in Ob(\mathcal{R})$. In other words, every hereditary torsion theory on $(\mathcal{R}^{op}, Ab)$ is differential.

Finally, we consider the "module of quotients" $Q^\tau(\mathcal{M})$ of $\mathcal{M}$ with respect to τ, constructed as in [7, §2.2]. We conclude with Theorem 2.13, where we show that every δ-derivation d on $\mathcal{M}$ extends uniquely to a δ-derivation $\overline{d}$ on $Q^\tau(\mathcal{M})$.

2 Hereditary Torsion Theories are Differential

Throughout, we let $\mathcal{R}$ be a small preadditive category, which we will see as a "ring with several objects," following the philosophy of Mitchell [14]. Given objects a, $b \in \mathcal{R}$, we will denote by $\mathcal{R}(a,b)$ the abelian group consisting of morphisms in $\mathcal{R}$ from a to b. The following notion of a derivation on a ring with several objects is implicit at several places in the literature.

Definition 2.1 Let $\mathcal{R}$ be a small preadditive category. A derivation δ on $\mathcal{R}$ is a family of homomorphisms

$$\delta(a,b) : \mathcal{R}(a,b) \longrightarrow \mathcal{R}(a,b) \qquad a, b \in Ob(\mathcal{R})$$

satisfying the following condition (for any a, b, $c \in Ob(\mathcal{R})$):

$$\delta(c,a)(f \circ g) = \big(\delta(b,a)(f)\big) \circ g + f \circ \big(\delta(c,b)(g)\big) \qquad f \in \mathcal{R}(b,a), g \in \mathcal{R}(c,b).$$

When there is no danger of confusion, for any morphism $f \in \mathcal{R}(b,a)$ in $\mathcal{R}$, we will denote $\delta(b,a)(f)$ simply by $\delta(f)$.

It is clear that if $\mathcal{R}$ is a preadditive category with a single object $*$, then a derivation on $\mathcal{R}$ is simply an ordinary derivation on the ring $\mathcal{R}(*,*)$. We now give some examples of derivations on small preadditive categories.

Example 2.2 (a) Let k be a commutative ring. Let (A, δ) be a pair consisting of a k-algebra A equipped with a k-linear derivation δ and let (M, δ_M) be a "pre-(A, δ)-module" in the sense of Tanaka [16, Definition 3.1]. In other words, M is a left A-module and $\delta_M : M \to M$ is a k-linear map satisfying

$$\delta_M(am) = \delta(a)m + a\delta_M(m) \quad \forall\ a \in A, m \in M.$$

We consider the category $\mathrm{Pre}_{(A,\delta)}$ whose objects are pre-(A, δ)-modules, with a morphism $f : (M, \delta_M) \to (N, \delta_N)$ being given by an ordinary A-module morphism $f : M \to N$. Then if $\mathcal{C} \subseteq \mathrm{Pre}_{(A,\delta)}$ is any subcategory that is small and full, it is evident that the homomorphisms

$$\begin{aligned} &\delta\big((M, \delta_M), (N, \delta_N)\big) : \\ &\mathrm{Pre}_{(A,\delta)}\big((M, \delta_M), (N, \delta_N)\big) \to \mathrm{Pre}_{(A,\delta)}\big((M, \delta_M), (N, \delta_N)\big) \\ &\qquad f \mapsto \delta_N \circ f - f \circ \delta_M \end{aligned}$$

for every (M, δ_M), $(N, \delta_N) \in Ob(\mathcal{C})$ define a derivation on $\mathcal{C}$ in the sense of Definition 2.1.

(b) If $\mathcal{H}$ is a Hopf algebra, the notion of an "$\mathcal{H}$-module category" or "$\mathcal{H}$-category" arises naturally in studying Galois coverings of categories (see Cibils and Solotar [5, §2]) and in noncommutative geometry (see Kaygun and Khalkhali [9, §6]). Explicitly speaking, an $\mathcal{H}$-category is a preadditive category $\mathcal{C}$ where each morphism set is an $\mathcal{H}$-module, and if f, g are two composable morphisms in $\mathcal{C}$, the action of $\mathcal{H}$ satisfies $h(g \circ f) = \sum h_{(1)}(g) \circ h_{(2)}(f)$ for each $h \in \mathcal{H}$. Here, the coproduct Δ on $\mathcal{H}$ is given in Sweedler notation by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for every $h \in \mathcal{H}$.

We now consider some $x \in \mathcal{H}$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. For example, $\mathcal{H}$ could be the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of a Lie algebra $\mathcal{L}$, and we could take any $x \in \mathcal{L}$. Then if $\mathcal{C}$ is a small $\mathcal{H}$-module category, it is clear that the action of the element x on each morphism set of $\mathcal{C}$ gives a derivation on $\mathcal{C}$ in the sense of Definition 2.1.

If $\mathcal{R}$ is a small preadditive category, a right $\mathcal{R}$-module is simply an additive functor $\mathcal{M} : \mathcal{R}^{\mathrm{op}} \to Ab$, where Ab is the category of abelian groups. As such, the category of right $\mathcal{R}$-modules will be denoted by $(\mathcal{R}^{\mathrm{op}}, Ab)$. For any $a \in Ob(\mathcal{R})$, we consider the representable functor

$$H_a : \mathcal{R}^{\mathrm{op}} \longrightarrow Ab \qquad r \longmapsto \mathcal{R}(r, a) \quad \forall\ r \in Ob(\mathcal{R}).$$

We now recall the following well known result.

Proposition 2.3 *Let $\mathcal{R}$ be a small preadditive category. Then the category $(\mathcal{R}^{\mathrm{op}}, Ab)$ of right $\mathcal{R}$-modules is a Grothendieck category with the representable functors $\{H_a\}_{a \in Ob(\mathcal{R})}$ forming a set of finitely generated projective generators.*

Proof We refer the reader to, for instance, [6, Lemma 2.2] for the fact that $(\mathcal{R}^{\mathrm{op}}, Ab)$ is a Grothendieck category with the representable functors $\{H_a\}_{a \in Ob(\mathcal{R})}$ being a set of projective generators. From the Yoneda lemma, we know that for any $\mathcal{M} \in (\mathcal{R}^{\mathrm{op}}, Ab)$, we have $\mathrm{Hom}_{(\mathcal{R}^{\mathrm{op}}, Ab)}(H_a, \mathcal{M}) = \mathcal{M}(a)$. Since colimits are computed componentwise in the functor category, it follows that

$$\varinjlim_{i\in I} \mathrm{Hom}_{(\mathcal{R}^{op},Ab)}(H_a,\mathcal{M}_i) = \varinjlim_{i\in I}\mathcal{M}_i(a) = \mathrm{Hom}_{(\mathcal{R}^{op},Ab)}(H_a,\varinjlim_{i\in I}\mathcal{M}_i),$$

where $\{\mathcal{M}_i\}_{i\in I}$ is a filtered system of objects in $(\mathcal{R}^{op}, Ab)$ connected by monomorphisms. It follows that each $H_a \in (\mathcal{R}^{op}, Ab)$ is a finitely generated object. ∎

Let $a \in Ob(\mathcal{R})$ and consider a subobject $I \subseteq H_a$ in $(\mathcal{R}^{op}, Ab)$. Then, we have $I(r) \subseteq H_a(r) = \mathcal{R}(r,a)$ for each $r \in Ob(\mathcal{R})$. If $h \in \mathcal{R}(b,a) = \mathrm{Hom}_{(\mathcal{R}^{op},Ab)}(H_b, H_a)$ is a morphism in $\mathcal{R}$, we set

$$h^{-1}(I)(r) := \{f \in \mathcal{R}(r,b) \mid h\circ f \in I(r)\} \qquad \forall\, r \in Ob(\mathcal{R}).$$

It is evident that $h^{-1}(I) \subseteq H_b$ as a right $\mathcal{R}$-module.

The notion of a Gabriel filter in a Grothendieck category with a set of finitely generated projective generators is due to Garkusha [7, §2.1]. Because of Proposition 2.3, we can consider Gabriel filters in $(\mathcal{R}^{op}, Ab)$.

Definition 2.4 Let $\mathcal{R}$ be a small preadditive category. A Gabriel filter $\mathfrak{G} = \{\mathfrak{G}_a\}_{a\in Ob(\mathcal{R})}$ on $(\mathcal{R}^{op}, Ab)$ is a collection such that the following hold:

(i) For each $a \in Ob(\mathcal{R})$, $\mathfrak{G}_a$ is a family of subobjects of $H_a \in (\mathcal{R}^{op}, Ab)$.
(ii) For each $a \in Ob(\mathcal{R})$, $H_a \in \mathfrak{G}_a$.
(iii) If $I \in \mathfrak{G}_a$ and $h \in \mathcal{R}(b,a) = \mathrm{Hom}_{(\mathcal{R}^{op},Ab)}(H_b, H_a)$ is a morphism in $\mathcal{R}$, then $h^{-1}(I) \in \mathfrak{G}_b$.
(iv) Let $J \in \mathfrak{G}_a$. If $K \subseteq H_a$ is such that $h^{-1}(K) \in \mathfrak{G}_b$ for every morphism $h \in \mathcal{R}(b,a) = \mathrm{Hom}_{(\mathcal{R}^{op},Ab)}(H_b, H_a)$ satisfying $\mathrm{Im}\left(\xrightarrow{h\circ}\right) \subseteq J(r)$ for each $r \in Ob(\mathcal{R})$, then $K \in \mathfrak{G}_a$.

It may be shown (see [7, §2.1]) that a Gabriel filter $\mathfrak{G} = \{\mathfrak{G}_a\}_{a\in Ob(\mathcal{R})}$ on $(\mathcal{R}^{op}, Ab)$ also satisfies the following property: for $I, J \subseteq H_a$ for some $a \in Ob(\mathcal{R})$ with $J \subseteq I$, we have

$$J \in \mathfrak{G}_a \implies I \in \mathfrak{G}_a.$$

We now consider the preadditive category $\mathcal{R}$ along with a derivation δ in the sense of Definition 2.1. Let $\mathfrak{G} = \{\mathfrak{G}_a\}_{a\in Ob(\mathcal{R})}$ be a Gabriel filter on $(\mathcal{R}^{op}, Ab)$. We will say that $\mathfrak{G}$ is δ-invariant if for each $a \in Ob(\mathcal{R})$ and each $I \in \mathfrak{G}_a$, there exists some $J \in \mathfrak{G}_a$ such that $\delta(J(r)) \subseteq I(r)$ as subsets of $H_a(r) = \mathcal{R}(r,a)$, for all $r \in Ob(\mathcal{R})$. This brings us to our first main result.

Theorem 2.5 *Let $\mathcal{R}$ be a small preadditive category equipped with a derivation δ. Then every Gabriel filter $\mathfrak{G} = \{\mathfrak{G}_a\}_{a\in Ob(\mathcal{R})}$ on $(\mathcal{R}^{op}, Ab)$ is δ-invariant.*

Proof We fix some $a \in Ob(\mathcal{R})$ and consider some $I \in \mathfrak{G}_a$. By definition, $I \subseteq H_a$ in $(\mathcal{R}^{op}, Ab)$. Since $I \hookrightarrow H_a$ is a morphism of functors, we note that for any morphism $g \in \mathcal{R}(r, r')$ in $\mathcal{R}$, we have a commutative diagram

$$\begin{array}{ccc} I(r') & \longrightarrow & H_a(r') = \mathcal{R}(r',a) \\ {\scriptstyle \_\circ g}\downarrow & & {\scriptstyle \_\circ g}\downarrow \\ I(r) & \longrightarrow & H_a(r) = \mathcal{R}(r,a). \end{array}$$

In other words, we must have

$$f \in I(r') \implies f \circ g \in I(r) \qquad \forall\, r, r' \in Ob(\mathcal{R}), g \in \mathcal{R}(r,r'). \tag{2.1}$$

We now set

$$J(r) := \{f \in I(r) \subseteq H_a(r) = \mathcal{R}(r,a) \mid \delta(f) \in I(r)\} \qquad \forall\, r \in Ob(\mathcal{R}).$$

It is clear that $J(r) \subseteq I(r) \subseteq H_a(r)$ for each $r \in Ob(\mathcal{R})$. We consider some $f \in J(r')$ and a morphism $g \in \mathcal{R}(r,r')$. Since $J(r') \subseteq I(r')$, it follows from (2.1) that the composition $f \circ g \in I(r)$. Additionally, we have

$$\delta(f \circ g) = \delta(f) \circ g + f \circ \delta(g). \tag{2.2}$$

Since $f \in J(r')$, we know that $\delta(f) \in I(r')$. Applying (2.1) to each of the two terms on the right-hand side of (2.2), it follows that $\delta(f \circ g) \in I(r)$. Hence, $f \circ g \in J(r)$, and we realize that J is also a functor, *i.e.*, $J \in (\mathcal{R}^{\mathrm{op}}, Ab)$. Clearly, $J \subseteq H_a$ in $(\mathcal{R}^{\mathrm{op}}, Ab)$.

We now consider a morphism $h \in \mathcal{R}(b,a) = \mathrm{Hom}_{(\mathcal{R}^{\mathrm{op}},Ab)}(H_b, H_a)$ satisfying $\mathrm{Im}\left(\xrightarrow{h\circ\_}\right) \subseteq I(r)$ for each $r \in Ob(\mathcal{R})$. We claim that

$$(\delta(h))^{-1}(I) \subseteq h^{-1}(J) \tag{2.3}$$

in the category $(\mathcal{R}^{\mathrm{op}}, Ab)$. Indeed, if $f \in (\delta(h))^{-1}(I)(r)$ for some $r \in Ob(\mathcal{R})$, we know that $\delta(h) \circ f \in I(r)$. On the other hand, the assumption on the morphism $h \in \mathcal{R}(b,a)$ guarantees that $h \circ \delta(f) \in I(r)$. Hence, $\delta(h \circ f) = \delta(h) \circ f + h \circ \delta(f)$ lies in $I(r)$. The assumption on the morphism h also guarantees that $h \circ f \in I(r)$. From the definition of $J(r)$, we now get $h \circ f \in J(r)$. In other words, we have $f \in h^{-1}(J)(r)$.

Finally, since $I \in \mathfrak{G}_a$, we notice that property (iii) of Gabriel filters in Definition 2.4 implies that $(\delta(h))^{-1}(I) \in \mathfrak{G}_b$. From (2.3), we know that $(\delta(h))^{-1}(I) \subseteq h^{-1}(J)$ and hence $h^{-1}(J) \in \mathfrak{G}_b$. It now follows from property (iv) of Gabriel filters in Definition 2.4 that $J \in \mathfrak{G}_a$. ∎

By definition, a torsion theory on $(\mathcal{R}^{\mathrm{op}}, Ab)$ (see, for instance, [2, §1.1]) is a pair $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ of strict and full subcategories of $(\mathcal{R}^{\mathrm{op}}, Ab)$ satisfying the following two conditions.

(a) For any $\mathcal{M} \in \mathcal{T}^\tau$ and $\mathcal{N} \in \mathcal{F}^\tau$, we have $\mathrm{Hom}_{(\mathcal{R}^{\mathrm{op}},Ab)}(\mathcal{M},\mathcal{N}) = 0$.
(b) For each $\mathcal{M} \in (\mathcal{R}^{\mathrm{op}}, Ab)$ we have a short exact sequence $0 \to \mathcal{M}^\tau \to \mathcal{M} \to \mathcal{M}/\mathcal{M}^\tau \to 0$, with $\mathcal{M}^\tau \in \mathcal{T}^\tau$ being a torsion object and $\mathcal{M}/\mathcal{M}^\tau \in \mathcal{F}^\tau$ being a torsion free object.

The subcategory $\mathcal{T}^\tau$ is called the *torsion class*, while $\mathcal{F}^\tau$ is called the *torsion free class*. Further, when the torsion class $\mathcal{T}^\tau$ is closed under subobjects, the torsion theory τ is said to be *hereditary*, and $\mathcal{T}^\tau$ becomes a hereditary torsion class. Since $(\mathcal{R}^{\mathrm{op}}, Ab)$ is a locally finitely presented Grothendieck category, hereditary torsion classes in $(\mathcal{R}^{\mathrm{op}}, Ab)$ are the same as localizing subcategories of $(\mathcal{R}^{\mathrm{op}}, Ab)$ (see, for instance, [4, Theorem 1.13.5]).

In the category $(\mathcal{R}^{\text{op}}, Ab)$, we know (see [7, Theorem 2.1]) that there is a one-to-one correspondence between hereditary torsion classes and Gabriel filters given as follows: if $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ is a hereditary torsion theory, we can associate a Gabriel filter $\mathfrak{G}^\tau = \{\mathfrak{G}^\tau_a\}_{a \in Ob(\mathcal{R})}$ by setting

$$\mathfrak{G}^\tau_a := \{I \subseteq H_a \mid H_a/I \in \mathcal{T}^\tau\}.$$

On the other hand, given a Gabriel filter $\mathfrak{G} = \{\mathfrak{G}_a\}_{a \in Ob(\mathcal{R})}$, we can associate a hereditary torsion class $\mathcal{T}^{\mathfrak{G}} \subseteq (\mathcal{R}^{\text{op}}, Ab)$, defined by setting

$$(2.4) \quad Ob(\mathcal{T}^{\mathfrak{G}}) := \Big\{\mathcal{M} \mid \text{Ker}(x : H_a \to \mathcal{M}) \in \mathfrak{G}_a \quad \text{for each } x \in \text{Hom}_{(\mathcal{R}^{\text{op}}, Ab)}(H_a, \mathcal{M}), a \in Ob(\mathcal{R})\Big\}.$$

In what follows, we will make the convention that if x is an element of $\mathcal{M}(a)$ for some $\mathcal{M} \in (\mathcal{R}^{\text{op}}, Ab)$ and $a \in Ob(\mathcal{R})$, we also denote by x the corresponding morphism $H_a \to \mathcal{M}$.

Lemma 2.6 *Let $\mathcal{M} \in (\mathcal{R}^{\text{op}}, Ab)$ be a right $\mathcal{R}$-module. Let $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ be a hereditary torsion theory on $(\mathcal{R}^{\text{op}}, Ab)$ and $\mathfrak{G}^\tau = \{\mathfrak{G}^\tau_a\}_{a \in Ob(\mathcal{R})}$ the corresponding Gabriel filter on $(\mathcal{R}^{\text{op}}, Ab)$. For each $a \in Ob(\mathcal{R})$, we now set*

$$(2.5) \qquad \mathcal{M}'(a) := \Big\{x \in \text{Hom}_{(\mathcal{R}^{\text{op}}, Ab)}(H_a, \mathcal{M}) \mid \text{Ker}(x : H_a \longrightarrow \mathcal{M}) \in \mathfrak{G}^\tau_a\Big\}.$$

Then the following hold.

(i) *The association $a \mapsto \mathcal{M}'(a)$ is a functor,* i.e., *$\mathcal{M}'$ is also a right $\mathcal{R}$-module with $\mathcal{M}' \subseteq \mathcal{M}$ in $(\mathcal{R}^{\text{op}}, Ab)$.*
(ii) *The right $\mathcal{R}$-module $\mathcal{M}'$ is torsion,* i.e., $\mathcal{M}' \in \mathcal{T}^\tau$.
(iii) *The module $\mathcal{M}'$ is the torsion subobject of $\mathcal{M}$,* i.e., $\mathcal{M}' = \mathcal{M}^\tau$.

Proof (i) We consider some $h \in \mathcal{R}(b, a)$ and the corresponding morphism $h : H_b \to H_a$ in $(\mathcal{R}^{\text{op}}, Ab)$. Then for any $x \in \mathcal{M}'(a) \subseteq \mathcal{M}(a)$ and any object $r \in Ob(\mathcal{R})$, we have

$$(2.6) \quad \begin{aligned} \text{Ker}(H_b \xrightarrow{h} H_a \xrightarrow{x} \mathcal{M})(r) &= \{f \in \mathcal{R}(r, b) \mid x(r)(h \circ f) = 0\} \\ &= \{f \in \mathcal{R}(r, b) \mid h \circ f \in \text{Ker}(H_a \xrightarrow{x} \mathcal{M})(r)\} \\ &= h^{-1}(\text{Ker}(H_a \xrightarrow{x} \mathcal{M}))(r). \end{aligned}$$

Since $x \in \mathcal{M}'(a)$, we know that $(\text{Ker}(H_a \xrightarrow{x} \mathcal{M})) \in \mathfrak{G}^\tau_a$. From (2.6) and property (iii) of Gabriel filters in Definition 2.4, we see that $\text{Ker}(H_b \xrightarrow{x \circ h} \mathcal{M}) \in \mathfrak{G}^\tau_b$, *i.e.*, $x \circ h \in \mathcal{M}'(b)$. This shows that $\mathcal{M}'$ is a functor on $\mathcal{R}^{\text{op}}$, and it is clear that $\mathcal{M}' \subseteq \mathcal{M}$.

(ii) We consider a morphism $x : H_a \to \mathcal{M}'$ for some $a \in Ob(\mathcal{R})$. Then $x \in \mathcal{M}'(a) \subseteq \mathcal{M}(a)$, and it follows from (2.5) that $\text{Ker}(x : H_a \to \mathcal{M}') = \text{Ker}(x : H_a \to \mathcal{M}' \hookrightarrow \mathcal{M}) \in \mathfrak{G}^\tau_a$. Applying (2.4), it is now clear that $\mathcal{M}' \in \mathcal{T}^\tau$.

(iii) From parts (i) and (ii), we know that $\mathcal{M}' \subseteq \mathcal{M}$ is a torsion object. Since $\mathcal{M}^\tau$ contains all torsion subobjects of $\mathcal{M}$, we must have $\mathcal{M}' \subseteq \mathcal{M}^\tau$. We now consider $x \in \mathcal{M}^\tau(a)$ for some $a \in Ob(\mathcal{R})$. Since $\mathcal{M}^\tau \in \mathcal{T}^\tau$, we know from (2.4) that $\text{Ker}(H_a \xrightarrow{x} \mathcal{M}^\tau \hookrightarrow \mathcal{M}) = \text{Ker}(H_a \xrightarrow{x} \mathcal{M}^\tau) \in \mathfrak{G}^\tau_a$. From the definition of $\mathcal{M}'(a)$ in (2.5), it now follows that $x \in \mathcal{M}'(a)$. Hence, $\mathcal{M}' = \mathcal{M}^\tau$. ■

https://doi.org/10.4153/S0008439518000656 Published online by Cambridge University Press

Definition 2.7 Let $\mathcal{R}$ be a small preadditive category equipped with a derivation δ. Let $\mathcal{M} \in (\mathcal{R}^{\text{op}}, Ab)$ be a right $\mathcal{R}$-module. A δ-derivation d on $\mathcal{M}$ is a family of abelian group homomorphisms

$$d = \{d(r) : \mathcal{M}(r) \to \mathcal{M}(r)\}_{r \in Ob(\mathcal{R})}$$

satisfying the condition

$$d(r) \circ \mathcal{M}(h) = \mathcal{M}(h) \circ d(a) + \mathcal{M}(\delta(h))$$

for any $h \in \mathcal{R}(r, a)$, r, $a \in Ob(\mathcal{R})$. Here, $\mathcal{M}(h)$ is the morphism $\mathcal{M}(h) : \mathcal{M}(a) \to \mathcal{M}(r)$ induced by $h \in \mathcal{R}(r, a)$. When there is no danger of confusion, we denote the morphism $d(r)$ simply by d.

For example, take any $x \in Ob(\mathcal{R})$ and consider the right module $H_x \in (\mathcal{R}^{\text{op}}, Ab)$. Then it can be easily verified that the family of homomorphisms $\{d(r) := \delta(r, x) : H_x(r) \to H_x(r)\}_{r \in Ob(\mathcal{R})}$ gives a δ-derivation on the right module H_x.

Definition 2.8 We will say that a hereditary torsion theory $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ on $(\mathcal{R}^{\text{op}}, Ab)$ is differential if for every module $\mathcal{M} \in (\mathcal{R}^{\text{op}}, Ab)$ and every δ-derivation d on $\mathcal{M}$, we have $d(\mathcal{M}^\tau(a)) \subseteq \mathcal{M}^\tau(a)$, $\forall\ a \in Ob(\mathcal{R})$.

For the category of modules over a given ring, the notion of a differential torsion theory was introduced by Bland [3, §1]. The notion we have introduced in Definition 2.8 extends this idea to the case of "rings with several objects."

In [10] it was shown that every hereditary torsion theory on the category of modules over an ordinary ring is differential. We are now ready to prove the main result of this paper.

Theorem 2.9 *Let $\mathcal{R}$ be a small preadditive category equipped with a derivation δ. Then every hereditary torsion theory on $(\mathcal{R}^{\text{op}}, Ab)$ is differential.*

Proof Let $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ be a hereditary torsion theory on $(\mathcal{R}^{\text{op}}, Ab)$ and let $\mathfrak{G}^\tau = \{\mathfrak{G}^\tau_a\}_{a \in Ob(\mathcal{R})}$ be the Gabriel filter corresponding to τ. We consider a right $\mathcal{R}$-module $\mathcal{M}$ equipped with a δ-derivation d and an element $x \in \mathcal{M}^\tau(a)$ for some $a \in Ob(\mathcal{R})$. We need to show that $d(x) \in \mathcal{M}^\tau(a)$.

Using Lemma 2.6, we know that

$$K := \text{Ker}(H_a \xrightarrow{x} \mathcal{M}) \in \mathfrak{G}^\tau_a.$$

From Theorem 2.5, we know that the Gabriel filter $\mathfrak{G}^\tau$ is δ-invariant. As such, there exists $J \in \mathfrak{G}^\tau_a$ such that $\delta(J(r)) \subseteq K(r)$ for each $r \in Ob(\mathcal{R})$. We now set $I := J \cap K \subseteq H_a$. Since $\mathfrak{G}^\tau = \{\mathfrak{G}^\tau_a\}_{a \in Ob(\mathcal{R})}$ is a Gabriel filter, it follows (see [7, §2.1]) that $I = J \cap K \in \mathfrak{G}^\tau_a$.

We consider the element $d(x) = d(a)(x) \in \mathcal{M}(a)$. We now pick a morphism $h \in I(r) \subseteq H_a(r) = \mathcal{R}(r, a)$. Then $h \in K(r)$, and hence $\mathcal{M}(h)(x) = 0$. Since $h \in J(r)$, it follows that $\delta(h) \in K(r)$. Hence, we also have $\mathcal{M}(\delta(h))(x) = 0$. Since d is a δ-derivation on $\mathcal{M}$, we know that

$$\big(d(r) \circ \mathcal{M}(h)\big)(x) = \big(\mathcal{M}(h) \circ d(a)\big)(x) + \big(\mathcal{M}(\delta(h))\big)(x).$$

This yields $\mathcal{M}(h)(d(x)) = 0$ for any $h \in I(r) \subseteq H_a(r)$. It follows that the composition $I \hookrightarrow H_a \xrightarrow{d(x)} \mathcal{M}$ is always zero. We now know that

$$I \subseteq \mathrm{Ker}(H_a \overset{d(x)}{\rightarrow} \mathcal{M})$$

as subobjects of H_a. Since $I \in \mathfrak{G}^\tau_a$, we must have $\mathrm{Ker}(H_a \overset{d(x)}{\rightarrow} \mathcal{M}) \in \mathfrak{G}^\tau_a$. It now follows from Lemma 2.6 that $d(x) \in \mathcal{M}^\tau(a)$. This proves the result. ∎

Suppose that R is an ordinary ring equipped with a derivation δ and that M is an object of the category $\mathrm{Mod}-R$ of right R-modules. If $\tau = (T, F)$ is a hereditary torsion theory on $\mathrm{Mod}-R$, we know that it must also be a differential torsion theory by [10]. As explained in the introduction, the significance of differential torsion theories in the literature is the fact that every δ-derivation on a module $M \in \mathrm{Mod}-R$ can be extended uniquely to a δ-derivation on "the module of quotients" $Q^\tau(M)$ of M with respect to τ.

We will now show that this property continues to hold in the category of modules over a "ring with several objects." We fix a hereditary torsion theory $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ on $(\mathcal{R}^{\mathrm{op}}, Ab)$. For any $\mathcal{M} \in (\mathcal{R}^{\mathrm{op}}, Ab)$, the 'module of quotients' $Q^\tau(\mathcal{M})$ of $\mathcal{M}$ with respect to τ is constructed by setting (see [7, §2.2])

$$Q^\tau(\mathcal{M})(a) := \varinjlim_{I \in \mathfrak{G}^\tau_a} \mathrm{Hom}_{(\mathcal{R}^{\mathrm{op}}, Ab)}(I, \mathcal{M}/\mathcal{M}^\tau) \qquad \forall\, a \in Ob(\mathcal{R}). \tag{2.7}$$

It is clear from the properties of the Gabriel filter $\mathfrak{G}^\tau = \{\mathfrak{G}^\tau_a\}_{a \in Ob(\mathcal{R})}$ that the colimit in (2.7) is filtered. Additionally, since $H_a \in \mathfrak{G}^\tau_a$ for each $a \in Ob(\mathcal{R})$, we have canonical morphisms

$$\begin{aligned}\mathcal{M}(a) &\longrightarrow (\mathcal{M}/\mathcal{M}^\tau)(a) \\ &= \mathrm{Hom}_{(\mathcal{R}^{\mathrm{op}}, Ab)}(H_a, \mathcal{M}/\mathcal{M}^\tau) \longrightarrow \varinjlim_{I \in \mathfrak{G}^\tau_a} \mathrm{Hom}_{(\mathcal{R}^{\mathrm{op}}, Ab)}(I, \mathcal{M}/\mathcal{M}^\tau) \\ &= Q^\tau(M)(a)\end{aligned}$$

determining a morphism $\Phi_{\mathcal{M}} : \mathcal{M} \to Q^\tau(\mathcal{M})$ in $(\mathcal{R}^{\mathrm{op}}, Ab)$.

Lemma 2.10 *Let $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ be a hereditary torsion theory on $(\mathcal{R}^{\mathrm{op}}, Ab)$. Let $\mathcal{N} \in (\mathcal{R}^{\mathrm{op}}, Ab)$ be a torsion free module equipped with a δ-derivation d. For some $a \in Ob(\mathcal{R})$, let $F \in Q^\tau(\mathcal{N})(a)$ be an element represented by $F : I \to \mathcal{N}$ for some $I \in \mathfrak{G}^\tau_a$. Let $K = I \cap J \in \mathfrak{G}^\tau_a$, where $J \in \mathfrak{G}^\tau_a$ is such that $\delta(J(r)) \subseteq I(r)$ for every $r \in Ob(\mathcal{R})$. Then the association*

$$f \in K(r) \longmapsto d\big(F(r)(f)\big) - F(r)\big(\delta(f)\big) \in \mathcal{N}(r) \tag{2.8}$$

is a morphism from K to $\mathcal{N}$ in $(\mathcal{R}^{\mathrm{op}}, Ab)$, giving an element $\overline{d}(F) \in Q^\tau(\mathcal{N})(a)$.

Proof We consider some $r' \in Ob(\mathcal{R})$ and some $h \in \mathcal{R}(r', r)$. In order to show that the association in (2.8) gives a morphism in $(\mathcal{R}^{\mathrm{op}}, Ab)$, we must verify that the following diagram is commutative:

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$$\begin{array}{ccc} K(r) & \xrightarrow{\overline{d}(F)(r)} & \mathcal{N}(r) \\ {\scriptstyle K(h)}\downarrow & & \downarrow{\scriptstyle \mathcal{N}(h)} \\ K(r') & \xrightarrow{\overline{d}(F)(r')} & \mathcal{N}(r'). \end{array}$$

For some $f \in K(r)$, on the one hand, we have

$$\begin{aligned} (2.9)\quad & \mathcal{N}(h)\big(\overline{d}(F)(r)(f)\big) \\ &= \mathcal{N}(h)\big(d(F(r)(f))\big) - \mathcal{N}(h)\big(F(r)(\delta(f))\big) \\ &= d\big(\mathcal{N}(h)(F(r)(f))\big) - \mathcal{N}\big(\delta(h)\big)\big(F(r)(f)\big) - \mathcal{N}(h)\big(F(r)(\delta(f))\big). \end{aligned}$$

In (2.9), we notice that since $K = I \cap J$ and J has been chosen so that $\delta(J(r)) \subseteq I(r)$ for every $r \in Ob(\mathcal{R})$, we know that $\delta(f) \in I(r)$, *i.e.*, $\delta(f)$ is in the domain of $F(r)$. On the other hand, we have

$$\begin{aligned} & \overline{d}(F)(r')\big(K(h)(f)\big) \\ &= \overline{d}(F)(r')(f \circ h) \\ &= d\big(F(r')(f \circ h)\big) - F(r')\big(\delta(f \circ h)\big) \\ &= d\big(\mathcal{N}(h)(F(r)(f))\big) - F(r')\big(\delta(f) \circ h\big) - F(r')\big(f \circ \delta(h)\big) \\ &= d\big(\mathcal{N}(h)(F(r)(f))\big) - \mathcal{N}(h)\big(F(r)(\delta(f))\big) - \mathcal{N}\big(\delta(h)\big)(F(r)(f)\big). \end{aligned}$$

This proves the result. ∎

Since $\mathfrak{G}^\tau = \{\mathfrak{G}^\tau_a\}_{a \in Ob(\mathcal{R})}$ is a Gabriel filter, for any $F \in Q^\tau(\mathcal{N})(a)$, it is clear that the element $\overline{d}(F) \in Q^\tau(\mathcal{N})(a)$ defined in Lemma 2.10 does not depend on the choice of $I, J \in \mathfrak{G}^\tau_a$. As such, we have a well-defined morphism

$$\overline{d}(a) : Q^\tau(\mathcal{N})(a) \longrightarrow Q^\tau(\mathcal{N})(a) \qquad F \longmapsto \overline{d}(F).$$

For the sake of convenience, we will almost always drop the mention of the object a and simply write $\overline{d} : Q^\tau(\mathcal{N})(a) \to Q^\tau(\mathcal{N})(a)$.

Lemma 2.11 *Let $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ be a hereditary torsion theory on $(\mathcal{R}^{\mathrm{op}}, Ab)$. Let $\mathcal{N} \in (\mathcal{R}^{\mathrm{op}}, Ab)$ be a torsion free module equipped with a δ-derivation d. Then, the family of morphisms*

$$\overline{d}(a) : Q^\tau(\mathcal{N})(a) \longrightarrow Q^\tau(\mathcal{N})(a) \qquad \forall\ a \in Ob(\mathcal{R})$$

defines a δ-derivation on $Q^\tau(\mathcal{N})$.

Proof We take a, $a' \in Ob(\mathcal{R})$ and some $h \in \mathcal{R}(a', a)$. We consider an element $F \in Q^\tau(\mathcal{N})(a)$ represented by a morphism $F : I \to \mathcal{N}$ for some $I \in \mathfrak{G}^\tau_a$. We choose $J \in \mathfrak{G}^\tau_a$ such that $\delta(J(r)) \subseteq I(r)$ for each $r \in Ob(\mathcal{R})$. We set $K = I \cap J \in \mathfrak{G}^\tau_a$ and let $I' \in \mathfrak{G}^\tau_{a'}$ be given by $I' = h^{-1}(K) \cap (\delta(h))^{-1}(K)$. Again, we consider $J' \in \mathfrak{G}^\tau_{a'}$ such that $\delta(J'(r)) \subseteq I'(r)$ for each $r \in Ob(\mathcal{R})$ and set $K' = I' \cap J' \in \mathfrak{G}^\tau_{a'}$. We put $\mathcal{Q} = Q^\tau(\mathcal{N})$. We claim that

$$\overline{d}\big(\mathcal{Q}(h)(F)\big) = \mathcal{Q}(h)\big((\overline{d}(F))\big) + \mathcal{Q}\big(\delta(h)\big)(F).$$

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For $r \in Ob(\mathcal{R})$ and $f \in K'(r)$, we have

$$\begin{aligned}\overline{d}\big(\mathcal{Q}(h)(F)\big)(r)(f) &= d\big(\mathcal{Q}(h)(F)(r)(f)\big) - \big(\mathcal{Q}(h)(F)\big)(r)\big(\delta(f)\big)\\ &= d\big(F(r)(h\circ f)\big) - F(r)\big(h\circ\delta(f)\big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathcal{Q}(h)\big((\overline{d}(F))\big)(r)(f) + \mathcal{Q}\big(\delta(h)\big)(F)(r)(f)\\ &\quad= \big(\overline{d}(F)\big)(r)(h\circ f) + F(r)\big(\delta(h)\circ f\big)\\ &\quad= d(F(r)\big(h\circ f)\big) - F(r)\big(\delta(h\circ f)\big) + F(r)\big(\delta(h)\circ f\big)\\ &\quad= d\big(F(r)(h\circ f)\big) - F(r)\big(h\circ\delta(f)\big).\end{aligned}$$

This proves the result. ∎

Lemma 2.12 *Let $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ be a hereditary torsion theory on $(\mathcal{R}^{op}, Ab)$. Let $\mathcal{N} \in (\mathcal{R}^{op}, Ab)$ be a torsion free module equipped with a δ-derivation d. Then, the δ-derivation $\overline{d}$ on $Q^\tau(\mathcal{N})$ lifts the δ-derivation d on $\mathcal{N}$. In other words, the following is a commutative diagram*

$$\begin{array}{ccc}\mathcal{N}(a) & \xrightarrow{\Phi_{\mathcal{N}}(a)} & Q^\tau(\mathcal{N})(a)\\ {\scriptstyle d(a)}\downarrow & & \downarrow{\scriptstyle \overline{d}(a)}\\ \mathcal{N}(a) & \xrightarrow{\Phi_{\mathcal{N}}(a)} & Q^\tau(\mathcal{N})(a)\end{array} \tag{2.10}$$

for each $a \in Ob(\mathcal{R})$. Additionally, $\overline{d}$ is the unique δ-derivation on $Q^\tau(\mathcal{N})$ lifting the δ-derivation d on $\mathcal{N}$.

Proof We consider $F \in \mathcal{N}(a)$. Then F corresponds to a morphism $F : H_a \to \mathcal{N}$, which gives an element of $Q^\tau(\mathcal{N})(a)$. For any $r \in Ob(\mathcal{R})$ and any $f \in H_a(r) = \mathcal{R}(r, a)$, we now have

$$\begin{aligned}\overline{d}(F)(r)(f) &= d\big(F(r)(f)\big) - F(r)(\delta(f))\\ &= d\big(\mathcal{N}(f)(F)\big) - F(r)\big(\delta(f)\big)\\ &= \mathcal{N}(f)\big(d(F)\big) + \mathcal{N}\big(\delta(f)\big)(F) - F(r)\big(\delta(f)\big)\\ &= \mathcal{N}(f)\big(d(F)\big) + F(r)\big(\delta(f)\big) - F(r)\big(\delta(f)\big)\\ &= d(F)(r)(f).\end{aligned}$$

This proves that the square (2.10) is commutative. To prove the uniqueness, suppose that $\overline{d}'$ is another δ-derivation on $Q^\tau(\mathcal{N})$ lifting the δ-derivation d on $\mathcal{N}$. We put $\mathcal{Q} = Q^\tau(\mathcal{N})$. For any morphism $h : b \to a$ in $\mathcal{R}$, we observe that

$$\begin{aligned}&\big(\overline{d}(b) - \overline{d}'(b)\big)\circ\mathcal{Q}(h)\\ &\quad= \big(\mathcal{Q}(h)\circ\overline{d}(a) + \mathcal{Q}(\delta(h))\big) - \big(\mathcal{Q}(h)\circ\overline{d}'(a) + \mathcal{Q}(\delta(h))\big)\\ &\quad= \mathcal{Q}(h)\circ\big(\overline{d}(a) - \overline{d}'(a)\big).\end{aligned} \tag{2.11}$$

It follows from (2.11) that $(\overline{d}-\overline{d}')$ is a morphism of functors, *i.e.*, a morphism $(\overline{d}-\overline{d}') : Q^\tau(\mathcal{N}) \to Q^\tau(\mathcal{N})$ in $(\mathcal{R}^{op}, Ab)$. Because $\overline{d}$ and $\overline{d}'$ both lift the δ-derivation d on $\mathcal{N}$, it follows that composing $(\overline{d}-\overline{d}') : Q^\tau(\mathcal{N}) \to Q^\tau(\mathcal{N})$ with the canonical morphism $\Phi_{\mathcal{N}} : \mathcal{N} \to Q^\tau(\mathcal{N})$ gives 0. As such, there is an induced morphism $\text{Coker}(\Phi_{\mathcal{N}}) \to Q^\tau(\mathcal{N})$ through which $(\overline{d}-\overline{d}') : Q^\tau(\mathcal{N}) \to Q^\tau(\mathcal{N})$ factors. But we know (see [7, Proposition 2.4 & Theorem 2.5]) that $\text{Coker}(\Phi_{\mathcal{N}}) \in \mathcal{T}^\tau$ and $Q^\tau(\mathcal{N}) \in \mathcal{F}^\tau$. Hence, the morphism $\text{Coker}(\Phi_{\mathcal{N}}) \to Q^\tau(\mathcal{N})$ must be zero, which shows that $0 = (\overline{d}-\overline{d}') : Q^\tau(\mathcal{N}) \to Q^\tau(\mathcal{N})$. ■

Theorem 2.13 *Let $\mathcal{R}$ be a small preadditive category equipped with a derivation δ. Let $\mathcal{M} \in (\mathcal{R}^{op}, Ab)$ be a right $\mathcal{R}$-module equipped with a δ-derivation d. Let $\tau = (\mathcal{T}^\tau, \mathcal{F}^\tau)$ be a hereditary torsion theory on $(\mathcal{R}^{op}, Ab)$. Then, there is a unique δ-derivation $\overline{d}$ on $Q^\tau(\mathcal{M})$ lifting d,* i.e., *we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(a) & \xrightarrow{\Phi_{\mathcal{M}}(a)} & Q^\tau(\mathcal{M})(a) \\ {\scriptstyle d(a)}\downarrow & & \downarrow{\scriptstyle \overline{d}(a)} \\ \mathcal{M}(a) & \xrightarrow{\Phi_{\mathcal{M}}(a)} & Q^\tau(\mathcal{M})(a) \end{array}$$

for every $a \in Ob(\mathcal{R})$.

Proof As before, let $\mathcal{M}^\tau \subseteq \mathcal{M}$ be the torsion subobject of $\mathcal{M}$. From Theorem 2.9, we know that $d(\mathcal{M}^\tau(a)) \subseteq \mathcal{M}^\tau(a)$ for all $a \in Ob(\mathcal{R})$. As such, $d(a)$ induces maps $\mathcal{M}(a)/\mathcal{M}^\tau(a) \to \mathcal{M}(a)/\mathcal{M}^\tau(a)$ that we continue to denote by $d(a)$. Since $\mathcal{M}/\mathcal{M}^\tau$ is torsion free, we can apply Lemma 2.12 to obtain a unique δ-derivation $\overline{d}$ on $Q^\tau(\mathcal{M}/\mathcal{M}^\tau) = Q^\tau(\mathcal{M})$ such that

$$\begin{array}{ccccc} \mathcal{M}(a) & \xrightarrow{p(a)} & (\mathcal{M}/\mathcal{M}^\tau)(a) & \xrightarrow{\Phi_{\mathcal{M}/\mathcal{M}^\tau}(a)} & Q^\tau(\mathcal{M}/\mathcal{M}^\tau)(a) = Q^\tau(\mathcal{M})(a) \\ {\scriptstyle d(a)}\downarrow & & {\scriptstyle d(a)}\downarrow & & \downarrow{\scriptstyle \overline{d}(a)} \\ \mathcal{M}(a) & \xrightarrow{p(a)} & (\mathcal{M}/\mathcal{M}^\tau)(a) & \xrightarrow{\Phi_{\mathcal{M}/\mathcal{M}^\tau}(a)} & Q^\tau(\mathcal{M}/\mathcal{M}^\tau)(a) = Q^\tau(\mathcal{M})(a) \end{array}$$

is commutative. Suppose that $\overline{d}'$ is another δ-derivation on $Q^\tau(\mathcal{M}/\mathcal{M}^\tau) = Q^\tau(\mathcal{M})$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}(a) & \xrightarrow{\Phi_{\mathcal{M}}(a)} & Q^\tau(\mathcal{M}/\mathcal{M}^\tau)(a) = Q^\tau(\mathcal{M})(a) \\ {\scriptstyle d(a)}\downarrow & & \downarrow{\scriptstyle \overline{d}'(a)} \\ \mathcal{M}(a) & \xrightarrow{\Phi_{\mathcal{M}}(a)} & Q^\tau(\mathcal{M}/\mathcal{M}^\tau)(a) = Q^\tau(\mathcal{M})(a). \end{array}$$

Then we have

$$\begin{aligned} \overline{d}'(a) \circ \Phi_{\mathcal{M}/\mathcal{M}^\tau}(a) \circ p(a) &= \overline{d'}(a) \circ \Phi_{\mathcal{M}}(a) = \Phi_{\mathcal{M}}(a) \circ d(a) \\ &= \Phi_{\mathcal{M}/\mathcal{M}^\tau}(a) \circ p(a) \circ d(a) \\ &= \overline{d}(a) \circ \Phi_{\mathcal{M}/\mathcal{M}^\tau}(a) \circ p(a). \end{aligned}$$

Since $p(a)$ is an epimorphism, it follows from the above that

$$\overline{d}'(a) \circ \Phi_{\mathcal{M}/\mathcal{M}^\tau}(a) = \overline{d}(a) \circ \Phi_{\mathcal{M}/\mathcal{M}^\tau}(a) = \Phi_{\mathcal{M}/\mathcal{M}^\tau}(a) \circ d(a).$$

From the uniqueness of the lifting in Lemma 2.12, we obtain $\overline{d}'(a) = \overline{d}(a)$. This proves the result. ■

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