

EXTENDING TORSORS UNDER QUASI-FINITE FLAT GROUP SCHEMES

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Abstract. Let R be a discrete valuation ring of field of fractions K and of residue field k of characteristic $p > 0$. In an earlier work, we studied the question of extending torsors over K -curves into torsors over R -regular models of the curves in the case when the structural K -group scheme of the torsor admits a finite flat model over R . In this paper, we first give a simpler description of the problem in the case where the curve is semistable using recent work in Holmes, Molcho, Orecchia, and Poirer (2023, *Journal für die Reine und Angewandte Mathematik [Crelle's Journal]* 230, 115–159) and Molcho and Wise (2022, *Compositio Mathematica* 158, 1477–1562). Second, if R is assumed to be Henselian and Japanese, we solve the problem of extending torsors by combining our previous work together with results in Antei and Emsalem (2018, *Nagoya Mathematical Journal* 230, 18–34) and Phung and Dos Santos (2023, *Algebraic Geometry* 230, 1–40), including the case where the structural group does not admit a finite flat R -model.

§1. Introduction

All over this paper, R denotes a discrete valuation ring with field of fractions K and residue field k of characteristic $p > 0$. In addition, schemes and log schemes are supposed to be locally Noetherian.

Let S be a regular scheme, and let $U \subseteq S$ be a dense open subset. Let $f : X \rightarrow S$ be a finite flat morphism of schemes, unramified over U . The Zariski–Nagata purity theorem, known as *purity of the branch locus*, says that the closed subset of S where f ramifies is either empty or of pure codimension 1. On the other hand, given a finite étale group scheme G/S , and an fppf G_U -torsor $X \rightarrow U$ (hence an étale torsor by fppf descent), if it extends into an fppf G -torsor over the whole S , it needs to be étale, hence unramified. But the purity theorem suggests that such an extension may not exist in general. Nevertheless, if the extension of the torsor $X \rightarrow U$ ramifies outside U , and if the ramification is tame, there might be a way to lift it into a *log* torsor over S . Indeed, assume that $D := S \setminus U$ is a normal crossing divisor, so that one can endow S with the divisorial log structure induced by D . Then, *logarithmic torsors* over X are, roughly speaking, tamely ramified over D . This approach of extending torsors into log torsors has been followed in [11], and the main purpose of this paper is to enhance their results. The paper is divided into two independent parts, which we explain below.

Part I:

Let C be a smooth projective curve over K , endowed with a K -point Q , and let J denote its Jacobian variety. Let \mathcal{C} be a regular model of C over R , such that its special

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fiber is a normal crossing divisor, and endow \mathcal{C} with the canonical log structure induced by this divisor; let \mathcal{Q} denote the R -section extending Q by properness. Let G be a finite commutative group scheme over K . It is well known that the Jacobian variety classifies fppf commutative torsors, which can be rephrased through the one-to-one correspondence (cf. [11, lemme 2.3]):

$$H_{fppf}^1(\mathcal{C}, \mathcal{Q}, G) \simeq \mathrm{Hom}(G^D, J), \quad (\star)$$

where the group on the left is the first cohomology group classifying fppf pointed G -torsors (relatively to \mathcal{Q}) over \mathcal{C} , and G^D is the Cartier dual of G . It is shown in [11, Rem. 1.11] that one has a similar correspondence for log torsors over \mathcal{C} . Indeed, if \mathcal{G} denotes a finite flat commutative R -group scheme, one has a one-to-one correspondence:

$$H_{klf}^1(\mathcal{C}, \mathcal{Q}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{G}^D, \mathrm{Pic}_{\mathcal{C}/R}^{\log}),$$

where the group on the left is the first cohomology group classifying Kummer log flat pointed \mathcal{G} -torsors (relatively to \mathcal{Q}) over \mathcal{C} , and $\mathrm{Pic}_{\mathcal{C}/R}^{\log}$ is the relative log Picard functor of \mathcal{C}/R . An immediate consequence of this is that, given a pointed fppf G -torsor over \mathcal{C} , it extends into a log torsor over \mathcal{C} if and only if there exists a finite flat R -model \mathcal{G} of G such that the K -morphism $G^D \rightarrow J$ from (\star) corresponding to the torsor extends into an R -morphism $\mathcal{G}^D \rightarrow \mathrm{Pic}_{\mathcal{C}/R}^{\log}$. Moreover, if \mathcal{J} is the Néron model of J over R , it is shown in the same paper that the canonical map $J \hookrightarrow \mathrm{Pic}_{C/R}$ extends uniquely into a map $\mathcal{J} \rightarrow \mathrm{Pic}_{\mathcal{C}/R}^{\log}$. In particular, if the morphism $G^D \rightarrow J$ extends into a morphism $\mathcal{G}^D \rightarrow \mathcal{J}$, the torsor extends into a \mathcal{G} -log torsor over \mathcal{C} . In this paper, we want to invest the converse. Given that \mathcal{J} is a smooth scheme, it is a *nicer* object to work with than the log Picard functor. Using the results of [12] on log curves, we give a partial answer to this question:

COROLLARY 3.4. *Let C be a smooth projective semistable and geometrically connected curve endowed with a K -point. Let \mathcal{C} be an R -regular model of C with normal crossing special fiber and endowed with the divisorial log structure (cf. example 2.1(3)). Let G be a finite commutative K -group scheme, and let \mathcal{G} be a finite flat R -model of G . Then a pointed fppf G -torsor over \mathcal{C} extends into a pointed \mathcal{G} -log torsor over \mathcal{C} if and only if the K -morphism $G^D \rightarrow J$ associated with the generic torsor (cf. (\star)) extends into an R -morphism $\mathcal{G}^D \rightarrow \mathcal{J}$.*

Part II:

In the second part of this paper, we would like to drop the assumption that G admits a finite flat R -model. Indeed, there exist groups which do not admit such a model (cf. Example 3.7). However, if G is finite, then what is true in general is that it admits a quasi-finite flat R -model (cf. [2, Th. 3.7]). In the latter, the authors took advantage of this to obtain partial answers to the problem of extending finite torsors. In particular, they showed that there exists a modification (a Néron blow-up) of the regular model of the curve over which the torsor extends under some quasi-finite flat group scheme. On the other hand, under additional assumptions on R , it is shown in [16] that a torsor under a quasi-finite flat group scheme reduces into a torsor under a finite flat group scheme. Combining these two results, together with our previous work, we prove the following:

THEOREM 4.4. *Let C be a smooth projective and geometrically connected K -curve with a K -point Q . Let \mathcal{C} be an R -regular model of C , and let G be a finite commutative K -group scheme. Let $Y \rightarrow \mathcal{C}$ be an fppf pointed G -torsor (relatively to Q). Then, there exists a*

quasi-finite flat group scheme \mathcal{G} over R with generic fiber G , and an fppf pointed \mathcal{G} -torsor $\mathcal{Y} \rightarrow \mathcal{C}$ that extends the G -torsor $Y \rightarrow C$.

§2. Preliminaries

2.1 Kummer log flat torsors

2.1.1. Log schemes.

Let (X, \mathcal{O}_X) denote a scheme. A logarithmic (log) structure on X is the data of a sheaf of monoids M_X on $X_{\text{ét}}$, together with a morphism $\alpha_X : M_X \rightarrow \mathcal{O}_X$ such that $\alpha_X^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times$. A scheme endowed with a log structure is said to be a logarithmic (log) scheme.

A morphism of log schemes is a morphism $f : X \rightarrow Y$ of the underlying schemes, together with a morphism $f^{-1}M_Y \rightarrow M_X$ such that the diagram

$$\begin{array}{ccc} f^{-1}M_Y & \longrightarrow & M_X \\ f^{-1}\alpha_Y \downarrow & & \downarrow \alpha_X \\ f^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \end{array}$$

commutes.

2.1.2. Charts.

If \underline{P} is the constant sheaf associated with a monoid P , and if we are given a morphism of sheaves $\underline{P} \rightarrow \mathcal{O}_X$, it induces a unique log structure on X [13, Prop. 1.1.5]. If (X, M_X) is a log scheme, it is said to have a *chart* on P if the log structure induced by P is isomorphic to M_X . All the log schemes in this paper are supposed to admit charts étale locally. Furthermore, if P is fine (finitely generated and integral, i.e., $P \hookrightarrow P^{gp}$) and saturated (i.e., if $a \in P^{gp}$ such that $a^n \in P$ for some nonzero integer n , then $a \in P$), X is said to be a *fine and saturated* log scheme; we refer to [13] for further details.

2.1.3. Inverse image log structure and strict morphisms.

If Y is a log scheme with underlying scheme \underline{Y} , and $f : X \rightarrow \underline{Y}$ is a morphism of schemes, then the composition $f^{-1}M_Y \xrightarrow{f^{-1}\alpha_Y} f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ is a prelog structure on X , and induces the *inverse image log structure* on X that we denote by f^*M_Y .

If $f : X \rightarrow Y$ is a morphism of log schemes, the map $f^{-1}M_Y \rightarrow M_X$ factors canonically through $f^*M_Y \rightarrow M_X$.

The morphism of log schemes $f : X \rightarrow Y$ is said to be *strict* if the induced map $f^*M_Y \rightarrow M_X$ is an isomorphism.

2.1.4. Direct image log structure.

If $f : X \rightarrow Y$ is a morphism of schemes and $\alpha_X : M_X \rightarrow \mathcal{O}_X$ is a log structure on X , then the natural map β in the diagram below

$$\begin{array}{ccc} f_*M_X \times_{f_*\mathcal{O}_X} \mathcal{O}_Y & \xrightarrow{\beta} & \mathcal{O}_Y \\ \downarrow & & \downarrow \\ f_*M_X & \xrightarrow{f_*\alpha_X} & f_*\mathcal{O}_X \end{array}$$

is a log structure on Y , called *the direct image log structure* induced by α_X . We denote it by $f_*^{log} \alpha_X : f_*^{log} M_X \rightarrow \mathcal{O}_Y$.

EXAMPLE 2.1.

1. Let X be a scheme. $M_X := \mathcal{O}_X^\times$ defines a fine and saturated log structure on X called the trivial log structure. X has a chart on the monoid $\{1\}$. If X is a log scheme, the largest Zariski open subset of X (possibly empty) on which the log structure is trivial is called the *open of triviality* of X .
2. Let X be a regular scheme, and let $j : U \hookrightarrow X$ be a dense open subset whose complementary is a normal crossing divisor D on X . Then the sheaf

$$M_X(V) := \{s \in \mathcal{O}_X(V) \mid s|_{V \cap U} \in \mathcal{O}_{V \cap U}(V \cap U)^\times\} \hookrightarrow \mathcal{O}_X(V)$$

defines the *divisorial log structure* on X .

It is the same as the direct image log structure on X of the trivial log structure on U , that is, $j_*^{log} \mathcal{O}_U^\times$ (cf. [8, §1.5]). It is a fine and saturated log structure on X .

Note that U is the open of triviality of the divisorial log structure on X . In particular, $\text{Spec}(R)$ can be seen as a fine and saturated log scheme with the log structure induced by $\text{Spec}(k)$ seen as a divisor. $\text{Spec}(R)$ has a chart on \mathbb{N} given by $\mathbb{N} \rightarrow R; 1 \mapsto \pi$, where π is the uniformizer of R . More generally, if X is a flat R -scheme such that its special fiber is a normal crossing divisor, then it can be seen as a fine and saturated log scheme with the log structure induced by its special fiber. The generic fiber X_K is the open of triviality of the log structure. Furthermore, it has (étale) locally a chart on \mathbb{N}^r .

3. If S is a fine and saturated log scheme, we say that $X \rightarrow S$ is a log curve if it is a proper, integral (cf. [7, Def. 2.3]), vertical,¹ log smooth morphism of (fine and saturated) log schemes with connected and reduced geometric fibers of pure dimension 1. Then, according to [6], the underlying scheme of X is a flat family of nodal curves over S and one has an explicit description of the log structure on the geometric points of X lying above geometric points of S . This is a fine and saturated log structure on X . Conversely, if S is the spectrum of a DVR R and $K = \text{Frac}(R)$, if C/K is a semistable curve with \mathcal{C} some regular model over S , there exist canonical log structures on \mathcal{C} and S making $\mathcal{C} \rightarrow S$ a log curve (cf. [14, §3]). Furthermore, if S is endowed with its divisorial log structure, and if the special fiber of \mathcal{C}/S is a normal crossing divisor, the so-mentioned canonical log structure on \mathcal{C} agrees with the divisorial one (cf. [12, §2 of proof of Lemma 2.2.5.1 and Theorem 2.4.1.3]).

We consider here the category of fine and saturated log schemes, endowed with the *Kummer log flat* topology (we sometimes write *klf* to refer to this topology for simplicity). We refer to [8] or [4, §2.2] for the definition of this Grothendieck topology. A torsor in this category, defined with respect to the klf topology, is called a *logarithmic torsor* (or a log torsor). The structural group of the torsor is always assumed to be endowed with the strict log structure (the inverse log structure of that of the base). Moreover, a Kummer log flat cover of a scheme endowed with the trivial log structure is just a cover for the fppf topology. So, in this paper, the category of schemes is endowed with the fppf topology.

¹ Vertical means that the curve does not have marked points (cf. [6, §1.8 (2)]).

EXAMPLE 2.2 [9, §1; 1.9.3]. Let A be a discrete valuation ring with uniformizer π . Assume that it contains a primitive n th root of unity and that $n \in \mathbb{N}$ is invertible in A . We set $B := A[\sqrt[n]{\pi}]$, $X := \operatorname{Spec}(A)$ and $Y := \operatorname{Spec}(B)$. We endow both these schemes with the divisorial log structure, making them into fine saturated log schemes. Let $G := \operatorname{Aut}_A(B) = \mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. Then $Y \rightarrow X$ is not an fppf G -torsor (because it is totally ramified while G is unramified), but it is a klf (more precisely, Kummer log étale) torsor.

2.2 Extension of torsors under a finite flat group scheme

We recall briefly in this section the main results of [11] on the problem of extending torsors under finite flat group schemes. From now on, $\operatorname{Spec}(R)$ is endowed with the divisorial log structure. Let $(fs/R)_{klf}$ denote the category of fine and saturated log schemes over R endowed with the Kummer log flat topology, and let $(Sch/R)_{fppf}$ be the category of schemes over R endowed with the fppf topology. The latter can be viewed as a full subcategory of $(fs/R)_{klf}$ by endowing an R -scheme with the inverse log structure of that of $\operatorname{Spec}(R)$. We recall the following definitions:

DEFINITION 2.3.

1. We define the following functor:

$$\begin{aligned} \mathbb{G}_{m, \log, R} : (fs/R)_{klf} &\rightarrow (Ab) \\ T &\mapsto \Gamma(T, M_T^{gp}), \end{aligned}$$

which is a sheaf in the klf site [8, Th. 3.2]. Note that generically, it is isomorphic to $\mathbb{G}_{m, K}$.

2. Let C be a smooth projective K -curve with an R -regular model \mathcal{C} endowed with the divisorial log structure. Using the embedding $(Sch/R)_{fppf} \hookrightarrow (fs/R)_{klf}$, consider the following functor:

$$\begin{aligned} (Sch/R)_{fppf} &\rightarrow (Sets) \\ T &\mapsto \{\mathbb{G}_{m, \log, \mathcal{C}} - \log \text{ torsors on } \mathcal{C}_T\}. \end{aligned}$$

The *log Picard functor*, denoted by $\operatorname{Pic}_{\mathcal{C}/R}^{\log}$, is defined to be the fppf sheaffication on $(Sch/R)_{fppf}$ of the previous functor. Furthermore, it is clear that its generic fiber is $\operatorname{Pic}_{C/K}$, the usual relative Picard functor of C/K .

THEOREM 2.4 [11, Rem. 1.11]. *Let C be a smooth projective and geometrically connected curve over K , endowed with a K -point Q , and let J denote its Jacobian variety. Let \mathcal{C} be an R -regular model of C such that its special fiber is a normal crossing divisor. Endow \mathcal{C} with the divisorial log structure, and let \mathcal{Q} be the R -section that extends Q over \mathcal{C} . Let G be a finite commutative K -group scheme with finite flat R -model \mathcal{G} , and let \mathcal{G}^D denote its Cartier dual. We have a canonical isomorphism:*

$$H_{klf}^1(\mathcal{C}, \mathcal{Q}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{G}^D, \operatorname{Pic}_{\mathcal{C}/R}^{\log}),$$

where $H_{klf}^1(\mathcal{C}, \mathcal{Q}, \mathcal{G})$ denotes the cohomology group that classifies logarithmic \mathcal{G} -torsors over \mathcal{C} , pointed relatively to \mathcal{Q} . In particular, a pointed fppf G -torsor (relatively to Q) over C extends into a pointed log \mathcal{G} -torsor (relatively to \mathcal{Q}) over \mathcal{C} if and only if the associated K -morphism $G^D \rightarrow J$ (cf. (\star)) extends into an R -morphism $\mathcal{G}^D \rightarrow \operatorname{Pic}_{\mathcal{C}/R}^{\log}$.

PROPOSITION 2.5 [11, Props. 1.12 and 1.18]. *With the assumptions of the previous theorem, if \mathcal{J} is the Néron model of J over R , then the closed immersion $J \hookrightarrow \mathrm{Pic}_{C/K}$ extends uniquely into an R -morphism $\mathcal{J} \rightarrow \mathrm{Pic}_{C/R}^{\log}$. In particular, if the associated K -morphism $G^D \rightarrow J$ of the generic torsor extends into an R -morphism $\mathcal{G}^D \rightarrow \mathcal{J}$, the torsor extends into a \mathcal{G} -log torsor over \mathcal{C} . Moreover, if \mathcal{J}^0 denotes the identity component of \mathcal{J} , the extended log torsor is fppf if and only if $\mathcal{G}^D \rightarrow \mathcal{J}$ factors through \mathcal{J}^0 .*

§3. Part I: Case of semistable curves

3.1 The Log Picard functor

Recently, the Picard log functor has been defined in a more general frame. Let S be a log regular scheme, and let $U \subseteq S$ be the open of triviality of the log structure on S (which is nonempty and even dense in S by log regularity). Let $X \rightarrow S$ be a logarithmic curve (hence smooth over U). In [12], following the ideas of Illusie and Kato, the authors constructed the analogue of the Picard functor in the logarithmic setting: the logarithmic Picard group that they denoted by $\mathrm{LogPic}_{X/S}$. It is the sheaf of isomorphism classes of the stack which parameterizes the logarithmic line bundles, that is, torsors under the group scheme $\mathbb{G}_{m, \log, S}$, which verify a certain condition called *the condition of bounded monodromy*.

Naturally, the logarithmic Picard group coincides with the ordinary Picard group over X_U , where the log structure is trivial. Furthermore, logarithmic line bundles have a natural notion of (total) degree extending the notion of degree of classical line bundles (cf. [12, §4.5]).

Using this notion of degree, it is defined in [5, De. 3.47] $\mathrm{LogPic}_{X/S}^0$, the subsheaf of $\mathrm{LogPic}_{X/S}$ consisting of log line bundles of total degree zero, which they called *the logarithmic Jacobian*. In fact, this provides the best possible extension of the Jacobian $\mathrm{Pic}_{X_U/U}^0$.

Furthermore, one can restrict the functor $\mathrm{LogPic}_{X/S}^0$ to the category of schemes via the embedding $(Sch)_{fppf} \hookrightarrow (fs/S)_{klf}$, and the resulting functor is called the *strict logarithmic Jacobian* and denoted by $\mathrm{sLogPic}_{X/S}^0$ (cf. [5, Def. 4.5]).

THEOREM 3.1 [5, Cor. 6.13]. *$\mathrm{sLogPic}_{X/S}^0$ is the Néron model of $\mathrm{Pic}_{X_U/U}^0$.*

REMARK 3.2. The condition of bounded monodromy is essential to get a log Picard group of X/S that is well behaved in families. For the purposes of this paper, we do not need to recall its definition in the general setting; we will simply recall it in the case where the base S is the spectrum of a discrete valuation ring endowed with the divisorial log structure. We will see that in this case, this condition is automatically satisfied.

So we take S to be $\mathrm{Spec}(R)$ endowed with the divisorial log structure. Let X denote an R -log curve, and let $s = \mathrm{Spec}(\bar{k})$ denote the geometric closed point of $\mathrm{Spec}(R)$. If Γ denotes the dual graph (which is assumed to be oriented) of X_s and if M_R denotes the divisorial log structure over the base $\mathrm{Spec}(R)$, one can define on Γ a *length map* $l : \Gamma \rightarrow \overline{M}_{R,s}$, where $\overline{M}_R := M_R/\mathcal{O}_R^*$. Since $M_R = \mathcal{O}_R \cap \mathcal{O}_K^*$, one finds that $\overline{M}_{R,s} \simeq \mathbb{N}$.

The data $\mathcal{X} = (\Gamma, l : \Gamma \rightarrow \mathbb{N})$ are called the tropical curve associated with X_s (cf. [12, §2.3]). In addition, one can define a topology on tropical curves (cf. [12, §3]) which allows to do homology on them. In particular, the length map l can be extended to $H_1(\mathcal{X})$.

To any log line bundle over X is associated a class of morphisms $H_1(\mathcal{X}) \rightarrow \overline{M}_{R,s}^{gp} = \mathbb{N}^{gp} = \mathbb{Z}$, called the *monodromy class* (cf. [12, §§3.5 and 4.1]). A logarithmic line bundle is

said to have bounded monodromy if for any $\gamma \in H_1(\mathcal{X})$, $\exists n \in \mathbb{N}$ such that $-nl(\gamma) \leq \alpha(\gamma) \leq nl(\gamma)$, where $\alpha : H_1(\mathcal{X}) \rightarrow \mathbb{Z}$ is some representative in the monodromy class of the line bundle (this condition does not depend on the choice of a representative). $\overline{M}_{R,s} \simeq \mathbb{N}$ being archimedean, it is clear that the monodromy condition is automatically satisfied in this setting.

Therefore, in the case where S is the spectrum of a discrete valuation ring endowed with its divisorial log structure, the condition of bounded monodromy is automatically satisfied, which means that log line bundles consist of all the $\mathbb{G}_{m,\log,S}$ -torsors. In particular, $\text{sLogPic}_{X/S}$ coincides with the log Picard functor we recalled in §2.2.

PROPOSITION 3.3. *Let C be a smooth projective semistable and geometrically connected curve endowed with a K -point. Let \mathcal{C} be an R -regular model of C with normal crossing special fiber and endowed with the divisorial log structure. Let \mathcal{G} be a finite flat R -group scheme. Then any R -morphism $\mathcal{G} \rightarrow \text{Pic}_{\mathcal{C}/R}^{\log}$ factors through $\text{sLogPic}_{\mathcal{C}/R}^0$.*

Proof. According to the previous remark, $\text{sLogPic}_{\mathcal{C}/R}^0$ is the subsheaf of $\text{Pic}_{\mathcal{C}/R}^{\log}$ of (total) degree zero log line bundles. On the other hand, since \mathcal{G} is of torsion, the morphism $\mathcal{G} \rightarrow \text{Pic}_{\mathcal{C}/R}^{\log}$ factors through the torsion of $\text{Pic}_{\mathcal{C}/R}^{\log}$. Now, given the (total) degree map $\text{Pic}_{\mathcal{C}/R}^{\log} \xrightarrow{\deg} \mathbb{Z}$ and the fact that \mathbb{Z} has no torsion, we deduce that torsion log line bundles have (total) degree zero. In addition, $\mathcal{G} \rightarrow \text{Spec}(R)$ being strict, we conclude that $\mathcal{G} \rightarrow \text{Pic}_{\mathcal{C}/R}^{\log}$ factors through $\text{sLogPic}_{\mathcal{C}/R}^0$. \square

COROLLARY 3.4. *Let C be a smooth projective semistable and geometrically connected curve endowed with a K -point. Let \mathcal{C} be an R -regular model of C with normal crossing special fiber and endowed with the divisorial log structure (cf. example 2.1(3)). Let G be a finite commutative K -group scheme, and let \mathcal{G} be a finite flat R -model of G . Then a pointed fppf G -torsor extends into a pointed \mathcal{G} -log torsor over \mathcal{C} if and only if the K -morphism $G^D \rightarrow J$ associated with the generic torsor (cf. (★)) extends into an R -morphism $\mathcal{G}^D \rightarrow \mathcal{J}$.*

Proof. This follows from Theorems 2.4 and 3.1 and Proposition 3.3. \square

3.2 On the existence of a finite flat model of the group scheme

We assume in this section that k is algebraically closed. Let C be a semistable smooth projective and geometrically connected K -curve with Jacobian variety J . Assume that G is a finite commutative subgroup scheme of J . In particular, the morphism $G \hookrightarrow J$ factors through $J[r]$, where r is the order of G . If \mathcal{J} is the Néron model of J , we let \overline{G} be the schematic closure of G inside $\mathcal{J}[r]$. Since C is semistable, $\mathcal{J}[r]$ is flat and quasi-finite (cf. [3, §7.3, Lem. 2]).

In a previous paper (cf. [11, §3]), we found a necessary and sufficient condition for $\mathcal{J}[r]$ to be finite and flat,² hence for G to admit a finite flat R -model, namely \overline{G} .

QUESTION. *If $\mathcal{J}[r]$ is not assumed to be finite flat anymore, can we still find a necessary and sufficient condition for the schematic closure \overline{G} of G to be finite and flat?*

² The conditions are: C is semistable, together with a combinatorial condition on the dual graph of the special fiber.

For the rest of the section, we assume that R is Henselian. Since $\mathcal{J}[r]$ is quasi-finite and R Henselian, according to [10, Lem. 1.1], we have an exact sequence

$$0 \rightarrow \mathcal{F}\mathcal{J}[r] \rightarrow \mathcal{J}[r] \rightarrow \mathcal{E}\mathcal{J}[r] \rightarrow 0, \quad (3.1)$$

where $\mathcal{F}\mathcal{J}[r]$ is a finite flat group scheme over R and $\mathcal{E}\mathcal{J}[r]$ is a étale group scheme over R , with trivial special fiber. In particular, it follows from [1, §IX, Lem. 2.2.3] that $\mathcal{F}\mathcal{J}[r]$ is the largest finite subgroup scheme in $\mathcal{J}[r]$.

The schematic closure \overline{G} of G in $\mathcal{J}[r]$ is flat and quasi-finite. Hence, we have as previously an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \overline{G} \rightarrow \mathcal{E} \rightarrow 0,$$

where \mathcal{F} is a finite flat group scheme over R and \mathcal{E} is a étale group scheme over R , with trivial special fiber. We would like to find a necessary and sufficient condition for \overline{G} to be finite.

We denote by $\mathcal{F}\mathcal{J}[r]_K$ the generic fiber of $\mathcal{F}\mathcal{J}[r]$.

LEMMA 3.5. *\overline{G} is finite if and only if $G \rightarrow J[r]$ factors through $\mathcal{F}\mathcal{J}[r]_K$.*

Proof. If \overline{G} is finite, since $\mathcal{F}\mathcal{J}[r]$ is the largest finite subgroup scheme inside $\mathcal{J}[r]$, then $\overline{G} \rightarrow \mathcal{J}[r]$ factors through $\mathcal{F}\mathcal{J}[r]$; hence, $G \rightarrow J[r]$ factors through $\mathcal{F}\mathcal{J}[r]_K$.

On the other hand, if $G \rightarrow J[r]$ factors through $\mathcal{F}\mathcal{J}[r]_K$, since $\mathcal{F}\mathcal{J}[r]$ is closed inside $\mathcal{J}[r]$ (it is a kernel), \overline{G} is the schematic closure of G in $\mathcal{F}\mathcal{J}[r]$; hence, it is finite (closed immersions are finite and the composition of two finite morphisms is finite). \square

COROLLARY 3.6. *Let C be a semistable smooth projective and geometrically connected curve with a K -point, let \mathcal{C} be an R -regular model of C with normal crossing special fiber and endowed with the divisorial log structure, and let J be the Jacobian of C and \mathcal{J} its Néron model. Let G be a finite subgroup scheme of J . Then, the corresponding fppf pointed G^D -torsor $Y \rightarrow C$ (cf. (\star)) extends into a log torsor over \mathcal{C} under a finite flat group scheme if and only if G is a subgroup of $\mathcal{F}\mathcal{J}[r]_K$, with r the order of G .*

Proof. If $G \hookrightarrow J[r]$ factors through $\mathcal{F}\mathcal{J}[r]_K$, then the schematic closure \overline{G} of G in $\mathcal{J}[r]$ is finite and flat by Lemma 3.5, and it follows from Corollary 3.4 that the torsor extends into a logarithmic \overline{G}^D -torsor over \mathcal{C} . On the other hand, if there exists a finite flat model \mathcal{G} of G such that the torsor extends into a log \mathcal{G}^D -torsor, then it follows from Corollary 3.4 again that the K -morphism $G \rightarrow J[r]$ extends into an R -morphism $\mathcal{G} \rightarrow \mathcal{J}[r]$. Since \mathcal{J} is separated, it follows that \mathcal{G} is necessarily the schematic closure of G in $\mathcal{J}[r]$; hence, by Lemma 3.5, it implies that $G \hookrightarrow J[r]$ factors through $\mathcal{F}\mathcal{J}[r]_K$. \square

COUNTER-EXAMPLE 3.7. Let A be a local Noetherian and complete ring, with fraction field K , and residue field of characteristics p . We find in [15, §5] the following bijection:

$$\{\text{isomorphism classes of } A\text{-group schemes of order } p\} \simeq \{(a, b) \in A^2 | ab = p\} / \sim,$$

where $(a, b) \sim (c, d)$ if and only if $\exists u \in A^\times$ such that $c = u^{p-1}a$ and $d = u^{1-p}b$. Considering the restriction morphism

$$\{(a, b) \in A^2 | ab = p\} / \sim \xrightarrow{\varphi} \{(a, b) \in K^2 | ab = p\} / \sim,$$

and taking, for example, $A = \mathbb{Z}_p$, it is easy to see that φ is not surjective in general, which means that there are \mathbb{Q}_p -group schemes that does not extend into a finite flat \mathbb{Z}_p -group schemes.

QUESTION. *More generally, it is natural to ask what happens if we do not assume that the structural group of the torsor admits a finite flat R -model. We investigate this question in the next section.*

§4. Part II: Extension of torsors under a quasi-finite flat group scheme

The following result by Antei says that there exists some regular model of the curve where the torsor extends into an fppf torsor.

THEOREM 4.1 [2, Th. 3.7]. *Let $\mathcal{X} \rightarrow \operatorname{Spec}(R)$ be a faithfully flat morphism of finite type, with \mathcal{X} a regular and integral scheme of absolute dimension 2 endowed with an R -section. Let G be a finite K -group scheme, and let $f: Y \rightarrow \mathcal{X}_K$ be an fppf pointed G -torsor. Then there exists an integral scheme \mathcal{X}_0 , faithfully flat and of finite type over R , a model map $\lambda: \mathcal{X}_0 \rightarrow \mathcal{X}$ and an fppf \mathcal{G} -torsor $\mathcal{Y} \rightarrow \mathcal{X}_0$ extending the given G -torsor Y for some quasi-finite and flat R -group scheme \mathcal{G} . Moreover, \mathcal{X}_0 can be obtained by \mathcal{X} after a finite number of Néron blow-ups.*

On the other hand, it is shown in [16] that an fppf torsor under a quasi-finite flat group scheme reduces into a torsor under a finite flat group scheme.

THEOREM 4.2 [16, Th. 12.1]. *Let R be a discrete valuation ring which is assumed to be Henselian Japanese, such that its residue field k is perfect. Let \mathcal{X} be a normal, irreducible, projective, and flat R -scheme with geometrically reduced fibers and with an R -section. Let \mathcal{G} be a quasi-finite flat R -group scheme, and let $\mathcal{Y} \rightarrow \mathcal{X}$ be a pointed fppf \mathcal{G} -torsor. Then, there exist a finite flat R -group scheme \mathcal{H} , a morphism $\mathcal{H} \rightarrow \mathcal{G}$, and a pointed fppf \mathcal{H} -torsor $\mathcal{Y}_0 \rightarrow \mathcal{X}$ such that $\mathcal{Y}_0 \times^{\mathcal{H}} \mathcal{G} \simeq \mathcal{Y}$ is pointed fppf \mathcal{G} -torsors.*

We deduce from it the following.

COROLLARY 4.3. *With the same notations and assumptions as in Theorem 4.2, if $\mathcal{Y} \rightarrow \mathcal{X}$ is a pointed fppf \mathcal{G} -torsor and \mathcal{F} the largest finite subgroup scheme inside \mathcal{G} , there exists a pointed fppf \mathcal{F} -torsor $\mathcal{Y}_0 \rightarrow \mathcal{X}$ such that $\mathcal{Y}_0 \times^{\mathcal{F}} \mathcal{G} \simeq \mathcal{Y}$ as pointed fppf \mathcal{G} -torsors.*

Proof. Let \mathcal{H} be the finite group scheme in Theorem 4.2. Since \mathcal{F} is the largest finite subgroup of \mathcal{G} , $\mathcal{H} \rightarrow \mathcal{G}$ factors through \mathcal{F} . Therefore, the surjective map

$$\begin{aligned} H_{fppf}^1(\mathcal{X}, \mathcal{H}) &\rightarrow H_{fppf}^1(\mathcal{X}, \mathcal{G}) \\ \mathcal{T} &\mapsto \mathcal{T} \times^{\mathcal{H}} \mathcal{G} \end{aligned}$$

factors through $H_{fppf}^1(\mathcal{X}, \mathcal{F})$ in an obvious way. \square

THEOREM 4.4. *Let C be a smooth projective and geometrically connected K -curve with a K -point. Let \mathcal{C} be a regular model of C , and let G be a finite commutative K -group scheme. Let $Y \rightarrow C$ be an fppf pointed G -torsor. Then, there exist a quasi-finite flat group scheme \mathcal{G} over R with generic fiber G and an fppf pointed \mathcal{G} -torsor over \mathcal{C} that extends the G -torsor $Y \rightarrow C$.*

Proof. By Theorem 4.1, there exists a quasi-finite flat R -group scheme \mathcal{G} that extends G , together with a regular model \mathcal{C}_0 of C such that the fppf pointed G -torsor $Y \rightarrow C$ extends into an fppf \mathcal{G} -torsor $\mathcal{Y} \rightarrow \mathcal{C}_0$. By Corollary 4.3, if \mathcal{F} is the largest finite subgroup scheme

of \mathcal{G} , there exists a pointed fppf \mathcal{F} -torsor $\mathcal{Y}_0 \rightarrow \mathcal{C}_0$ such that $\mathcal{Y}_0 \times^{\mathcal{F}} \mathcal{G} \simeq \mathcal{Y}$. Hence, the pointed fppf \mathcal{F}_K -torsor $\mathcal{Y}_{0,K} \rightarrow C$ extends into the pointed fppf \mathcal{F} -torsor $\mathcal{Y}_0 \rightarrow \mathcal{C}_0$. If J denotes the Jacobian of C and \mathcal{J}^0 the identity component of its Néron model, this is equivalent by Proposition 2.5 to the fact that the associated K -morphism $\mathcal{F}_K^D \rightarrow J$ (cf. (\star)) extends into an R -morphism $\mathcal{F}^D \rightarrow \mathcal{J}^0$. But this implies by Proposition 2.5 again that the fppf \mathcal{F}_K -torsor $\mathcal{Y}_{0,K} \rightarrow C$ extends into an fppf \mathcal{F} -torsor $\mathcal{Y}' \rightarrow \mathcal{C}$. Hence, the G -torsor $Y \rightarrow C$ extends into the fppf \mathcal{G} -torsor $\mathcal{Y}' \times^{\mathcal{F}} \mathcal{G} \rightarrow \mathcal{C}$. \square

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