CUBATURE METHOD FOR THE NUMERICAL SOLUTION OF THE CHARACTERISTIC INITIAL VALUE PROBLEM $u_{xy} = f(x, y, u, u_x, u_y)$

M. K. JAIN and K. D. SHARMA

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1. Introduction

The resemblance of the Goursat problem

$$
\begin{aligned}
&u_{xy} = f(x, y, u, u_x, u_y)\\
&u(x, 0) = \varphi(x),\ u(0, y) = \psi(y),\ \varphi(0) = \psi(0) \qquad (1.1)\\
&0 \leqq x \leqq a,\ 0 \leqq y \leqq b
\end{aligned}
$$

for the hyperbolic partial differential equations to the initial value problem

$$
\begin{aligned}
&\frac{dy(x)}{dx} = f(x, y(x)) \qquad (1.2)\\
&y(x_0) = y_0 \qquad 0 < x \leqq a
\end{aligned}
$$

for the ordinary differential equations has suggested the extension of many well known numerical methods existing for (1.2) to the numerical treatment of (1.1). Day [2] discusses the quadrature methods while Diaz [3] generalizes the simple Euler-method. Moore [6] gives an analogue to the fourth order Runge-Kutta-method and Tornig [7] generalizes the explicit and implicit Adams-methods.

The subject of this paper is the development of a numerical procedure for finding an approximate solution of the Goursat problem (1.1) and is based upon the idea of using cubature formulae which are exact for polynomials of known degrees. In case the function $f(x, y, u, u_x, u_y)$ happens to be a polynomial of certain known finite degree, the method may be used to obtain exact numerical results. The paper has been divided into five sections. In Section 2, methods for obtaining the initial numerical approximation to the exact solution have been discussed. Section 3 deals with an iteration scheme for improving the values obtained in Section 2. The convergence of the numerical approximations in Section 3 to the exact solution has been discussed in Section 4. Section 5 considers a non-linear example to show a good agreement of the computed values with the exact values.

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2. Calculation of starting values

Before proceeding to obtain a numerical approximation to the initial field, we shall assume that

(i) the real-valued function $f(x, y, u, u_x, u_y)$ is defined for all values of (x, y, u, u_x, u_y) which satisfy the inequalities

$$x_0 \leqq x \leqq x_0+a, y_0 \leqq y \leqq y_0+b, -\infty < u, u_x, u_y < +\infty.$$

(ii) $f(x, y, u, u_x, u_y)$ is continuous and bounded in absolute value, so that for a certain non-negative constant M

$$|f(x, y, u, u_x, y_y)| \leqq M.$$

(iii) $f(x, y, u, u_x, u_y)$ satisfies the Lipschitz condition in the three arguments u, u_x and u_y, i.e., there is a constant $L \geqq 0$ such that

$$|f(x, y, u, u_x, u_y)-f(x, y, u^*, u_x^*, u_y^*)| \leqq L[|u-u^*|+|u_x-u_x^*|+|u_y-u_y^*|]$$

for any (u, u_x, u_y) and (u^*, u_x^*, u_y^*), whenever (x, y) lies in the rectangle

$$R: x_0 \leqq x \leqq x_0+a,\ y_0 \leqq y \leqq y_0+b.$$

(iv) The real valued function $\varphi(x)$ is defined for all x in the interval $x_0 \leqq x \leqq x_0+a$ and possesses a continuous first order derivative $\varphi'(x)$ in this interval while the real valued function $\psi(y)$ is defined for all y in the interval $y_0 \leqq y \leqq y_0+b$ and possesses a continuous first order derivative $\psi'(y)$ in this interval.

Having ensured the unique existence of the solution of the Goursat problem (1.1), it is suggestive to consider the following system of integral equations equivalent to (1.1):

$$\begin{aligned} u(x, y) &= \varphi(x)+\psi(y)-\varphi(0)+\int_0^x\int_0^y f(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta))d\xi d\eta \\ p(x, y) &= \varphi'(x)+\int_0^y f(x, \eta, u(x, \eta), p(x, \eta), q(x, \eta))d\eta \\ q(x, y) &= \psi'(y)+\int_0^x f(\xi, y, u(\xi, y), p(\xi, y), q(\xi, y))d\xi \end{aligned} \tag{2.1}$$

where

$$p(x, y) = \frac{\partial u(x, y)}{\partial x}, \qquad q(x, y) = \frac{\partial u(x, y)}{\partial y}.$$

The procedure to be described can be viewed as a technique of stepwise approximate integration of these equations. Let us consider a subdivision of the rectangle

$$R: x_0 \leqq x \leqq x_0+a, y_0 \leqq y \leqq y_0+b$$

given by

$$x_0 \equiv x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m \equiv x_0+a,$$
$$y_0 \equiv y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n \equiv y_0+b,$$

where

$$x_{i+1} = x_i + h,\ h > 0 \qquad (i = 0, 1, 2, \cdots, m-1)$$
$$y_{j+1} = y_j + k,\ k > 0 \qquad (j = 0, 1, 2, \cdots, n-1).$$

The straight lines which are parallel to the axis and are passing through these points subdivide the rectangle R into mn closed subrectangles

$$R^{m,n}_{r,s} \qquad (r = 0, 1, \cdots, m,\ s = 0, 1, \cdots, n).$$

If we denote

$$u_{r,s} = u(x_r, y_s); \quad P_{r,s} = p(x_r, y_s); \quad q_{r,s} = q(x_r, y_s),$$
$$f(x, y, u, p, q) = F(x, y); \quad f(x_r, y_s, u_{r,s}, p_{r,s}, q_{r,s}) = f_{r,s},$$

the system of integral equations (2.1) can be written in the form

$$u_{r+1,s+1} = u_{r+1,s} + u_{r,s+1} - u_{r,s} + \int_{x_r}^{x_{r+1}} \int_{y_s}^{y_{s+1}} F(\xi, \eta)\, d\xi\, d\eta, \tag{2.2}$$

$$p_{r+1,s+1} = p_{r+1,s} + \int_{y_s}^{y_{s+1}} F(x_{r+1}, \eta)\, d\eta, \tag{2.3}$$

$$q_{r+1,s+1} = q_{r,s+1} + \int_{x_r}^{x_{r+1}} F(\xi, y_{s+1})\, d\xi. \tag{2.4}$$

This representation of the system (2.1) is more suitable from computational point of view.

Before making use of these equations for the starting values, we shall further assume that the exact values of the partial derivatives $p(x, y)$ and $q(x, y)$ along the axis of y and x, respectively, are known to us (see example in Section 5). In case the exact values of these quantities are not available, an approximation to them may be obtained, by using Adam's-methods (see [5]) as follows:

Denoting the backward difference operator by ∇, the equations (2.3) and (2.4) for the set of points $(x_r, 0)$, $(0, y_s)$, $(r = 0, 1, 2, \cdots m;$ $s = 0, 1, 2, \cdots n)$ may be expressed in the form

$$p_{0,s+1} = p_{0,s} + k \sum_{\rho=0}^{n} \alpha_\rho \nabla^\rho_x F(x_r, y_0) + R_{n+1}, \tag{2.5}$$

$$q_{r+1,0} = q_{r,0} + h \sum_{\rho=0}^{m} \alpha_\rho \nabla^\rho_x F(x_r, y_0) + R^*_{m+1}, \tag{2.6}$$

where

$$\alpha_\rho = \frac{1}{\rho!} \int_0^1 u(u+1)(u+2) \cdots (u+\rho-1)\, du,$$

$$|R_{n+1}| \leqq k^{n+2} |\alpha_{n+1}| \left| \frac{\partial^{n+1} F(x_0, y)}{\partial y^{n+1}} \right|_{\mathrm{Max}},$$

$$|R^*_{m+1}| \leqq h^{m+2} |\alpha_{m+1}| \left| \frac{\partial^{m+1} F(x, y_0)}{\partial x^{m+1}} \right|_{\mathrm{Max}}.$$

If we neglect the remainder terms in (2.5) and (2.6) and express all the differences in terms of the values of the function at the pivotal points, then the approximations to the values of

$$p_{0,s+1},\ q_{r+1,0} \qquad (r = 0, 1, \cdots m-1;\ s = 0, 1, \cdots, n-1)$$

may be obtained from

$$p_{0,s+1} = p_{0,s} + k \sum_{\rho=0}^{n} \beta_{n,\rho} f_{0,s-\rho}, \tag{2.7}$$

$$q_{r+1,0} = q_{r,0} + h \sum_{\rho=0}^{m} \beta_{m,\rho} f_{r-\rho,0}, \tag{2.8}$$

where

$$\beta_{\mu,\nu} = (-1)^{\nu} \sum_{\rho=\nu}^{\mu} \alpha_{\rho} \binom{\rho}{\nu}.$$

The coefficients $\beta_{\mu,\nu}$ depend upon μ as well as ν, which makes it more difficult to change the number of differences employed in (2.7) and (2.8). Some numerical values of the coefficients $\beta_{\mu,\nu}$ have been listed in Table 1 of this paper.

TABLE 1. Values of $\beta_{\mu,\nu}$

ν	μ = 0	1	2	3	4	5
0	1	$\frac{3}{2}$	$\frac{23}{12}$	$\frac{55}{24}$	$\frac{1901}{720}$	$\frac{4277}{1440}$
1		$-\frac{1}{2}$	$-\frac{16}{12}$	$-\frac{59}{24}$	$-\frac{2774}{720}$	$-\frac{7923}{1440}$
2			$\frac{5}{12}$	$\frac{37}{24}$	$\frac{2616}{720}$	$\frac{9982}{1440}$
3				$-\frac{9}{24}$	$-\frac{1274}{720}$	$-\frac{7298}{1440}$
4					$\frac{251}{720}$	$\frac{2877}{1440}$
5						$-\frac{475}{1440}$

Since the accuracy of the values of the initial field depends upon the accuracy of the values of $p_{0,s+1}$ and $q_{r+1,0}$ computed above, therefore, before making any attempt for the calculation of the starting values it is advisible to refine them by some iterative scheme. If we use the modified multistep method [1] the convergence of which is already ensured, then for the set of points under consideration, the iterative scheme shall read

$$p_{0,s} = \sum_{\rho=0}^{l} a_{\rho} p_{0,s-\rho} + k \left[\sum_{\rho=0}^{l} b_{\rho} f_{0,s-\rho} + b_{l+1} F(x_0, y_{s-c}) \right], \tag{2.9}$$

$$q_{s,0} = \sum_{\rho=0}^{l} a_\rho q_{s-\rho,0} + h\,[\sum_{\rho=0}^{l} b_\rho f_{s-\rho} + b_{l+1} F(x_{s-c}, y_0)], \tag{2.10}$$

where the values of the parameters a_ρ, b_ρ and c are chosen in such a way so as to yield stable processes of orders as high as possible. For a particular value (used in example of Section 5 we list down in Table 2, the value of a_ρ and b_ρ for a particular value of $c = \frac{1}{2}$.

TABLE 2

$\rho \backslash l$	1		2		3	
	a_ρ	b_ρ	a_ρ	b_ρ	a_ρ	b_ρ
0	0	$\frac{1}{6}$	0	$\frac{15}{93}$	0	$\frac{465}{3085}$
1	1	$\frac{1}{6}$	$\frac{32}{31}$	$\frac{12}{93}$	$\frac{783}{617}$	$\frac{135}{3085}$
2	0	$\frac{2}{3}$	$-\frac{1}{31}$	$-\frac{1}{93}$	$-\frac{135}{617}$	$-\frac{495}{3085}$
3				$\frac{64}{93}$	$-\frac{31}{617}$	$-\frac{39}{3085}$
4					0	$\frac{2304}{3085}$

We shall evaluate the double integral in (2.2) by cubature formulae similar to those discussed in [8]. These formulae shall be based upon the idea of approximating the function $F(x, y)$ occurring in (2.2) by a Langrange polynomial $P(x, y)$ of the type

$$P(x, y) = \sum_{i=0}^{M} \sum_{j=0}^{M} C_{ij} x^{2i} y^{2j} \tag{2.11}$$

which coincides with the function $F(x, y)$ at a set of suitably predetermined $(M^2+3M+2)/2$ points. If we now demand the cubature formula

$$\begin{aligned} I_N &= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} P(x, y)\,dx\,dy \\ &= 4hk\,[\sum_{\tau=1}^{N} P(x_\tau, y_\tau)] \end{aligned} \tag{2.12}$$

to be exact for all polynomials of degree N, we get a system of equations which determines the weights w_τ and the points x_τ and y_τ.

For example the formulae

$$I_3 = hk\,[\sum P(x_i \pm h/\sqrt{3}, y_i \pm k/\sqrt{3})] \tag{2.13}$$

$$\begin{aligned} I_4 = \frac{hk}{49} \Bigg[9 \sum P\left(x_i \pm \frac{\sqrt{7}}{3} h, y_i \pm \frac{\sqrt{7}}{3} k\right) \Bigg] \\ + 40 \sum P(x_i \pm \sqrt{\tfrac{7}{15}} h, y_i) + 40 \sum P(x_i, y_i \pm \sqrt{\tfrac{7}{15}} k) \end{aligned} \tag{2.14}$$

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(where summations extend over all distinct combinations of signs) are exact for 3rd and 5th degree polynomials. Using these cubature formulae in (2.2) and making approximations to the integrals involved in (2.3) and (2.4) by suitable quadrature formulae, we obtain a system of equations which may be used for the calculation of the starting values. It may however be noted that the determination of the coefficients C_{ij} in (2.11) can be avoided by choosing a particular Langrange polynomial which coincides with the function values of $F(x, y)$ at the set of points involved in the cubature formula (2.13) and (2.14) and also coincides with $F(x, y)$ at a few other points. For in this case values of the polynomial $P(x, y)$ at this set of points shall be identical with the values of the function $F(x, y)$ which may be calculated by making use of the following approximate formulae (in which $\sigma, \tau \leqq 1$)

$$\begin{aligned}(2.15)\quad u(x_0+\sigma h, y_0+\tau k) &= [1+2(\sigma^3+\tau^3)-3(\sigma^2+\tau^2)]u(x_0, y_0)+[3\tau^2-2\tau^3]\\ &\quad u(x_0, y_0+k)+[3\sigma^2-2\sigma^3]u(x_0+h, y_0)\\ &\quad +h[\sigma^3-2\sigma^2+\sigma-\sigma\tau^2]p(x_0, y_0)\\ &\quad +h\sigma\tau^2 p(x_0, y_0+k)+h(\sigma^3-\sigma^2)p(x_0+h, y_0)\\ &\quad +k[\tau^3-2\tau^2-\sigma^2\tau+\tau]q(x_0, y_0)+k(\tau^3-\tau^2)q(x_0, y_0+k)\\ &\quad +k\sigma^2\tau q(x_0+h, y_0)+hk\sigma\tau(1-\sigma-\tau)F(x_0, y_0)+O(h^4)\end{aligned}$$

$$\begin{aligned}(2.16)\quad p(x_0+\sigma h, y_0+\tau k) &= \frac{6}{h}\,[(\sigma^2-\sigma)\{u(x_0, y_0)-u(x_0+h, y_0)\}]\\ &\quad +\frac{2k\sigma\tau}{h}\,[q(x_0+h, y_0)-q(x_0, y_0)]\\ &\quad +[1+3\sigma^2-4\sigma-\tau^2]p(x_0, y_0)\\ &\quad +\tau^2 p(x_0, y_0+k)+(3\sigma^2-2\sigma)p(x_0+h, y_0)\\ &\quad +k\tau(1-\tau-2\sigma)F(x_0, y_0)+O(h^3)\end{aligned}$$

$$\begin{aligned}(2.17)\quad q(x_0+\sigma h, y_0+\tau k) &= \frac{6}{k}\,[(\tau^2-\tau)\{u(x_0, y_0)-u(x_0, y_0+k)\}]\\ &\quad +\frac{2h\sigma\tau}{k}\,[p(x_0, y_0+k)-p(x_0, y_0)]\\ &\quad +(1+3\tau^2-4\tau-\sigma^2)q(x_0, y_0)\\ &\quad +(3\tau^2-2\tau)q(x_0, y_0+k)+\sigma^2 q(x_0+h, y_0)\\ &\quad +h\sigma(1-2\tau-\sigma)F(x_0, y_0)+O(h^3).\end{aligned}$$

These approximations are valid in each subrectangle with its corners denoted by (x_0, y_0), (x_0+h, y_0), (x_0, y_0+k) and (x_0+h, y_0+k).

For evaluating the double integral in (2.2) one may however use the cartesian product formulae as well. For example, if in particular we use the

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cartesian product formulae corresponding to the Lobatto four-point rule in one dimension

$$(2.18)\quad \int_{x_0}^{x_0+h} \varphi(\tau)d\tau = \frac{h}{2}\sum_{\rho=1}^{4} w_\rho \varphi(\tau_\rho) - \frac{4h^7\varphi^{(6)}(\xi)}{2^7.47250}\; x_0 < \xi < x_0+h,$$

where

$$\tau_1 = x_0,\quad \tau_2 = x_0 + \left(\frac{5-\sqrt{5}}{10}\right)h,\quad \tau_3 = x_0 + \left(\frac{5+\sqrt{5}}{10}\right)h,\quad \tau_4 = x_0+h,$$

$$w_1 = w_4 = \tfrac{1}{6};\quad w_2 = w_3 = \tfrac{5}{6};$$

we get for the system of equations (2.2), (2.3) and (2.4), the following equivalent system:

$$(2.19)\quad u_{r+1,s+1} = u_{r+1,s} + u_{r,s+1} - u_{r,s} + \frac{hk}{4}\sum_{\mu=1}^{4}\sum_{\nu=1}^{4} w_\mu w_\nu F(\tau_\mu, \tau_\nu) + E,$$

$$(2.20)\quad P_{r+1,s+1} = P_{r+1,s} + \frac{k}{2}\sum_{\mu=1}^{4} w_\mu F(x_{r+1}, \tau_\mu) + \bar{E},$$

$$(2.21)\quad q_{r+1,s+1} = q_{r,s+1} + \frac{h}{2}\sum_{\mu=1}^{4} w_\mu F(\tau_\mu, y_{s+1}) + \bar{\bar{E}},$$

where

$$E = \frac{-4}{2^7.47250}\,[h^7 D_x^5 F(x_r, y_s) + k^7 D_y^5 F(x_r, y_s)] + O(h^8) + O(k^8),$$

with

$$D_x f = f_x + pf_u + u_{xx} f_p + ff_q,$$
$$D_y f = f_y + qf_u + ff_p + u_{yy} f_q,$$

$$(2.22)\quad \begin{cases} \bar{E} = \dfrac{-4k^7}{2^7.47250}\left(\dfrac{\partial^6 F(x_r, \xi)}{\partial \xi^6}\right) y_s < \xi < y_{s+1}, \\[2ex] \bar{E} = \dfrac{-4h^7}{2^7.47250}\left(\dfrac{\partial^6 F(\eta, y_s)}{\partial \eta^6}\right) x_s < \eta < x_{s+1}. \end{cases}$$

We prefer the use of Lobatto quadrature formula since the values of the function $F(x, y)$, at the set of points needed for the evaluation of the integrals in (2.3) and (2.4), are already made available in the process of evaluating the double integral in (2.2). For the calculation of starting values one may neglect the error terms in (2.16), (2.17) and (2.18) and use the resulting formulae for the approximate calculation of the quantities

$$u_{r+1,s+1},\ p_{r+1,s+1},\ q_{r+1,s+1} \qquad (r = 0, 1, \cdots m-1,\ s = 0, 1, \cdots, n-1).$$

In order to have some idea about the applicability of the results discussed

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in this section we performed calculations for the Liouville differential equation [8]:

$$u_{xy} = e^{2u}$$

with initial condition

$$u(x, 0) = \frac{x}{2} - \log(1+e^x),$$

$$u(0, y) = \frac{y}{2} - \log(1+e^y).$$

In this case, as is obvious, it is not possible to get the exact values of $p(0, y)$ and $q(x, 0)$ by mere inspection of the differential equation, therefore a recourse was made to the equations (2.7) to (2.10). The results obtained with $h = k = 0.05$ for the initial field have been listed in Table 3 of this section.

These results have been compared with the exact values obtained from the exact solution of this problem given by

$$u(x, y) = (x+y)/2 - \log(e^x+e^y).$$

TABLE 3[1]. Values of the initial field in the partial differential equation $\frac{\partial^2 u}{\partial x \partial y} = e^{2u}$

	$u(x, y)$				
$y \backslash x$	0.1	0.2	0.3	0.4	0.5
0.1	−.69314727	−.69439673	−.69813908	−.70435542	−.71301531
	−.69314716	−.69439662	−.69813885	−.70435520	−.71301520
0.2	−.69439673	−.69314747	−.69439723	−.69813968	−.70435592
	−.69439662	−.69314713	−.69439663	−.69813881	−.70435510
0.3	−.69813908	−.69439723	−.69314827	−.69439813	−.69814048
	−.69813885	−.69439663	−.69314717	−.69439660	−.6981880
0.4	−.70435542	−.69813968	−.69439813	−.69314927	−.69439903
	−.70435520	−.69813881	−.69439660	−.69314710	−.69439660
0.5	−.71301531	−.70435592	−.69814048	−.69439903	−.69315007
	−.71301520	−.70435510	−.69813880	−.69439660	−.69314710

[1] The upper value in each row gives the computed value, the lower value in each row gives the exact value.

3. Refining the initial field by an iteration scheme

In Section 2, while deriving the cubature formulae we had used irrational points to obtain the values of the function at the pivotal points. Using the method discussed below one can derive some cubature formulae which involve the values of the function at the pivotal points only and are helpful in deriving the iteration scheme. Let us suppose that this time the function $F(x, y)$ is replaced by the Langrange polynomial $Q(x, y)$ of the form

$$Q(x, y) = \sum_{i=0}^{2n} \sum_{j=0}^{2n} C'_{ij} x^i y^j \qquad (i+j \leqq 2n),$$

which coincides with the values of function $F(x, y)$ at a set of suitably predetermined points. If we now demand that the relation

$$I = \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} Q(x, y)\,dx\,dy = 4hk \sum_{\alpha_1}^{m} R_\alpha Q(x_\alpha, y_\alpha)$$

be true for all polynomials of certain finite degree (say M), then we get a system of equations which determines the values of R_α, x_α, and y_α. For example, if one takes $2n = 1, 3, 5$ etc. in the Langrange polynomial taken above one gets the following cubature formulae

$$I_1 = hk[Q_{i+1,j+1} + Q_{i-1,j+1} + Q_{i+1,j-1} + Q_{i-1,j-1}], \tag{3.1}$$

$$I_3 = \tfrac{2}{3}hk[2Q_{i,j} + Q_{i+1,j-1} + Q_{i-1,j} + Q_{i,j+1} + Q_{i,j-1}], \tag{3.2}$$

$$I_5 = hk[-112s_1 + 4s_2 + 5s_3 + 64s_4], \tag{3.3}$$

where

$$s_1 = Q_{i,j},$$
$$s_2 = Q_{i,j+1} + Q_{i+1,j} + Q_{i,j-1} + Q_{i-1,j},$$
$$s_3 = Q_{i+1,j+1} + Q_{i+1,j-1} + Q_{i-1,j+1} + Q_{i-1,j-1},$$
$$s_4 = Q_{i,j+\frac{1}{2}} + Q_{i+\frac{1}{2},j} + Q_{i,j-\frac{1}{2}} + Q_{i-\frac{1}{2},j}.$$

As discussed in the last section the polynomial $Q(x, y)$ should be made to coincide with the values of the function $F(x, y)$ at a set of suitably predetermined points. One may obtain quite a variety of such formulae giving us the values of the double integral. If we denote by I_N, a cubature formula exact for all polynomials of degree $\leqq N$, then the equation (2.2) reads

$$u_{r+1,s+1} = u_{r+1,s} + u_{r,s+1} - u_{r,s} + I_N + E_1$$

where

$$E_1 = \frac{1}{2\pi i} \int_c \left[\prod_{\mu=1}^{m} \prod_{\nu=1}^{n} (s - u_{\mu,\nu}) F(t) \Big/ \prod_{\mu=1}^{m} \prod_{\nu=1}^{n} (t - u_{\mu,\nu})(t - z) \right] dt$$

c being a simple closed rectifiable curve containing R in its interior. Again making use of the modified multistep method for the integrals in

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(2.3) and (2.4), the equations iterating the values of p and q are given by

$$(3.5)\quad p_{r+1,s+1} = \sum_{\rho=0}^{l} a_\rho p_{r+1,s+1-\rho} + k\sum_{l=0}^{l} b_\rho f_{r+1,s-\rho} + kb_{l+1} f_{r+1,s-c} + E_2,$$

$$(3.6)\quad q_{r+a,s+1} = \sum_{\rho=0}^{l} a_\rho q_{r+1-\rho,s+1} + h\sum_{\rho=0}^{l} b_\rho f_{r-\rho,s+1} + hb_{l+1} f_{r-c,s+1} + E_3,$$

where

$$E_2 = k^* \frac{\partial^{2l+2}F}{\partial y^{2l+2}} + O((2k)^{2l+3}),$$

$$(3.7)\quad E_3 = k^* \frac{\partial^{2l+2}F}{\partial x^{2l+2}} + O((2h)^{2l+3}),$$

$$k^* = 2b_{l+1}\left[\frac{c(1-c)\cdots(l-c)}{l!}\right]^2\left[-\frac{1}{c}+\frac{1}{1-c}+\frac{1}{2-c}+\cdots+\frac{1}{l-c}\right]\frac{l!}{(2l+2)!}.$$

Here we have assumed that the values of the quantities otbained from the previous steps are exact. If we neglect the error terms in the above equations, the first set of iterated values will be given by

$$(3.8)\quad u_{r,s}^{[1]} = u_{r,s-1} + u_{r-1,s} - u_{r-1,s-1} + hk\sum_\mu\sum_\nu A_{\mu,\nu} f_{\mu,\nu}^{[0]}$$

(summation is taken over all nodal points occurring in the Langrange polynomial)

$$(3.9)\quad p_{r,s}^{[1]} = p_{r,s-1} + \sum_{\rho=0}^{l} a_\rho p_{r,s-\rho} + k\sum_{\rho=0}^{l} b_\rho f_{r,s-\rho}^{[0]} + kb_{l+1} f_{r,s-c}^{[0]},$$

$$(3.10)\quad q_{r,s}^{[1]} = q_{r-1,s} + \sum_{\rho=0}^{l} a_\rho q_{r-\rho,s} + h\sum_{\rho=0}^{l} b_\rho f_{r-\rho,s}^{[0]} + hb_{l+1} f_{r-c,s}^{[0]},$$

$$f_{r,s}^{[1]} = f(x_r, y_s, u_{r,s}^{[1]}, p_{r,s}^{[1]}, q_{r,s}^{[1]}),$$

where the weights $A_{\rho,\sigma}$ are to be read from the equations (3.2), (3.3), $\cdots$ and the constants a's and b's occurring in the last two equations for different values of c are to be taken from Table 2. The iteration scheme corresponding to $\rho \geqq 1$ is as follows

$$(3.11)\quad u_{r,s}^{[\rho+1]} = u_{r,s}^{[\rho]} + hk\sum_\mu\sum_\nu A_{\mu,\nu}(f_{\mu,\nu}^{[\rho]} - f_{\mu,\nu}^{[\rho-1]}),$$

$$(3.12)\quad p_{r,s}^{[\rho+1]} = p_{r,s}^{[\rho]} + k\sum_{\mu=0}^{l} b_\mu (f_{r,s-\mu}^{[\rho]} - f_{r,s-\mu}^{[\rho-1]}) + kb_{l+1}(f_{r,s-c}^{[\rho]} - f_{r,s-c}^{[\rho-1]}),$$

$$(3.13)\quad q_{r,s}^{[\rho+1]} = q_{r,s}^{[\rho]} + h\sum_{\nu=0}^{l} b_\nu (f_{r-\nu,s}^{[\rho]} - f_{r-\nu,s}^{[\rho-1]}) + hb_{l+1}(f_{r-c,s}^{[\rho]} - f_{r-c,s}^{[\rho-1]}),$$

$$f_{r,s}^{[\rho+1]} = f(x_r, y_s, u_{r,s}^{[\rho+1]}, p_{r,s}^{[\rho+1]}, q_{r,s}^{[\rho+1]}),$$

$$(r = 1, 2, \cdots, m;\ s = 1, 2, \cdots, n).$$

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The convergence of this iteration scheme has been discussed in Section 4. Having obtained the refined values of $u(x, y)$, $p(x, y)$, $q(x, y)$ at the pivotal points, one is now in a position to write down a Hermite polynomial expression approximating the analytic solution of $u(x, y)$ of the problem (1.1) in the entire region R.

4. Convergence of the iteration scheme

For discussing the convergence of the numerical approximations obtained in section (3), we shall write

$$\begin{aligned} u_{i,j}^{[\rho+1]}-u_{i,j}^{[\rho]} &= d_{i,j}^{[\rho]}, \\ p_{i,j}^{[\rho+1]}-p_{i,j}^{[\rho]} &= \bar{d}_{i,j}^{[\rho]}, \\ q_{i,j}^{[\rho+1]}-q_{i,j}^{[\rho]} &= \bar{\bar{d}}_{i,j}^{[\rho]}, \end{aligned} \tag{4.1}$$

$$|d_{i,j}^{[\rho]}|+|\bar{d}_{i,j}^{[\rho]}|+|\bar{\bar{d}}_{i,j}^{[\rho]}| = S_{i,j}^{[\rho]}; \tag{4.2}$$

then we may write down the following estimates for the equations (3.11), (3.12) and (3.13):

$$\begin{aligned} |d_{i,j}^{[\rho]}| &\leqq hkL \sum_{\mu}\sum_{\nu} A_{\mu,\nu}\, S_{i,j}^{[\rho-1]}, \\ |\bar{d}_{i,j}^{[\rho]}| &\leqq kL\, [\sum_{\nu=0}^{l} b_\nu S_{i,j-\nu}^{[\rho-1]}+b_{l+1}S_{i,j-c}^{[\rho-1]}], \\ |\bar{\bar{d}}_{i,j}^{[\rho]}| &\leqq hL\, [\sum_{\mu=0}^{l} b_\mu S_{i-\mu,j}^{[\rho-1]}+b_{l+1}S_{i-c,j}^{[\rho-1]}]. \end{aligned} \tag{4.3}$$

The addition of these three equations gives

$$\begin{aligned} S_{i,j}^{[\rho]} \leqq{} & hkL \sum_{i}\sum_{j} A_{i,j} S_{i,j}^{[\rho-1]} \\ & +kL\sum_{\mu=1}^{l} b_\mu S_{i-\mu,j}^{[\rho-1]}+hL\sum_{\mu=1}^{l} b_\mu S_{i,j-\mu}^{[\rho-1]}+kLb_{l+1}S_{i,j-c}^{[\rho-1]}+hLb_{l+1}S_{i-c,j}^{[\rho-1]}. \end{aligned}$$

This can easily be seen to be of the form

$$S_{i,j}^{[\rho]} \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} |B_{i,j}^{m,n}|\, S_{i,j}^{[\rho-1]}. \tag{4.4}$$

If all the eigen-value of the matrix

$$B = (|B_{i,j}|)$$

are numerically < 1, then the convergence of the series

$$\sum_{\rho=0}^{\infty} d_{i,j}^{[\rho]},\ \sum_{\rho=0}^{\infty} \bar{d}_{i,j}^{[\rho]},\ \sum_{\rho=0}^{\infty} \bar{\bar{d}}_{i,j}^{[\rho]},$$

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is ensured. From this it follows that

$$\max_{i,j} \{\sum_{i=1}^{m} \sum_{j=1}^{n} |B_{i,j}^{m,n}|\} < 1 \quad \text{(partial sum criterion)},$$

$$\max_{m,n} \{\sum_{i=1}^{m} \sum_{j=1}^{n} |B_{i,j}^{m,n}|\} < 1 \quad \text{(split sum criterion)}.$$

These inequalities give us the upper bound for the interval lengths h and k. For $h = k, N = m = n = 2$ and $l = 1$, both these conditions are satisfied if

$$h < \tfrac{1}{4}\left[\sqrt{\left(1+\frac{4}{L}\right)}-1\right].$$

Similarly we may obtain upper bounds for any other set of values of N, m, n and l.

5. Example

As an example we shall consider the non-linear partial differential equation

$$u_{xy} = pq/u+u \tag{5.1}$$

which involves all the variables u, p and q and is associated with the boundary conditions

$$u(x, 0) = e^{x^2}, \quad u(0, y) = \cos y. \tag{5.2}$$

If we write down the given differential equation (5.1) for points lying on the axis, then

$$\begin{aligned} \frac{\partial p(0, y)}{\partial y} &= p(0, y)q(0, y)/u(0, y)+u(0, y), \\ \frac{\partial q(x, 0)}{\partial x} &= p(x, 0)q(x, 0)/u(x, 0)+u(x, 0). \end{aligned} \tag{5.3}$$

From which it easily follows that

$$p(0, y) = y \cos y, \quad q(x, 0) = xe^{x^2}. \tag{5.4}$$

For obtaining the numerical results we wrote programmes for the IBM 1620 (Model 1) computer. The starting values were calculated by using the equations (2.19) to (2.21) with $h = k = 0.05$. These values were iterated two times with the help of the equations (3.11), (3.12) and (3.13) written for $l = 1, 2$, $N = 5$, $h = 0.05$. The exact solution of the partial differential equation (5.1) is

$$u(x, y) = e^{x^2+xy} \cos y.$$

TABLE 4*

y \ x	iteration number	.1	.2	.3	.4	.5
		$\varepsilon(x, y)$				
.1	0	$-.200\times10^{-6}$	$-.400\times10^{-6}$	$-.500\times10^{-6}$	$-.500\times10^{-6}$	$-.800\times10^{-6}$
	1	$-.226\times10^{-7}$	$-.512\times10^{-7}$	$-.962\times10^{-7}$	$-.170\times10^{-6}$	$-.279\times10^{-6}$
	2	$-.226\times10^{-7}$	$-.258\times10^{-7}$	$-.938\times10^{-7}$	$-.162\times10^{-6}$	$-.264\times10^{-6}$
	0	$-.200\times10^{-6}$	$-.500\times10^{-6}$	$-.800\times10^{-6}$	$-.100\times10^{-5}$	$-.140\times10^{-5}$
.2	1	$-.456\times10^{-7}$	$-.353\times10^{-8}$	$-.873\times10^{-7}$	$-.131\times10^{-6}$	$-.340\times10^{-6}$
	2	$-.453\times10^{-7}$	$-.353\times10^{-8}$	$-.538\times10^{-7}$	$-.354\times10^{-7}$	$-.149\times10^{-6}$
	0	$-.700\times10^{-7}$	$-.600\times10^{-6}$	$-.900\times10^{-6}$	$-.110\times10^{-5}$	$-.120\times10^{-5}$
.3	1	$-.683\times10^{-7}$	$-.644\times10^{-7}$	$-.230\times10^{-6}$	$-.444\times10^{-6}$	$-.969\times10^{-6}$
	2	$-.675\times10^{-7}$	$-.466\times10^{-7}$	$-.144\times10^{-6}$	$-.192\times10^{-6}$	$-.409\times10^{-6}$
	0	$-.200\times10^{-6}$	$-.500\times10^{-6}$	$-.700\times10^{-6}$	$-.700\times10^{-6}$	$-.200\times10^{-6}$
.4	1	$-.862\times10^{-7}$	$-.195\times10^{-7}$	$-.270\times10^{-6}$	$-.586\times10^{-6}$	$-.364\times10^{-6}$
	2	$-.853\times10^{-7}$	$-.129\times10^{-7}$	$-.109\times10^{-6}$	$-.836\times10^{-7}$	$-.210\times10^{-6}$
	0	$-.900\times10^{-7}$	$-.300\times10^{-6}$	$-.400\times10^{-6}$	$-.200\times10^{-6}$	$-.160\times10^{-5}$
.5	1	$-.908\times10^{-7}$	$-.538\times10^{-7}$	$-.416\times10^{-6}$	$-.103\times10^{-6}$	$-.681\times10^{-6}$
	2	$-.908\times10^{-7}$	$-.258\times10^{-7}$	$-.180\times10^{-6}$	$-.103\times10^{-6}$	$-.428\times10^{-6}$

* The first row gives the errors in the values of the initial field, the second row gives the errors in the values obtained after first iteration. The third row gives the errors in the values obtained after second iteration.

TABLE 5*. Solution of the partial differential equation $u_{xy} = pq/u+u$

y \ x	0.1	0.2	0.3	0.4	0.5
	$U(x, y)$				
0.1	1.0151046	1.0565318	1.1218641	1.2153001	1.3431154
	1.0151045	1.0565317	1.1218640	1.2153007	1.3431151
0.2	1.0099140	1.0616934	1.1386749	1.245908	1.3907808
	1.0099138	1.0616933	1.1386748	1.245907	1.3907806
0.3	0.99432457	1.0558101	1.1437456	1.2640343	1.4251947
	0.99432443	1.0558100	1.1437453	1.2640341	1.4251944
0.4	0.96828488	1.0384934	1.1362928	1.2684188	1.4445115
	0.96828470	1.0384933	1.1362926	1.2684186	1.4445120
0.5	0.93184932	1.0094603	1.1156263	1.2578651	1.446890
	0.93184919	1.0094601	1.1156260	1.2578648	1.446889

* Upper value in each row gives the computed value.
Lower value in each row gives the exact value.

In order to study the rate of convergence of the iteration process we list in Table 4, the errors in the computed values of $u(x, y)$ for the initial field, obtained after first iteration and second iteration, respectively. The results obtained after two iterations have been listed in Table 5.

As is expected, the work involved in the case when u, p and q are all present in the function f is more than when p, q are absent (refer [2]).

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Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-29
India