Canad. Math. Bull. Vol. **36** (2) 1993, pp. 251–256

LIMIT CYCLES OF LIÉNARD EQUATIONS WITH NON LINEAR DAMPING

A.M. URBINA, G. LEÓN DE LA BARRA,
M. LEÓN DE LA BARRA AND M. CAÑAS

ABSTRACT. We consider the Liénard's equation $\ddot{x} = f(x)\dot{x} - x^{2n-1}$, $n \in \mathbb{N}$, with $f(x)$ polynomial. By using the generalized polar coordinates we establish the maximum possible number of small amplitude limit cycles of such equation in terms of the degree of $f(x)$.

1. **Introduction.** The Liénard's equation

$$\ddot{x} = f(x)\dot{x} + g(x) \tag{1}$$

or in the equivalent vector field form

$$X : \begin{cases} \dot{x} = y + F(x) \\ \dot{y} = g(x) \end{cases} \tag{2}$$

where $F(x) = \int_0^x f(t)\,dt$, has been widely studied and arise frequently in applications. See Staude's survey [6] for references.

We are interested in establishing the existence and number of periodic solutions of system (2).

In this context, for $g(x) = -x$, Lins, de Melo and Pugh [3] proved: "Given an integer m with $0 \leq m \leq n$, there is a polynomial $F(x)$ of degree $N = 2n + 1$ or $N = 2n + 2$ such that the system (2) has exactly m closed orbits". They also conjectured that in this case the equation has no more than n closed orbits.

Blows and Lloyd [1] showed for the same system, using a suitable Lyapunov function, that there are at most n small amplitude limit cycles.

Recently, Dumortier and Rousseau [2] presented an extensive qualitative study of the phase portrait for the case of a linear damping $f(x)$ and a cubic restoring force $g(x)$. They found limit cycles surrounding one singularity and limit cycles surrounding two or three singularities and proved that the equation can have at most one limit cycle of the first kind and that such a limit cycle can never be surrounded by a limit cycle of the second kind.

Finally we have to mention the work of R. Moussu [5] concerning symmetry and normal form of degenerated centers and foci.

Partially supported by FONDECYT, under grant # 0243-91 and U.T.F.S.M., under grant # 911204.
Received by the editors January 29, 1992.
AMS subject classification: 34C05, 34C25.

In this paper we consider the case $g(x) = -x^{2n-1}$ and $F(x) = \sum_{i=n}^{N} a_i x^i$, $n \in \mathbb{N}$, generalizing the results obtained by Lins *et al.*, Blows and Lloyd.

The results obtained here are important in order to increase the knowledge on limit cycles of polynomial vector fields in the context of Hilbert's 16-th problem.

2. **Small amplitude limit cycles.** Let us consider the vector field X defined by the system:

$$\begin{aligned} \dot{x} &= y + \sum_{i=m}^{N} a_i x^i \\ \dot{y} &= -x^{2n-1} \end{aligned} \tag{3}$$

LEMMA 1. *X has separatrices if*

i) $m < n$ or

ii) $m = n$ and $|a_n| \geq \frac{2}{\sqrt{n}}$

PROOF. Let us consider the blowing-up

$$\phi \colon (x, y) \longrightarrow (x, x^m y) \text{ with } m \leq n$$

Then, the vector field $\phi^* X$ (up a factor x^{m-1}) is given by:

$$\phi^* X : \begin{cases} \dot{x} = xy + \sum_{k=m}^{N} a_k x^{k-m+1} \\ \dot{y} = -my^2 - my \sum_{k=m}^{N} a_k x^{k-m} - x^{2(n-m)} \end{cases}$$

i) For $m < n$, $\phi^*(0, y) = (0, -my^2 - m\, a_m y)$ has two singularities $(0, 0)$ and $(0, -a_m)$. Then X has separatrices.

ii) Let $m = n$. Then $\phi^*(0, y) = (0, -ny^2 - na_n y - 1)$ has at least one singularity if and only if $|a_n| \geq 2/\sqrt{n}$. Again, X has separatrices. ∎

For $m > n$ or $m = n$ and $|a_n| < 2/\sqrt{n}$ the vector field X has no separatrices because in these cases, the Poincaré mapping exists in a neighborhood of the origin.

We will only consider the case $m = n$ and $|a_n| < 2/\sqrt{n}$, since $m > n$ implies $a_n = 0$.

To prove the existence of the return mapping, let us consider the functions $C_s\theta$ and $S_n\theta$ given by Lyapunov [4] and defined by the Cauchy problem:

$$\begin{aligned} \frac{d}{d\theta}(Cs\theta) &= -Sn\theta \quad Cs0 = 1 \\ \frac{d}{d\theta}(Sn\theta) &= Cs^{2n-1}(\theta) \quad Sn0 = 0 \end{aligned}$$

These functions satisfy the identity:

$$Cs^{2n}(\theta) + nSn^2(\theta) = 1$$

Furthermore, they are w-periodic with

$$w = 4\sqrt{n} \int_0^1 (1 - x^{2n})^{-1/2}\, dx$$

We consider the coordinate change:

$$x = rCs\theta$$
$$y = -r^n Sn\theta$$

and we obtain that the system (3) becomes (up a factor r^{n-1})

$$\dot r = rCs^{2n-1}(\theta)\sum_{i=n}^{N} a_i r^{i-n} Cs^i(\theta)$$
$$\dot\theta = 1 - nSn\theta\sum_{i=n}^{N} a_i r^{i-n} Cs^i(\theta)$$

and then

$$\frac{dr}{d\theta} = \frac{rCs^{2n-1}(\theta)\sum_{i=n}^{N} a_i r^{i-n} Cs^i(\theta)}{1 - na_nCs^n(\theta)Sn\theta - nSn\theta\sum_{i=n+1}^{N} a_i r^{i-n} Cs^i(\theta)} \tag{4}$$

If $|a_n| < 2n^{-1/2}$, (4) is an analytic function for r small enough.
In fact

$$|na_nCs^n(\theta)Sn\theta| \leq \frac{\sqrt{n}}{2}|a_n||Cs^{2n}(\theta) + nSn^2\theta| < 1$$

We have two cases:

i) $a_n = 0$. The right side of (4) is represented by the series:

$$\sum_{i=2}^{\infty} R_i r^i$$

Let the function r, which satisfies the equation (4), be written in the form of a series:

$$r(\theta, r_0) = r_0 + u_2(\theta)r_0^2 + u_3(\theta)r_0^3 + \cdots$$

with $r(0, r_0) = r_0$, and $r(w, r_0)$ representing the Poincaré mapping and $u_i(w)$ corresponding, up a numerical factor, to its derivative of order i.

We obtain, by recurrence that

$$\begin{aligned} R_j = R_j(\theta) &= p_{n+(j-1)}(\theta)Cs^{2n-1}(\theta) \\ &\quad + nSn\theta[p_{n+(j-2)}(\theta)R_2 + p_{n+(j-3)}(\theta)R_3 + \cdots + p_{n+1}(\theta)R_{j-1}] \end{aligned} \tag{5}$$

where

$$p_i(\theta) = a_i Cs^i(\theta), \quad n \leq i \leq N, \quad j \geq 2$$

($R_0 = R_1 = 0$) and

$$u_\ell(\theta) = \int_0^\theta \sum_{i_1+\cdots+i_j=\ell} (u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_j})(\tau)R_j(\tau)\, d\tau \tag{6}$$

where $u_1(\theta) \equiv 1$.

ii). If $a_n \neq 0$ the right side of the equation (3) is represented by the series expansion: $R_1 r + R_2 r^2 + \cdots$, which is uniformly convergent for all θ and for sufficiently small r.

The coefficients R_s are rational functions of $Sn\theta$ and $Cs\theta$ with denominators equal to different powers of the function $1 - np_n(\theta)Sn\theta$.

In fact:

$$R_1 = \frac{p_n(\theta)Cs^{2n-1}\theta}{1 - np_n(\theta)Sn(\theta)}$$

and

$$R_s = \frac{1}{1 - np_n(\theta)Sn(\theta)}\Big[p_{n+s-1}(\theta)Cs^{2n-1}(\theta) + nSn\theta\big(p_{n+1}(\theta)R_{s-1} + p_{n+2}(\theta)R_{s-2} + \cdots + p_{n+s-1}(\theta)R_1\big)\Big]$$

Now, let $r = \rho\exp\big(\int_0^\theta R_1(\tau)\,d\tau\big)$, then

$$\frac{d\rho}{d\theta} = \exp\Big(-\int_0^\theta R_1(\tau)\,d\tau\Big)\cdot\frac{dr}{d\theta} - \rho R_1 = T_2\rho^2 + T_3\rho^3 + \cdots \tag{7}$$

where

$$T_s = R_s\exp\Big[(s-1)\int_0^\theta R_1(\tau)\,d\tau\Big]$$

As in the preceeding case the $u_\ell(\theta)$ are given by (6), replacing $R_j(\tau)$ by $T_j(\tau)$.

LEMMA 2. *Let $\tilde{X}$ be the vector field (3) with $a_i = 0$ for all odd i and $|a_n| < 2n^{-1/2}$. Then, the origin is a centre for $\tilde{X}$.*

PROOF. $|a_n| < 2n^{-1/2}$ implies that the vector field has no separatrices to the origin, and by symmetry the singularity is a centre. ∎

LEMMA 3. *If $|a_n| < 2n^{-1/2}$, the type of stability of the origin for the system (3) is determined by the first $a_i \neq 0$ with odd i.*

PROOF. We consider two cases:

CASE i). n even. Let us suppose $a_{n+1} = \cdots = a_{n+2j-1} = 0$ and $a_{n+2j+1} \neq 0$.

Taking into account formula (6) we have that $u_k(\theta) = \tilde{u}_k(\theta)$ for $1 \leq k \leq 2j+1$, and

$$u_{2j+2}(\theta) = \tilde{u}_{2j+2}(\theta) + H(\theta)$$

where $\tilde{u}_\ell(\theta)$ are the corresponding functions for $\tilde{X}$. Then by Lemma 2, $u_k(w) = 0$, $1 < k \leq 2j+1$ and

$$u_{2j+2}(w) = H(w) = \begin{cases} a_{n+2j+1}\int_0^w Cs^{3n+2j}(\theta)\,d\theta, & \text{if } a_n = 0 \\ a_{n+2j+1}\int_0^w \exp[(2j+1)\int_0^\theta R_1(\tau)\,d\tau]\frac{Cs^{3n+2j}(\theta)\,d\theta}{(1-np_n(\theta)Sn\theta)^2} & \text{if } a_n \neq 0 \end{cases}$$

therefore $\text{sgn}\big(u_{2j+2}(w)\big) = \text{sgn}\, a_{n+2j+1}$ in both cases.

CASE ii). n odd. Let us suppose $a_n = a_{n+2} = \cdots = a_{n+2(j-1)} = 0$ and $a_{n+2j} \neq 0$. Analogously, by Lemma 2

$$\begin{aligned} u_k(w) &= 0 \text{ for } 1 < k \leq 2j \quad \text{and} \\ u_{2j+1}(w) &= \int_0^w p_{n+2j}(\theta) Cs^{2n-1}(\theta)\, d\theta \\ &= a_{n+2j} \int_0^w Cs^{3n+2j-1}\, d(\theta)\, d\theta \end{aligned}$$

therefore $\operatorname{sgn}\big(u_{2j+1}(w)\big) = \operatorname{sgn} a_{n+2j}$

Then the origin is an attracting (repelling) focus if $a_{n+2j+1} < 0$ ($a_{n+2j+1} > 0$) for even n or $a_{n+2j} < 0$ ($a_{n+2j} > 0$) for odd n. ∎

REMARK. In the case n even we have the additional property $u_{2k} \equiv 0$ for $1 \leq k \leq j$.

THEOREM. *Consider the system (3) with* $|a_n| < 2n^{-1/2}$ *and*

$$N = \begin{cases} n + 2k + 1 & \text{or } n + 2k + 2 \text{ if } n \text{ is even} \\ n + 2k & \text{or } n + 2k + 1 \text{ if } n \text{ is odd} \end{cases}$$

i) There are at most k small amplitude limit cycles.

ii) If

$$0 < |a_{n+1}| \ll |a_{n+3}| \ll \cdots \ll |a_{n+2k+1}| \text{ and}$$
$$a_{n+2j-1} \cdot a_{n+2j+1} < 0, \quad j = 1, \ldots, k \ (n \text{ even})$$

or

$$0 < |a_n| \ll |a_{n+2}| \ll \cdots \ll |a_{n+2k}| \text{ and}$$
$$a_{n+2(j-1)} \cdot a_{n+2j} < 0, \quad j = 1, \ldots, k \ (n \text{ odd})$$

then there are exactly k small amplitude limit cycles.

PROOF. In order to illustrate, let us assume n even, $a_{n+1} = a_{n+3} = \cdots = a_{n+2k-1} = 0$ and $a_{n+2k+1} \neq 0$

If $a_{n+2k+1} > 0$ by Lemma 2, the origin is unstable. Now choose $a_{2n+2k-1} < 0$, the origin becomes stable and if $|a_{n+2k-1}| \ll |a_{n+2k+1}|$ we have $u_{2j-2}(w) < 0$ and $u_{2j}(w) > 0$, then a repelling limit cycle is generated around the origin.

By choosing successively $a_{n+2k-3}, \ldots, a_{n+1}$ so that each a_{n+2j-1} is of the opposite sign to a_{n+2j+1} and small enough, the stability of the origin is reversed k times, and each time a small amplitude limit cycle bifurcates out of the origin.

The same argument works if $a_{n+2k+1} < 0$. ∎

COROLLARY. *Let N be as in the theorem. Given an integer s with* $0 \leq s \leq k$ *there is a polynomial* $\sum_{i=n}^{N} a_i x^i$ *of degree N such that the system (3) has exactly s small amplitude limit cycles.*

REFERENCES

1. T. R. Blows and N. G. Lloyd, *The number of small-amplitude limit cycles of Liénard equations*, Math. Proc. Camb. Phil. Soc. **95**(1984), 359–366.
2. F. Dumortier and C. Rousseau, *Cubic Liénard equations with linear damping*, pre-print, 1989.
3. A. Lins, W. de Melo and C. Pugh, *On Liénard's equation*. Lecture Notes in Mathematics, **597** Springer-Verlag, 1977, 335–357.
4. A. Lyapunov, *Stability of motion*. Mathematics in Science and Engineering, **30**, Academic Press, New York, London, 1966.
5. R. Moussu, *Symétrie et Forme Normale des Centres et Foyers Dégénérés*. Ergod. Th. and Dynam. Systems. **2**(1982), 241–251.
6. U. Staude, *Uniqueness of periodic solutions of the Liénard equations*. Recent Advances in Differential Equations, Academic Press, 1981.

Departamento de Matemática
Universidad Técnica Federico Santa María
Casilla 110 - V. Valparaíso
Chile