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ON DARBOUX AND MEAN VALUE PROPERTIES

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ABSTRACT. In this paper we extend and greatly generalize, with some new information, the well known results that an approximately continuous function is Darboux, and that a finite approximate derivative has the mean value property and is Darboux. Our theorems on Darboux and mean value properties of derivatives include also those of selective derivatives and I-approximate derivatives.

1. **Introduction**. Denjoy [2] showed that an approximately continuous function on a linear interval is Darboux (that is, has the intermediate value property). Later Khintchine [5] showed that a finite approximate derivative has the mean value property and is Darboux. Goffman and Neugebauer [3] obtained the same results with new methods of proofs. Recently Sinharoy [12] has obtained some generalizations of these results, which cover also the notions of proximal continuity and derivative introduced in [10]. The purpose of this paper is to prove more extensive results, with some additional information, using very simple proofs. Our theorems on Darboux and mean value properties of derivatives include also those of selective derivatives (O'Malley [7]) and I-approximate derivatives (Wilczynski [14]).

For the sake of wider coverage, we shall consider derivations and approximations relative to a given strictly increasing function ω: $R \to R$ and relative to the Lebesgue-Stieltjes outer measure ω^* induced by ω on R, respectively, where R denotes the real line.

2. **Definitions and lemmas**.

DEFINITION 2.1. *Let $E \subset R$, and let $I_{x,y}$ denote any closed interval on R with end points x and y. We define*

$$d(E, x, y) = d(E, y, x) = \omega^*(E \cap I_{x,y})/\omega^*(I_{x,y}),$$

$$d^+(E, x) = \lim \sup_{y \to x^+} d(E, x, y),$$

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$$d^-(E, x) = \lim \sup_{y \to x^-} d(E, x, y),$$

$$d_\infty(E, x) = \min\{d^+(E, x), d^-(E, x)\}.$$

If $\lim_{y \to x} d(E, x, y)$ *exists, then this limit is called the* ω*-density,* $d(E, x)$, *of* E *at* x.

NOTE. The above definition of ω-density appears in [9], Definition 2.1, p. 130. The ω*-Vitali covering theorems ([9], Theorems 1.1, 1.2, pp. 129–130) imply in the usual way the ω-density theorem, namely that, if $\omega^*(E) > 0$, then there is at least one point $x \in E$ such that $d(E, x) = 1$.

DEFINITION 2.2. *A subset* $E \subset R$ *is said to be bilaterally* ω*-dense in itself if, for every open interval* I *with* $\emptyset \neq E \cap \bar{I}$ *(bar denoting closure), we have* $\omega^*(E \cap I) > 0$.

In the sequel we shall need the following lemma relating to the above notion of ω*-denseness, which implies a strong connectivity property of intervals. It is worth comparing this result with Lemma 2.3 in [10].

LEMMA 2.1. *Let an open interval* I *be the union of two nonempty disjoint sets* A *and* B, *each of which is bilaterally* ω*-dense in itself. Then there is at least one point* $c \in I$ *such that* $d_\infty(A, c) = 1 = d_\infty(B, c)$.

PROOF. We first note that, by ω*-denseness, each of the sets A and B is obviously bilaterally dense in itself. Therefore, if an open interval $J \subset I$ intersects both A and B, then $\inf(A \cap J) < \sup(B \cap J)$ and $\inf(B \cap J) < \sup(A \cap J)$.

Now we assert that, given any $\epsilon > 0$ and any open interval $J \subset I$ intersecting both A and B, there are tetrads of points $a_1 < p_1 < p_2 < a_2$ and $b_1 < q_1 < q_2 < b_2$ with $a_1, p_2, a_2, q_1 \in A \cap J$ and $b_1, q_2, b_2, p_1 \in B \cap J$, such that

(1) $\qquad a_2 - a_1 < \epsilon, d(A, x, a_j) > 1 - \epsilon \quad$ if $\quad x \in (p_1, p_2), j = 1, 2,$

and

(2) $\qquad b_2 - b_1 < \epsilon, d(B, x, b_j) > 1 - \epsilon \quad$ if $\quad x \in (q_1, q_2), j = 1, 2.$

Because of symmetry, it is enough to show the existence of a tetrad of the first kind. To this end, we will first find points $a_1 < a_3 < a_4 < a_1 + \epsilon$ with $a_1, a_4 \in A \cap J$ and $a_3 \in B \cap J$, such that $d(A, x, a_1) > 1 - \epsilon$ for all $x \in (a_3, a_4)$.

Let B^- and B^+ denote the sets of points of I which are limit points of B on the left and on the right, respectively. Since $\inf(A \cap J) < \sup(B \cap J)$, there are points $a_0 \in A \cap J$ and $b_0 \in B \cap J$ such that $a_0 < b_0$. We note that $[a_0, b_0] \subset J \subset A \cup B$ and, by denseness of B at b_0, $B \cap (a_0, b_0) \neq \emptyset$. Let now $x_0 = \inf(B \cap (a_0, b_0))$. Since $A \cap B = \emptyset$ and $a_0 \in A$, the condition $x_0 \in B$ would mean that $a_0 < x_0$ and that $B \cap (a_0, x_0) = \emptyset$, contrary to the denseness of B at x_0. So $x_0 \in A$, $a_0 \leq x_0 < b_0$ and $x_0 \in B^+$; in particular, therefore, we can find a point $x_1 \in B \cap (x_0, x_0 + \epsilon) \cap (x_0, b_0)$.

Now, first suppose $A \cap (x_0, x_1)$ has nonvoid interior, having a component (p, q), say. Then, since $A \cap B = \emptyset$, denseness of B implies that $p \in A \cap B^-$ and $q \in A \cap B^+$; also $x_0 < p < q < x_1$, since $x_0 \in B^+$ and $x_1 \in B$. Take $a_1 = p$ and note that $[a_1, q] \subset A$. Clearly then, by denseness of A at q, there exist points $a_4 \in A \cap (q, x_1)$ and $a_3 \in B \cap (q, a_4)$ such that the triple (a_1, a_3, a_4) fulfils the required conditions.

Next suppose $A \cap (x_0, x_1)$ has void interior. Then each point of $A \cap (x_0, x_1)$ is a bilateral limit point of both A and B. Now, since $x_0 \in A$, by ω^*-denseness of A we have $\omega^*(A \cap (x_0, x_1)) > 0$. So by ω-density theorem, there is $a_1 \in A \cap (x_0, x_1)$ such that $d(A, a_1) = 1$. Clearly then we can find $a_4 \in A \cap (a_1, x_1)$ and $a_3 \in B \cap (a_1, a_4)$ such that the triple (a_1, a_3, a_4) fulfils the required conditions.

Thus we always have a triple (a_1, a_3, a_4) as desired. Let now $y_0 = \sup(B \cap (a_3, a_4))$. Then, since $a_3 \in B$ and $a_4 \in A$, arguing analogously as before we get that $a_3 < y_0 < a_4$ and $y_0 \in A \cap B^-$. Repeating the preceding arguments in this analogous fashion, with $A \cap (a_3, y_0)$ in place of $A \cap (x_0, x_1)$, we now find points $p_2, a_2 \in A \cap (a_3, y_0)$ and $p_1 \in B \cap (a_3, y_0)$, $p_1 < p_2 < a_2$, such that $d(A, x, a_2) > 1 - \epsilon$ for all $x \in (p_1, p_2)$. Then the tetrad (a_1, p_1, p_2, a_2) thus obtained evidently fulfils all the required conditions.

We note further that, by denseness of A and B, each of the open intervals (p_1, p_2) and (q_1, q_2) intersects both A and B, and, hence, the above process can be repeated with either of them in place of J.

Now, starting with the fact that I intersects both A and B, and applying (1) and (2) alternately, we can find a contracting sequence of intervals $(a_{11}, a_{21}) \supset [p_{11}, p_{21}] \supset (b_{11}, b_{21}) \supset [q_{11}, q_{21}] \supset (a_{12}, a_{22}) \supset [p_{12}, p_{22}] \supset (b_{12}, b_{22}) \supset [q_{12}, q_{22}] \supset \cdots$ such that, for each positive integer n, we have

$$b_{2n} - b_{1n} < a_{2n} - a_{1n} < \frac{1}{n}, \tag{3}$$

$$d(A, x, a_{jn}) > 1 - \frac{1}{n} \quad \text{if} \quad x \in (p_{1n}, p_{2n}), j = 1, 2, \tag{4}$$

and

$$d(B, x, b_{jn}) > 1 - \frac{1}{n} \quad \text{if} \quad x \in (q_{1n}, q_{2n}), j = 1, 2. \tag{5}$$

Evidently, there is a unique point c belonging to all the intervals (p_{1n}, p_{2n}) and (q_{1n}, q_{2n}), such that $\lim_{n\to\infty} a_{jn} = c = \lim_{n\to\infty} b_{jn}$ for $j = 1, 2$. Then for $x = c$, (4) and (5) both hold for all n. Hence $d_\infty(A, c) = 1 = d_\infty(B, c)$, which completes the proof.

DEFINITION 2.3. *Given* $f: I = [a, b] \to R$, *let* $E_r = \{y \in I \mid f(y) \leq r\}$ *and* $E^r = \{y \in I \mid f(y) \geq r\}$, $r \in R$. *For* $x \in (a, b)$, *we define*

$$L_1 f(x) = \sup\{r \mid d_\infty(E_r, x) < 1\}$$

and

$$L^1 f(x) = \inf\{r \mid d_\infty(E^r, x) < 1\}.$$

If $L^1 f(x) = L_1 f(x)$, *then this common equal value is called the* (1_ω)*-limit*, $L(1_\omega)f(x)$, *of* f *at* x. *If* $L^1 f(x) \le f(x) \le L_1 f(x)$, *then the function* f *is said to be* (1_ω)*-continuous at the point* x.

REMARK. The function f is said to be ω-approximately continuous at the point x, if $d(E_r, x) = 0 = d(E^s, x)$ whenever $r < f(x) < s$ ([9], Definition 2.2, p. 131). Clearly then the notion of (1_ω)-continuity is more general then ω-approximate continuity. For $\omega(x) = x$, the reader can easily verify that (1_ω)-continuity is even more general than proximal continuity ([10], Definition 4.1; cf. Corollary 3.1.1, p. 32). The (1_ω)-continuity of f at a point x does not necessarily imply the existence of $L(1_\omega)f(x)$.

DEFINITION 2.4. *Let* f: $[a, b] \to R$ *and* $x \in [a, b]$. *Defining the ordinary Dini* ω*-derivates* $D^+ f_\omega(x)$ *and etc. of* f *at* x *in the usual way, by considering the ordinary extreme unilateral limits of* $(f(y) - f(x))/(\omega(y) - \omega(x))$ *as* $y \to x$, *we further define*

$$D_\infty f_\omega(x) = \min\{D^+ f_\omega(x), D^- f_\omega(x)\}$$

and

$$D^\infty f_\omega(x) = \max\{D_+ f_\omega(x), D_- f_\omega(x)\},$$

ignoring the upper and lower ω*-derivatives on the left when* x *is the left end point* a, *and those on the right when* x *is the right end point* b.

If $D^\infty f_\omega(x) = D_\infty f_\omega(x)$, then this common equal value is called the (∞)-ω-derivative, $D(\infty)f_\omega(x)$, of f at x. If $D^\infty f_\omega(x) \le D_\infty f_\omega(x)$, then the function f is said to be (∞)-ω-derivable at the point x.

REMARK. The meaning of (∞)-ω-derivability is simply that the upper ω-derivate on either side is not less than the lower ω-derivate on the opposite side. From the obvious ω-analogs of the relations between Dini derivatives ([4], section 292, p. 392), it follows that every function is (∞)-ω-derivable n.e. (except for a countable set of points). The (∞)-ω-derivability of f at a point x does not necessarily imply the existence of $D(\infty)f_\omega(x)$.

DEFINITION 2.5. *Given* f: $[a, b] \to R$ *and* $x \in [a, b]$, *we define*

$$L_\infty f(x) = \min\{\overline{f(x_+)}, \overline{f(x_-)}\}$$

and

$$L^\infty f(x) = \max\{\underline{f(x_+)}, \underline{f(x_-)}\},$$

ignoring the upper and lower limits on the left when x *is the left end point* a, *and those on the right when* x *is the right end point* b.

If $L^\infty f(x) = L_\infty f(x)$, then this common equal value is called the (ω)-limit, $L(\infty)f(x)$, of f at x. If $L^\infty f(x) \le f(x) \le L_\infty f(x)$, then the function f is said to be (∞)-continuous at the point x.

REMARK. The meaning of (∞)-continuity is simply that f lies between its unilateral lower and upper limits on either side. Every Darboux function is obviously (∞)-continuous. By Young's theorem on the relations between a function and its extreme unilateral limits ([4], section 228, p. 304), every function is (∞)-continuous n.e. The (∞)-continuity of f at a point x does not necessarily imply the existence of $L(\infty)f(x)$.

Sargent [8] calls a function f: $[a, b] \to R$ continuous in the generalized sense, (CG), if $[a, b]$ is the union of a sequence of closed sets E_n, such that $f|E_n$ is continuous for each n. Generalizing this notion, we shall say that the function f is semi-continuous in the generalized sense, (SCG), if $[a, b]$ is the union of a sequence of closed sets E_n, such that $f|E_n$ is either lower semi-continuous or upper semi-continuous (depending on n) for each n. A point $c \in [a, b]$ will be called a point of semi-continuity of f, if there is a neighborhood I of c such that $f|I$ is either lower semi-continuous or upper semi-continuous.

The following lemma relating to the above notion of (SCG) will be crucial in our proofs of Darboux and mean value properties of derivatives.

LEMMA 2.2 *Let f: $[a, b] \to R$ be (∞)-continuous and (SCG) on $[a, b]$. Then, either f is strictly monotone and continuous on $[a, b]$, or f has a local extremum at some point of semi-continuity of f in (a, b).*

PROOF. Let E denote the set of points of $[a, b]$ having no neighborhood in $[a, b]$ on which f is strictly monotone. Obviously E is closed. Routine arguments show that f is strictly monotone on every component of $(a, b)\backslash E$, and then (∞)-continuity of f implies that f is strictly monotone and continuous on the closure of each such component. It follows at once that, if E has an isolated point, c say, then f is continuous in a neighborhood of c, $c \in (a, b)$ and f has a local extremum at c; but, if $E = \emptyset$, then f is strictly monotone and continuous on $[a, b]$.

Suppose, on the other hand, that E is nonempty and perfect. Then, since f is (SCG) on $[a, b]$, using Baire's category theorem we can find points $p, q \in E$, with $p < q$ and $E \cap (p, q) \neq \emptyset$, such that $f|E \cap [p, q]$ is semi-continuous (in a fixed sense). Since, as shown above, f is monotone and continuous on each closed interval contiguous to $E \cap [p, q]$ in $[p, q]$, it readily follows that in fact $f|[p, q]$ is semi-continuous. In particular, therefore, f is Baire 1 on $[p, q]$. Since, further, by hypothesis f is (∞)-continuous on $[p, q]$, it follows by a result of Sen ([11], Theorem III, p. 21) that f is Darboux on $[p, q]$. But, since $E \cap (p, q) \neq \emptyset$, f is not strictly monotone on $[p, q]$. Hence, recalling that a one-to-one Darboux function is necessarily strictly monotone ([1], Theorem 5.2, sqq., p. 101), we must have $f(r) = f(s)$ for some $[r, s] \subset [p, q]$. Then, if $f|[r, s]$ is continuous, f must have a local extremum at some point $c \in (r, s)$, and we are finished.

Suppose now that $f|[r, s]$ is not continuous. Then, recalling that f is (∞)-continuous, either there is a point $t \in (r, s]$ such that $\underline{f(t_-)} < \overline{f(t_-)}$, or there is a point $t \in [r, s)$ such that $\underline{f(t_+)} < \overline{f(t_+)}$. Since f is Darboux on $[r, s]$, it follows that in the first case

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we can find successively points $u, t_1, t_2, v \in (r, t)$, $u < t_1 < t_2 < v$, such that $\underline{f(t_-)} < f(u) < \overline{f(t_-)}$, $f(t_1) < f(u)$, $f(t_2) > f(u)$ and $f(v) = f(u)$. Similarly, in the second case we can find successively points $v, t_2, t_1, u \in (t, s)$, $v > t_2 > t_1 > u$, such that $\underline{f(t_+)} < f(v) < \overline{f(t_+)}$, $f(t_2) > f(v)$, $f(t_1) < f(v)$ and $f(u) = f(v)$. Thus, in any case we can find an interval $[u, v] \subset (r, s)$ such that

$$(*) \qquad f(t_1) < f(u) = f(v) < f(t_2) \qquad \text{for some } t_1, t_2 \in (u, v).$$

But $f | [u, v]$ being either lower semi-continuous or upper semi-continuous, it must have either a least value or a greatest value. Hence it follows from (*) that f has a local extremum at some point $c \in (u, v)$ which completes the proof.

3. **Main results**.

THEOREM 3.1. *Let f: $[a, b] \to R$ be (∞)-continuous on $[a, b]$ and (1_ω)-continuous on (a, b). Then f is Darboux on $[a, b]$.*

More critically, if $L^\infty f(p) < r < L_\infty f(q)$ for some $r \in R$ and $p, q \in [a, b]$ (possibly $p = q$), then, for every open interval $I \subset (a, b)$ with $p, q \in \bar{I}$, there is at least one point $c \in I$ such that, either (i) $L(\infty)f(c) = f(c) = r$ or (ii) $L(1_\omega)f(c) = f(c) = r$.

PROOF. Suppose (i) is false for all $c \in I$. Then, setting

$$A_0 = \{x \in I | f(x) < r\},\ B_0 = \{x \in I | f(x) > r\},$$

$$A = A_0 \cup \{x \in I | L^\infty f(x) < f(x) = r \leq L_\infty f(x)\}$$

and

$$B = B_0 \cup \{x \in I | L^\infty f(x) = f(x) = r < L_\infty f(x)\},$$

we have $A \cup B = I$, $A \cap B = \emptyset$ and, by (∞)-continuity of f at p and q, both A and B are nonempty.

Again, since $L^\infty f < r$ n A, each point of A is a limit point of A_0 on both sides; also, since $L^1 f \leq f < r$ on A_0, for every $x \in A_0$ we have $d_\infty(B, x) < 1$. These imply that A is bilaterally ω^*-dense in itself. Similarly, B is bilaterally ω^*-dense in itself.

Hence, by Lemma 2.1, there is a point $c \in I$ such that $d_\infty(A, c) = 1 = d_\infty(B, c)$. Clearly then $L_1 f(c) \leq r \leq L^1 f(c)$, which in conjunction with the hypothesis $L^1 f(c) \leq f(c) \leq L_1 f(c)$ gives (ii), completing the proof.

THEOREM 3.2. *Suppose $\omega | [a, b]$ is continuous, and f: $[a, b] \to R$ is (∞)-continuous and (SCG) on $[a, b]$ and is (∞)-ω-derivable at the points of semi-continuity of f in (a, b). Then we have:*

(i) *if $D^\infty f_\omega(p) < r < D_\infty f_\omega(q)$ for some $r \in R$ and $p, q \in [a, b]$ (possibly $p = q$), then, for every open interval $I \subset (a, b)$ with $p, q \in \bar{I}$, f has a point of semi-continuity $c \in I$ such that $D(\infty)f_\omega(c)$ exists and equals r;*

(ii) *if E is the cocountable subset of $[a, b]$ where f is (∞)-ω-derivable, then every extended real valued function u, satisfying $D^\infty f_\omega(x) \leq u(x) \leq D_\infty f_\omega(x)$ for all $x \in E$, is Darboux on E; in particular, the functions $D^\infty f_\omega$* and *$D_\infty f_\omega$ are Darboux on E;*

(iii) *the function* $D(\infty)f_\omega$ *is Darboux on the subset* E_0 *of* $[a, b]$ *where it exists, and also n the subset of* E_0 *consisting of the points of semi-continuity of* f.

PROOF. Clearly, (i) implies both (ii) and (iii).

Now, to prove (i), we define

$$g(x) = f(x) - r \cdot \omega(x) \qquad \text{for all } x \in [a, b].$$

Then we have

$$D^\infty g_\omega(p) = D^\infty f_\omega(p) - r < 0$$

and

$$D_\infty g_\omega(q) = D_\infty f_\omega(q) - r > 0,$$

which together imply that g is not monotone on $\bar{I}$. But, g is evidently (∞)-continuous and (SCG) on $\bar{I}$. Hence, by Lemma 2.2, there is a $c \in I$, a point of semi-continuity of g (and, hence, also of f), such that $g(c)$ is a local extremum of g, wherefore $D^\infty g_\omega(c) \geq 0$ and $D_\infty g_\omega(c) \leq 0$. Then, since f is (∞)-ω-derivable at c, we have

$$0 \leq D^\infty g_\omega(c) = D^\infty f_\omega(c) - r \leq D_\infty f_\omega(c) - r = D_\infty g_\omega(c) \leq 0,$$

whence $D(\infty)f_\omega(c) = r$. This completes the proof of (i).

THEOREM 3.3. *Under the hypotheses of Theorem 3.2, f has a point of semi-continuity* $c \in (a, b)$, *such that* $D(\infty)f_\omega(c)$ *exists and equals the mean value*

$$r = (f(b) - f(a))/(\omega(b) - \omega(a)).$$

PROOF. Let us define

$$g(x) = f(x) - r \cdot \omega(x) \qquad \text{for all } x \in [a, b].$$

Evidently, g is (∞)-continuous and (SCG) on $[a, b]$ and, further, $g(a) = g(b)$. So, by Lemma 2.2, there is a $c \in (a, b)$, a point of semi-continuity of g (and, hence, also of f), such that $g(c)$ is a local extremum of g. Then the proof ends by arguing exactly as in the last part of the proof of the preceding theorem.

CONCLUDING REMARKS. As simple consequences of the various definitions, continuity in the ordinary or approximate or proximal or Cesaro (C_λ) [8] or I-approximate ([14], Definition 3, p. 248) sense implies (∞)-continuity; also, a function having a finite selective derivative ([7], p. 77) is necessarily (∞)-continuous.

Kulbacka [6] showed that an approximate derivative (finite or infinite) of an approximately continuous function f is Darboux. Here, by a result of Tolstoff ([13], pp. 499–500), then function f is necessarily (CG). A proximally continuous function having proximal derivative can also be shown to be (CG). Again, Sargent [8] proved the mean value property for C_λ-derivatives of C_λ-continuous functions f, by first showing that such an f is necessarily (CG). Also, O'Malley ([7], Theorem 11, p. 87) showed that a function having a finite selective derivative is (CG), and the derivative has the Darboux property, suggesting also the mean value property (p. 88, supra). As reported by Wilczynski ([14], Theorems 35, 41, pp. 260, 261), a function having a finite I-approximate derivative is also (CG), and the derivative has the mean value property.

Thus, the hypotheses of (∞)-continuity and (SCG) in our Theorem 3.2, 3.3 are quite general.

The extent of Theorems 3.2, 3.3 can be surmised further from the facts that, if the function f has an ordinary or approximate or proximal or C_λ or selective or I-approximate derivative at a point c, then, for $\omega(x) = x$, f is trivially (∞)-ω-derivable at c and, further, this derivative of f equals $D(\infty)f_\omega(c)$, whenever the latter exists.

Finally, Bruckner and Cedar ([1], section 3, p. 96, infra) asked if the hypothesis of approximate continuity in the result of Kulbacka mentioned above can be relaxed. Theorem 3.2 certainly gives a respectable affirmative answer to this query.

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