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ABSTRACT

Fix $\alpha > 0$. Then by Fejér's theorem $(\alpha(\log n)^A \bmod 1)_{n \geq 1}$ is uniformly distributed if and only if $A > 1$. We sharpen this by showing that all correlation functions, and hence the gap distribution, are Poissonian provided $A > 1$. This is the first example of a deterministic sequence modulo 1 whose gap distribution and all of whose correlations are proven to be Poissonian. The range of A is optimal and complements a result of Marklof and Strömbergsson who found the limiting gap distribution of $(\log(n) \bmod 1)$, which is necessarily not Poissonian.

1. Introduction

A sequence $(x(n))_{n \geq 1} \subseteq [0, 1)$ is *uniformly distributed modulo 1* if the proportion of points in any subinterval $I \subseteq [0, 1)$ converges to the size of the interval: $\#\{n \leq N : x(n) \in I\} \sim N|I|$, as $N \rightarrow \infty$. The theory of uniform distribution dates back to 1916, to a seminal paper of Weyl [Wey16], and constitutes a simple test of pseudo-randomness. A well-known result of Fejér (see [KN74, p. 13]) implies that for any $A > 1$ and any $\alpha > 0$ the sequence

$$(\alpha(\log n)^A \bmod 1)_{n > 0}$$

is uniformly distributed, while for $A = 1$ the sequence is *not* uniformly distributed. In this paper, we study stronger, local tests for pseudo-randomness for this sequence.

Given an increasing \mathbb{R} -valued sequence, $(\omega(n)) = (\omega(n))_{n > 0}$, we denote the sequence modulo 1 by

$$x(n) := \omega(n) \bmod 1.$$

Furthermore, let $u_N(n) \subset [0, 1)$ denote the ordered sequence, thus $u_N(1) \leq u_N(2) \leq \dots \leq u_N(N)$. With that, we define the *gap distribution* of $(x(n))$ as the limiting distribution (if it exists): for $s > 0$,

$$P(s) := \lim_{N \rightarrow \infty} \frac{\#\{n \leq N : N\|u_N(n) - u_N(n+1)\| < s\}}{N},$$

where $\|\cdot\|$ denotes the distance to the nearest integer, and $u_N(N+1) = u_N(1)$. Thus, $P(s)$ represents the limiting proportion of (scaled) gaps between (spatially) neighboring elements in the sequence which are less than s . We say that a sequence has *Poissonian gap distribution* if $P(s) = 1 - e^{-s}$, the expected value for a uniformly distributed sequence on the unit interval.

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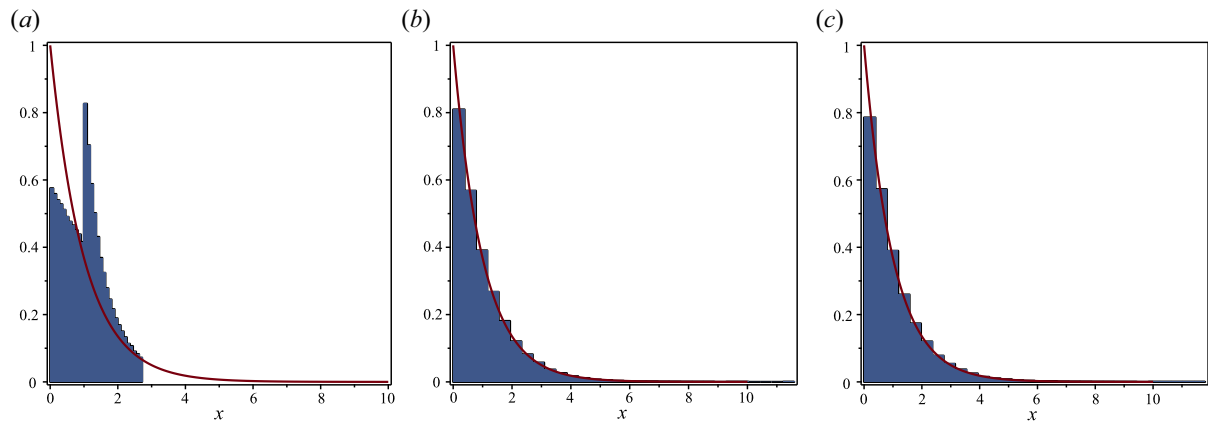


FIGURE 1. From left to right, the histograms represent the gap distribution *density* at time N of $(\log n)_{n \geq 1}$, $((\log n)^2)_{n > 0}$, and $((\log n)^3)_{n > 0}$ when $N = 10^5$ and the curve is the graph of $x \mapsto e^{-x}$. Note that $(\log n)$ is not uniformly distributed, and the gap distribution is not Poissonian.

Our main theorem is the following result.

THEOREM 1.1. *Let $\omega(n) := \alpha(\log n)^A$ for $A > 1$ and any $\alpha > 0$. Then $x(n)$ has Poissonian gap distribution.*

In fact, this theorem follows (via the method of moments) from Theorem 1.2 (below) which states that for every $m \geq 2$ the m -point correlation function of this sequence is Poissonian. By that we mean the following. Let $m \geq 2$ be an integer, and let $f \in C_c^\infty(\mathbb{R}^{m-1})$ be a compactly supported function which can be thought of as a stand-in for the characteristic function of a Cartesian product of compact intervals in \mathbb{R}^{m-1} . Let $[N] := \{1, \dots, N\}$ and define the m -point correlation of $(x(n))$, at time N , to be

$$R^{(m)}(N, f) := \frac{1}{N} \sum_{\mathbf{n} \in [N]^m}^* f(N\|x(n_1) - x(n_2)\|, N\|x(n_2) - x(n_3)\|, \dots, N\|x(n_{m-1}) - x(n_m)\|), \quad (1.1)$$

where \sum^* denotes a sum over distinct m -tuples. Thus, the m -point correlation measures how correlated points are on the scale of the average gap between neighboring points (which is N^{-1}). We say that $(x(n))$ has *Poissonian m -point correlation* if

$$\lim_{N \rightarrow \infty} R^{(m)}(N, f) = \int_{\mathbb{R}^{m-1}} f(\mathbf{x}) d\mathbf{x} =: \mathbf{E}(f) \quad \text{for any } f \in C_c^\infty(\mathbb{R}^{m-1}). \quad (1.2)$$

That is, if the m -point correlation converges to the expected value if the sequence was uniformly distributed on the unit interval.

THEOREM 1.2. *Let $\omega(n) := \alpha(\log n)^A$ for $A > 1$ and any $\alpha > 0$. Then $x(n)$ has Poissonian m -level correlations for all $m \geq 2$.*

It should be noted that Theorem 1.2 is far stronger than Theorem 1.1. In addition to the gap distribution, Theorem 1.2 allows us to recover a wide variety of statistics such as the i th nearest neighbor distribution for any $i \geq 1$.

For our analysis, it is more useful to have good control on the support of the Fourier transform of f , and not crucial that f be compactly supported. Thus, we restate Theorem 1.2 here for this class of f . We prove that Theorem 1.2 follows from Theorem 1.3 in § 8.

THEOREM 1.3. Let $\omega(n) := \alpha(\log n)^A$ for $A > 1$ and any $\alpha > 0$. Then, for any $m \geq 2$, we have

$$\lim_{N \rightarrow \infty} R^{(m)}(N, f) = \mathbf{E}(f)$$

for any positive function $f \in C^\infty(\mathbb{R}^{m-1})$, whose Fourier transform \widehat{f} has compact support.

Here we use \widehat{f} to denote the Fourier transform of f . To avoid carrying a constant through we assume the support of \widehat{f} is contained in $(-1, 1)$.

Previous work

The study of the uniform distribution and fine-scale local statistics of sequences modulo 1 has a long history which we outlined in more detail in a previous paper [LST24]. If we consider the sequence $(\alpha n^\theta \bmod 1)_{n \geq 1}$, there have been many attempts to understand the local statistics, in particular the pair correlation (when $m = 2$). It is known that for any $\theta \neq 1$, if α belongs to a set of full measure, then the pair correlation function is Poissonian [RS98, AEM21, RT22]. However, there are very few explicit (i.e. non-metric) results. When $\theta = 2$ Heath-Brown [Hea10] gave an algorithmic construction of certain α for which the pair correlation is Poissonian; however, this construction did not give an exact number. When $\theta = 1/2$ and $\alpha^2 \in \mathbb{Q}$ the problem lends itself to tools from homogeneous dynamics. This was exploited by Elkies and McMullen [EM04] who showed that the gap distribution is *not* Poissonian, and by El-Baz, Marklof, Vinogradov [EMV15] who showed that the sequence $(\alpha n^{1/2} \bmod 1)_{n \in \mathbb{N} \setminus \square}$, where \square denotes the set of squares, does have Poissonian pair correlation.

With these sparse exceptions, the only explicit results occur when the exponent θ is small. If $\theta \leq 14/41$ the authors and Sourmelidis [LST24] showed that the pair correlation function is Poissonian for all values of $\alpha > 0$. This was later extended by the authors [LT21] to show that these monomial sequences exhibit Poissonian m -point correlations (for $m \geq 3$) for any $\alpha > 0$ if $\theta < 1/(m^2 + m - 1)$. To the best of our knowledge the former is the only explicit result proving Poissonian pair correlations for sequences modulo 1, and the latter result is the only result proving convergence of the higher-order correlations to any limit.

The authors' previous work motivates the natural question: what about sequences which grow slower than any power of n ? It is natural to hypothesize that such sequences might exhibit Poissonian m -point correlations for all m . However, there is a constraint. Marklof and Strömbergsson [MS13] have shown that the gap distribution of $((\log n)/(\log b) \bmod 1)_{n \geq 1}$ exists for $b > 1$, and is *not* Poissonian (thus the correlations cannot all be Poissonian). However, they also showed that in the limit as b tends to 1, this limiting distribution converges to the Poissonian distribution (see [MS13, (74)]). Thus, the natural question becomes: what can be said about sequences growing faster than $\log(n)$ but slower than any power of n ?

With that context in mind, our result has several implications. First, it provides the only example at present of an explicit sequence whose m -point correlations can be shown to converge to the Poissonian limit (and thus whose gap distribution is Poissonian). Second, it answers the natural question implied by our previous work on monomial sequences. Finally, it answers the natural question implied by Marklof and Strömbergsson's result on logarithmic sequences.

1.1 Plan of the paper

The proof of Theorem 1.2 adds several new ideas to the method developed in [LT21], which is insufficient for the definitive results established here. Broadly, we argue in three steps, detailing the difficulties and innovations in each step.

In the remainder of this work we take $\alpha = 1$; exactly the same proof applies to general α , leaving straightforward adaptations aside. Fix $m \geq 2$ and assume the sequence has Poissonian j -point correlation for $2 \leq j < m$.

Step 1. Remove the distinctness condition in the m -point correlation by relating the completed correlation to the m th moment of a random variable. This will add a new frequency variable, with the benefit of decorrelating the sequence elements. Then we perform a Fourier decomposition of this moment and, using a combinatorial argument from [LT21, §3], we reduce the problem of convergence for the moment to convergence of one particular term to an explicit ‘target’. This step works quite similarly to what we did in [LT21].

Step 2. Using various partitions of unity, we further reduce the problem to an asymptotic evaluation of the $L^m([0, 1])$ -norm of a two-dimensional exponential sum. Then we apply van der Corput’s B -process to each of these variables. In contrast to our argument in [LT21], we can no longer use the form of the sequence to perform explicit computations throughout. Instead, a more fundamental understanding of how the two B -processes work is now required. In fact, after the first application of the B -process we end up with an implicitly defined phase function. Surprisingly, after the second application of the B -process (in the other variable) we can show that a manageable phase function arises! This is the content of Lemma 4.10, and we believe this by-product of our investigation to be of some independent interest. Being able to understand the phase function arising is crucial to performing the next step. Furthermore, a simple computation yields that if we stop at this step and apply the triangle inequality the resulting error term is of size $O((\log N)^{(A+1)m})$.

Step 3. Finally, we expand the $L^m([0, 1])$ -norm giving an oscillatory integral. Then, using a localized version of Van der Corput’s lemma, we achieve an extra saving to bound the error term by $o(1)$. In [LT21] we used classical theorems from linear algebra to justify that that localized version of Van der Corput’s lemma is applicable, by showing that Wronskians of a family of relevant curves are uniformly bounded from below. In the present situation, the underlying geometry and Wronskians are considerably more involved. After several initial manipulations we boil matters down to determinants of generalized Vandermonde matrices. To handle those we rely on recent work of Khare and Tao [KT21], which is precise enough to (by a small logarithmic gain) single out the largest contribution to the Wronskian and thereby complete the argument.

Notation

Throughout, we use the usual Bachmann–Landau notation: for functions $f, g : X \rightarrow \mathbb{R}$, defined on some set X , we write $f \ll g$ (or $f = O(g)$) to denote that there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in X$. Moreover, let $f \asymp g$ denote $f \ll g$ and $g \ll f$, and let $f = o(g)$ denote that $f(x)/g(x) \rightarrow 0$.

Given a Schwartz function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, let \widehat{f} denote the m -dimensional Fourier transform:

$$\widehat{f}(\mathbf{k}) := \int_{\mathbb{R}^m} f(\mathbf{x}) e(-\mathbf{x} \cdot \mathbf{k}) d\mathbf{x}, \quad \text{for } \mathbf{k} \in \mathbb{Z}^m.$$

Here, and throughout we let $e(x) := e^{2\pi i x}$. Moreover, we use the convention $0/0 = 0$ to avoid extra conditions on summations.

All of the sums which appear range over integers, in the indicated interval. We will frequently be taking sums over multiple variables, thus if \mathbf{u} is an m -dimensional vector, for brevity, we write

$$\sum_{\mathbf{k} \in [f(\mathbf{u}), g(\mathbf{u})]} F(\mathbf{k}) = \sum_{k_1 \in [f(u_1), g(u_1))} \cdots \sum_{k_m \in [f(u_m), g(u_m))} F(\mathbf{k}).$$

Moreover, all L^p norms are taken with respect to Lebesgue measure; we often do not include the domain when it is obvious. Let

$$\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}.$$

For ease of notation, $\varepsilon > 0$ may vary from line to line by a bounded constant.

2. Preliminaries

The following stationary phase principle is derived from the work of Blomer, Khan and Young [BKY13, Proposition 8.2]. In application we will not make use of the full asymptotic expansion, but this will give us a particularly good error term which is essential to our argument.

PROPOSITION 2.1 (Stationary phase expansion). *Let $w \in C_c^\infty(\mathbb{R})$ be supported in a compact interval J of length $\Omega_w > 0$ so that there exists an $\Lambda_w > 0$ for which*

$$w^{(j)}(x) \ll_j \Lambda_w \Omega_w^{-j}$$

for all $j \in \mathbb{N}$. Suppose ψ is a smooth function on J so that there exists a unique critical point x_0 with $\psi'(x_0) = 0$. Suppose there exist values $\Lambda_\psi > 0$ and $\Omega_\psi > 0$ such that

$$\psi''(x) \gg \Lambda_\psi \Omega_\psi^{-2}, \quad \psi^{(j)}(x) \ll_j \Lambda_\psi \Omega_\psi^{-j}$$

for all $j > 2$. Moreover, let $\delta \in (0, 1/10)$, and $Z := \Omega_w + \Omega_\psi + \Lambda_w + \Lambda_\psi + 1$. If

$$\Lambda_\psi \geq Z^{3\delta} \quad \text{and} \quad \Omega_w \geq \frac{\Omega_\psi Z^{\delta/2}}{\Lambda_\psi^{1/2}} \tag{2.1}$$

hold, then

$$I := \int_{-\infty}^{\infty} w(x) e(\psi(x)) dx$$

has the asymptotic expansion

$$I = \frac{e(\psi(x_0))}{\sqrt{\psi''(x_0)}} \sum_{0 \leq j \leq 3C/\delta} p_j(x_0) + O_{C,\delta}(Z^{-C})$$

for any fixed $C \in \mathbb{Z}_{\geq 1}$; here

$$p_n(x_0) := \frac{e(1/8)}{n!} \left(\frac{i}{2\psi''(x_0)} \right)^n G^{(2n)}(x_0),$$

where

$$G(x) := w(x) e(H(x)), \quad H(x) := \psi(x) - \psi(x_0) - \frac{1}{2} \psi''(x_0)(x - x_0)^2.$$

In a slightly simpler form we have the following lemma.

LEMMA 2.2 (First-order stationary phase). *Let ψ and w be smooth, real-valued functions defined on a compact interval $[a, b]$. Let $w(a) = w(b) = 0$. Suppose there exist constants $\Lambda_\psi, \Omega_w, \Omega_\psi \geq 3$*

satisfying (2.1), with Z as in Proposition 2.1 and $\Lambda_w = 1$ so that

$$\psi^{(j)}(x) \ll \frac{\Lambda_\psi}{\Omega_\psi^j}, \quad w^{(j)}(x) \ll \frac{1}{\Omega_w^j} \quad \text{and} \quad \psi^{(2)}(x) \gg \frac{\Lambda_\psi}{\Omega_\psi^2} \quad (2.2)$$

for all $j = 0, 1, 2, \dots$ and all $x \in [a, b]$. If $\psi'(x_0) = 0$ for a unique $x_0 \in [a, b]$, and if $\psi^{(2)}(x) > 0$, then

$$\int_a^b w(x) e(\psi(x)) dx = \frac{e(\psi(x_0) + 1/8)}{\sqrt{|\psi^{(2)}(x_0)|}} w(x_0) + O\left(\frac{\Omega_\psi^3}{\Lambda_\psi^{3/2} \Omega_w^2} + \frac{1}{Z}\right).$$

If instead $\psi^{(2)}(x) < 0$ on $[a, b]$ then the same equation holds with $e(\psi(x_0) + 1/8)$ replaced by $e(\psi(x_0) - 1/8)$.

Proof. We apply Proposition 2.1 with $\Lambda_w = 1$ and $C = 1$. In which case the first error term comes from the term p_1 in the expansion. All higher-order terms give a smaller contribution; see [BKY13, p. 20] for a more detailed explanation. \square

Moreover, we also need the following version of van der Corput's lemma. ([Ste93, Chapter VIII, Proposition 2]).

LEMMA 2.3 (van der Corput's lemma). *Let $[a, b]$ be a compact interval. Let $\psi, w : [a, b] \rightarrow \mathbb{R}$ be smooth functions. Assume ψ'' does not change sign on $[a, b]$ and that for some $j \geq 1$ and $\Lambda > 0$ the bound*

$$|\psi^{(j)}(x)| \geq \Lambda$$

holds for all $x \in [a, b]$. Then

$$\int_a^b e(\psi(x)) w(x) dx \ll \left(|w(b)| + \int_a^b |w'(x)| dx \right) \Lambda^{-1/j}$$

where the implied constant depends only on j .

Generalized Vandermonde matrices

One of the primary difficulties which we will encounter in § 6 is the fact that taking derivatives of exponentials (which arise as the inverse of the logs defining our sequence) increases in complexity as we take derivatives. To handle this we appeal to a recent result of Khare and Tao [KT21] which requires us to set up some notation. Given an M -tuple $\mathbf{u} \in \mathbb{R}^M$, let

$$V(\mathbf{u}) := \prod_{1 \leq i < j \leq M} (u_j - u_i)$$

denote the Vandermonde determinant. Furthermore, given two tuples \mathbf{u} and \mathbf{n} , we define

$$\mathbf{u}^{\mathbf{n}} := u_1^{n_1} \cdots u_M^{n_M} \quad \text{and} \quad \mathbf{u}^{\mathbf{cn}} := \begin{pmatrix} u_1^{n_1} & u_1^{n_2} & \cdots & u_1^{n_M} \\ u_2^{n_1} & u_2^{n_2} & \cdots & u_2^{n_M} \\ \vdots & \vdots & \ddots & \vdots \\ u_M^{n_1} & u_M^{n_2} & \cdots & u_M^{n_M} \end{pmatrix},$$

the latter being a generalized Vandermonde matrix. Finally, let $\mathbf{n}_{\min} := (0, 1, \dots, M-1)$. Then Khare and Tao established the following lemma.

LEMMA 2.4 [KT21, Lemma 5.3]. Let K be a compact subset of the cone

$$\{(n_1, \dots, n_M) \in \mathbb{R}^M : n_1 < \dots < n_M\}.$$

Then there exist constants $C, c > 0$ such that

$$cV(\mathbf{u})\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}} \leq \det(\mathbf{u}^{\circ \mathbf{n}}) \leq CV(\mathbf{u})\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}} \quad (2.3)$$

for all $\mathbf{u} \in (0, \infty)^M$ with $u_1 \leq \dots \leq u_M$ and all $\mathbf{n} \in K$.

3. Combinatorial completion

The proof of Theorem 1.3 follows an inductive argument. Thus, fix $m \geq 2$ and assume $(x(n))$ has j -point correlations for all $j < m$. Let f be a positive $C^\infty(\mathbb{R})$ function with $\hat{f} \in C_c^\infty(\mathbb{R})$, and define

$$S_N(s, f) = S_N := \sum_{n \in [N]} \sum_{k \in \mathbb{Z}} f(N(\omega(n) + k + s)).$$

Note that if f was the indicator function of an interval I , then S_N would count the number of points in $(x_n)_{n \leq N}$ which land in the shifted interval $I/N + s/N$. Now consider the m th moment of S_N . Then one can show that (see [LT21, § 3])

$$\begin{aligned} \mathcal{M}^{(m)}(N) &:= \int_0^1 S_N(s, f)^m ds \\ &= \int_0^1 \sum_{\mathbf{n} \in [N]^m} \sum_{\mathbf{k} \in \mathbb{Z}^m} f(N(\omega(n_1) + k_1 + s)) \cdots f(N(\omega(n_m) + k_m + s)) ds \\ &= \frac{1}{N} \sum_{\mathbf{n} \in [N]^m} \sum_{\mathbf{k} \in \mathbb{Z}^{m-1}} F(N(\omega(n_1) - \omega(n_2) + k_1), \dots, N(\omega(n_{m-1}) - \omega(n_m) + k_{m-1})), \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} &F(z_1, z_2, \dots, z_{m-1}) \\ &:= \int_{\mathbb{R}} f(s)f(z_1 + z_2 + \dots + z_{m-1} + s)f(z_2 + \dots + z_{m-1} + s) \cdots f(z_{m-1} + s) ds. \end{aligned}$$

As such we can relate the m th moment of S_N to the m -point correlation of F . Note that since \hat{f} has compact support, \hat{F} also has compact support. To recover the m -point correlation in full generality, we replace the moment $\int S_N(s, f)^m ds$ with the mixed moment $\int \prod_{i=1}^m S_N(s, f_i) ds$ for some collection of $f_i : \mathbb{R} \rightarrow \mathbb{R}$. The proof below can be applied in this generality; however, for ease of notation we only explain the details in the former case.

3.1 Combinatorial target

We will need the following combinatorial definitions to explain how to prove convergence of the m -point correlation from (3.1). Given a partition \mathcal{P} of $[m]$, we say that $j \in [m]$ is *isolated* if j belongs to a partition element of size 1. A partition is called *non-isolating* if no element is isolated (and otherwise we say it is *isolating*). For our example $\mathcal{P} = \{\{1, 3\}, \{4\}, \{2, 5, 6\}\}$ we have that 4 is isolated, and thus \mathcal{P} is isolating.

Now consider the middle line of (3.1). We apply Poisson summation to each of the k_i sums. That is, we insert

$$\sum_{k \in \mathbb{Z}} f(N(\omega(n) + k + s)) = \frac{1}{N} \sum_{k \in \mathbb{Z}} e(k(\omega(n) + s)) \hat{f}\left(\frac{k}{N}\right) \quad (3.2)$$

yielding

$$\mathcal{M}^{(m)}(N) = \frac{1}{N^m} \int_0^1 \sum_{\mathbf{n} \in [N]^m} \sum_{\mathbf{k} \in \mathbb{Z}^m} \hat{f}\left(\frac{\mathbf{k}}{N}\right) e(\mathbf{k} \cdot \omega(\mathbf{n}) + \mathbf{k} \cdot \mathbf{1}s) ds, \quad (3.3)$$

where $\omega(\mathbf{n}) := (\omega(n_1), \omega(n_2), \dots, \omega(n_m))$ and where $\hat{f}(\mathbf{k}/N) = \prod_{i=1}^m \hat{f}(k_i/N)$.

In [LT21, § 3] we showed that, if

$$\mathcal{E}(N) := \frac{1}{N^m} \int_0^1 \sum_{\mathbf{n} \in [N]^m} \sum_{\mathbf{k} \in (\mathbb{Z}^*)^m} \hat{f}\left(\frac{\mathbf{k}}{N}\right) e(\mathbf{k} \cdot \omega(\mathbf{n}) + \mathbf{k} \cdot \mathbf{1}s) ds,$$

then for fixed m , and assuming the inductive hypothesis, Theorem 1.3 reduces to the following lemma.

LEMMA 3.1. *Let \mathcal{P}_m denote the set of non-isolating partitions of $[m]$. We have that*

$$\lim_{N \rightarrow \infty} \mathcal{E}(N) = \sum_{\mathcal{P} \in \mathcal{P}_m} \mathbf{E}(f^{|P_1|}) \dots \mathbf{E}(f^{|P_d|}). \quad (3.4)$$

where the partition $\mathcal{P} = (P_1, P_2, \dots, P_d)$, and $|P_i|$ is the size of P_i .

3.2 Dyadic decomposition

It is convenient to decompose the sums over n and k within $S_N(s, f)$ into (nearly) dyadic ranges in a smooth manner. Given N , we let $Q > 1$ be the unique integer with $e^Q \leq N < e^{Q+1}$. Now, we describe a smooth partition of unity which approximates the indicator function of $[1, N]$. Strictly speaking, these partitions depend on Q , but we suppress this from the notation. Furthermore, since we want asymptotics of $\mathcal{E}(N)$, we need to take a bit of care at the right end point of $[1, N]$, and so a tighter than dyadic decomposition is needed. Let us make this precise, and point out that a detailed construction can be found in the Appendix. For $0 \leq q < Q$ we let $\mathfrak{N}_q : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function for which

$$\text{supp}(\mathfrak{N}_q) \subset [e^q/2, 3e^q], \quad \text{for } 0 \leq q < Q,$$

and such that $\mathfrak{N}_q(x) + \mathfrak{N}_{q+1}(x) = 1$ for $x \in [e^q, e^{q+1})$. Now for $q \geq Q$ we let \mathfrak{N}_q form a smooth partition of unity for which

$$\sum_{q=0}^{2Q-1} \mathfrak{N}_q(x) = \begin{cases} 1 & \text{if } 1 < x < e^Q, \\ 0 & \text{if } x < 1/2 \text{ or } x > N + \frac{3N}{\log(N)} \end{cases} \quad (3.5)$$

and

$$\text{supp}(\mathfrak{N}_q) \subset \left[e^Q + (q - Q - 1.1) \frac{e^Q}{Q}, e^Q + (3 + q - Q) \frac{e^Q}{Q} \right] \quad \text{for } Q < q \leq 2Q - 1, \quad (3.6)$$

while $\text{supp}(\mathfrak{N}_Q) \subset (0.9 \cdot e^{Q-1}, 1.1 \cdot e^Q)$. Let $\|g\|_\infty$ denote the sup norm of a function $g : \mathbb{R} \rightarrow \mathbb{R}$. We impose the following condition on the derivatives:

$$\|\mathfrak{N}_q^{(j)}\|_\infty \ll \begin{cases} e^{-qj} & \text{for } q < Q, \\ (e^Q/Q)^{-j} & \text{for } Q < q, \end{cases} \quad (3.7)$$

for $j \geq 1$. For technical reasons, assume $\mathfrak{N}_q^{(1)}$ changes sign only once.

Notice that (3.5) implies

$$\sum_{n \in \mathbb{Z}} \sum_{q=0}^{2Q-2} \mathfrak{N}_q(n) \sum_{k \in \mathbb{Z}} f(N(\omega(n) + k + s)) \leq S_N(s, f) \leq \sum_{n \in \mathbb{Z}} \sum_{q=0}^{2Q-1} \mathfrak{N}_q(n) \sum_{k \in \mathbb{Z}} f(N(\omega(n) + k + s)).$$

Ignoring the lower bound, which can be treated similarly, applying Poisson summation for every fixed n and using the positivity of the inner sum, we then have

$$S_N(s, f) \leq \frac{1}{N} \sum_{n \in \mathbb{Z}} \sum_{q=0}^{2Q-1} \mathfrak{N}_q(n) \sum_{k \in \mathbb{Z}} \hat{f}\left(\frac{k}{N}\right) e(k(\omega(n) + s)).$$

Next, by positivity, we have that

$$\mathcal{M}^{(m)}(N) \leq \int_0^1 \left(\frac{1}{N} \sum_{q=0}^{2Q-1} \sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) \sum_{k \in \mathbb{Z}} \hat{f}\left(\frac{k}{N}\right) e(k\omega(n) + ks) \right)^m ds. \quad (3.8)$$

All frequencies \mathbf{k} for which $k_j = 0$ for at least some index $1 \leq j \leq m-1$ contribute to $\mathcal{M}^{(m)}(N)$ exactly

$$\sum_{i=1}^{m-1} \binom{m-1}{i} \hat{f}(0)^i \int_0^1 \left(\frac{1}{N} \sum_{n \in [N]} \sum_{k \neq 0} \hat{f}\left(\frac{k}{N}\right) e(k\omega(n) + ks) \right)^{m-i} ds.$$

Subtracting exactly the above term from both sides of (3.8), while using our inductive assumption that the $\mathcal{M}^{m-i}(N)$ converge (for $1 \leq i \leq m-2$), then yields

$$\mathcal{E}(N) \leq \int_0^1 \left(\frac{1}{N} \sum_{q=0}^{2Q-1} \sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) \sum_{k \neq 0} \hat{f}\left(\frac{k}{N}\right) e(k\omega(n) + ks) \right)^m ds + o(1). \quad (3.9)$$

The same argument can be used to yield

$$\mathcal{E}(N) + o(1) \geq \int_0^1 \left(\frac{1}{N} \sum_{q=0}^{2Q-2} \sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) \sum_{k \neq 0} \hat{f}\left(\frac{k}{N}\right) e(k\omega(n) + ks) \right)^m ds.$$

We similarly decompose the k sums, although thanks to the compact support of \hat{f} we do not need to worry about $k \geq N$. Let $\mathfrak{K}_u : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that, for $U := \lceil \log N \rceil$,

$$\sum_{u=-U}^U \mathfrak{K}_u(k) = \begin{cases} 1 & \text{if } |k| \in [1, N), \\ 0 & \text{if } |k| < 1/2 \text{ or } |k| > 2N, \end{cases}$$

and the symmetry $\mathfrak{K}_{-u}(k) = \mathfrak{K}_u(-k)$ holds true for all $u, k > 0$. Similarly,

$$\text{supp}(\mathfrak{K}_u) \subset [e^u/3, 3e^u] \quad \text{if } u \geq 0, \quad \text{and}$$

$$\|\mathfrak{K}_u^{(j)}\|_\infty \ll e^{-|u|j}, \quad \text{for all } j \geq 1.$$

As for \mathfrak{N}_q , we also assume $\mathfrak{N}_u^{(1)}$ changes sign exactly once.

Therefore, a central role is played by the smoothed exponential sums

$$\mathcal{E}_{q,u}(s) := \frac{1}{N} \sum_{k \in \mathbb{Z}} \mathfrak{K}_u(k) \hat{f}\left(\frac{k}{N}\right) e(ks) \sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) e(k\omega(n)). \quad (3.10)$$

Notice that (3.9) and the compact support of \widehat{f} imply

$$\mathcal{E}(N) \ll \left\| \sum_{u=-U}^U \sum_{q=0}^{2Q-1} \mathcal{E}_{q,u} \right\|_{L^m(\mathbb{R})}^m.$$

Now write

$$\mathcal{F}(N) := \frac{1}{N^m} \sum_{\mathbf{q} \in [0, 2Q-1]^m} \sum_{\mathbf{u} \in [-U, U]^m} \sum_{\mathbf{k}, \mathbf{n} \in \mathbb{Z}^m} \mathfrak{K}_{\mathbf{u}}(\mathbf{k}) \mathfrak{N}_{\mathbf{q}}(\mathbf{n}) \int_0^1 \widehat{f}\left(\frac{\mathbf{k}}{N}\right) e(\mathbf{k} \cdot \omega(\mathbf{n}) + \mathbf{k} \cdot \mathbf{1}s) ds,$$

where $\mathfrak{N}_{\mathbf{q}}(\mathbf{n}) := \mathfrak{N}_{q_1}(n_1) \mathfrak{N}_{q_2}(n_2) \cdots \mathfrak{N}_{q_m}(n_m)$ and $\mathfrak{K}_{\mathbf{u}}(\mathbf{k}) := \mathfrak{K}_{u_1}(k_1) \mathfrak{K}_{u_2}(k_2) \cdots \mathfrak{K}_{u_m}(k_m)$. Our goal will be to establish that $\mathcal{F}(N)$ is equal to the right-hand side of (3.4) up to an $o(1)$ term. Then, since we can establish the same asymptotic for the lower bound, we may conclude the asymptotic for $\mathcal{E}(N)$. Since the details are identical, we will only focus on $\mathcal{F}(N)$.

Fixing \mathbf{q} and \mathbf{u} , we let

$$\mathcal{F}_{\mathbf{q}, \mathbf{u}}(N) = \frac{1}{N^m} \int_0^1 \sum_{\mathbf{n}, \mathbf{k} \in \mathbb{Z}^m} \mathfrak{N}_{\mathbf{q}}(\mathbf{n}) \mathfrak{K}_{\mathbf{u}}(\mathbf{k}) \widehat{f}\left(\frac{\mathbf{k}}{N}\right) e(\mathbf{k} \cdot \omega(\mathbf{n}) + \mathbf{k} \cdot \mathbf{1}s) ds.$$

Remark. In the following sections, we will fix \mathbf{q} and \mathbf{u} . Because of the way we have defined \mathfrak{N}_q , this implies two cases: $q < Q$ and $q \geq Q$. The only real difference in these two cases are the bounds in (3.7), which differ by a factor of $Q = \log(N)$. To keep the notation simple, we will assume we have $q < Q$ and work with the first bound. In practice the logarithmic correction does not affect any of the results or proofs.

4. Applying the B -process

4.1 Degenerate regimes

Fix $\delta = 1/(m+1)$. We say that $(q, u) \in [2Q] \times [-U, U]$ is *degenerate* if either

$$|u| < q^{(A-1)/2} \quad \text{or} \quad q \leq \delta Q$$

holds. Otherwise (q, u) is called *non-degenerate*. Let $\mathcal{G}(N)$ denote the set of all non-degenerate pairs (q, u) . In this section it is enough to suppose that $u > 0$ (and therefore $k > 0$). Next, we show that degenerate (q, u) contribute a negligible amount to $\mathcal{F}(N)$.

First, assume $q \leq \delta Q$. Expanding the m th power, evaluating the s -integral and trivial estimation yield

$$\begin{aligned} \|\mathcal{E}_{q,u}\|_{L^m}^m &= \int_0^1 \left(\frac{1}{N} \sum_{k \in \mathbb{Z}} \mathfrak{K}_u(k) \widehat{f}\left(\frac{k}{N}\right) e(ks) \sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) e(k\omega(n)) \right)^m ds \\ &\ll \frac{1}{N^m} \sum_{\substack{k_i \asymp e^u, \\ i=1, \dots, m}} \max_x \left(\widehat{f}\left(\frac{x}{N}\right) \right)^m \sum_{\substack{n_i \asymp e^q, \\ i=1, \dots, m}} \left| \int_0^1 e((k_1 + \cdots + k_m)s) ds \right| \\ &\ll \frac{1}{N^m} \#\{k_1, \dots, k_m \asymp e^u : k_1 + \cdots + k_m = 0\} N^{m\delta} \ll N^{m\delta-1}. \end{aligned}$$

If $u < q^{(A-1)/2}$ and $q > \delta Q$, then we can apply the Euler summation formula, followed by van der Corput's lemma with $j = 1$, to conclude that

$$\sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) e(k\omega(n)) \ll \frac{e^q}{kq^{A-1}},$$

where the numerator is the size of the support of \mathfrak{N}_q and the denominator is the maximum value of $k\omega'(x)$ for x in that support. Hence,

$$\|\mathcal{E}_{q,u}\|_\infty \ll \frac{1}{N} \sum_{k \asymp e^u} \frac{e^q}{kq^{A-1}} \ll \frac{1}{N} \frac{e^q}{q^{A-1}}.$$

Note

$$\sum_{q \leq Q} \frac{e^q}{q^{A-1}} \ll \int_1^Q \frac{e^q}{q^{A-1}} dq = \int_1^{Q/2} \frac{e^q}{q^{A-1}} dq + \int_{Q/2}^Q \frac{e^q}{q^{A-1}} dq \ll e^{Q/2} + \frac{1}{Q^{A-1}} \int_{Q/2}^Q e^Q dq \ll \frac{e^Q}{Q^{A-1}}.$$

Thus,

$$\left\| \sum_{\delta Q \leq q \leq Q} \sum_{u \leq q^{(A-1)/2}} \mathcal{E}_{q,u} \right\|_\infty \ll \frac{1}{N} \sum_{q \leq Q} \sum_{u \leq q^{(A-1)/2}} \sum_{k \asymp e^u} \frac{1}{k} \frac{e^q}{q^{A-1}} \ll \frac{1}{N} \frac{e^Q}{Q^{A-1}} \sum_{u \leq Q^{(A-1)/2}} 1 \leq \frac{1}{Q^{(A-1)/2}}.$$

Taking the L^m -norm then yields

$$\left\| \sum_{(q,u) \in [2Q] \times [-U,U] \setminus \mathcal{G}(N)} \mathcal{E}_{q,u} \right\|_{L^m}^m \ll_\delta \log(N)^{-\rho},$$

for some $\rho > 0$. Hence, the triangle inequality implies

$$\mathcal{F}(N) = \left\| \sum_{(u,q) \in \mathcal{G}(N)} \mathcal{E}_{q,u} \right\|_{L^m}^m + O(N^{-\rho}). \quad (4.1)$$

Next, to dismiss the degenerate regimes, let w, W denote strictly positive numbers satisfying $w < W$. Consider

$$g_{w,W}(x) := \min \left(\frac{1}{\|xw\|}, \frac{1}{W} \right),$$

where $\|\cdot\|$ denotes the distance to the nearest integer. We shall need (as in [LT21, Proof of Lemma 4.1]) the following lemma.

LEMMA 4.1. *If $W < 1/10$, then*

$$\sum_{e^u \leq |k| < e^{u+1}} g_{w,W}(k) \ll \left(e^u + \frac{1}{w} \right) \log(1/W)$$

where the implied constant is absolute.

Proof. The proof is elementary, hence we only sketch the main idea. If $e^u w < 1$ then we achieve the bound $(1/w) \log(1/W)$, and otherwise we get the bound $e^u \log(1/W)$. Focusing on the latter, first make a case distinction between those x which contribute $1/\|xw\|$ and those that contribute $1/W$. Then count how many contribute the latter. For the former, since the spacing between consecutive points is small, we can convert the sum into $e^u w$ many integrals of the form $(1/w) \int_{W/w}^{1/w} (1/x) dx$. \square

With the previous lemma at hand, we can show that an additional degenerate regime is negligible. Specifically, when we apply the B -process, the first step is to apply Poisson summation. Depending on the new summation variable there may, or may not, be a stationary point. The following lemma allows us to dismiss the contribution when there is no stationary point.

Fix $k \asymp e^u$ and let $[a, b] := \text{supp}(\mathfrak{N}_q)$. Consider

$$\text{Err}(k) := \sum_{\substack{r \in \mathbb{Z} \\ m_q(r) > 0}} \int_{\mathbb{R}} e(\Phi_r(x)) \mathfrak{N}_q(x) dx,$$

where

$$\Phi_r(x) := k\omega(x) - rx, \quad m_q(r) := \min_{x \in [a, b]} |\Phi_r(x)|.$$

Our next aim is to show that the smooth exponential sum

$$\text{Err}_u(s) := \sum_{k \in \mathbb{Z}} e(ks) \text{Err}(k) \mathfrak{K}_u(k) \hat{f}\left(\frac{k}{N}\right)$$

is small on average over s .

LEMMA 4.2. *Fix any constant $C > 0$. Then the bound*

$$I_{\mathbf{u}} := \int_0^1 \prod_{i=1}^m \text{Err}_{u_i}(s) ds \ll Q^{-C} N^m \quad (4.2)$$

holds uniformly in $Q^{(A-1)/2} \leq \mathbf{u} \ll Q$.

Proof. Let $\mathcal{L}_{\mathbf{u}}$ denote the truncated sub-lattice of \mathbb{Z}^m defined by gathering all $\mathbf{k} \in \mathbb{Z}^m$ so that $k_1 + \dots + k_m = 0$ and $|k_i| \asymp e^{u_i}$ for all $i \leq m$. The quantity $\mathcal{L}_{\mathbf{u}}$ arises from

$$I_{\mathbf{u}} = \sum_{\substack{|k_i| \asymp e^{u_i} \\ i \leq m}} \left(\left(\prod_{i \leq m} \text{Err}(k_i) \mathfrak{K}_u(k_i) \hat{f}\left(\frac{k_i}{N}\right) \right) \int_0^1 e((k_1 + \dots + k_m)s) ds \right) \ll \sum_{\mathbf{k} \in \mathcal{L}_{\mathbf{u}}} \left(\prod_{i \leq m} \text{Err}(k_i) \right). \quad (4.3)$$

Partial integration and the dyadic decomposition allow one to show that the contribution of $|r| \geq Q^{O(1)}$ to $\text{Err}(k_i)$ can be bounded by $O(Q^{-C})$ for any $C > 0$. Hence, from van der Corput's lemma (Lemma 2.3) with $j = 2$ and the assumption $m_q(r) > 0$, we infer

$$\text{Err}(k) \ll Q^{O(1)} \min \left(\frac{1}{\|k\omega'(a) - r\|}, \frac{1}{(k\omega''(a))^{1/2}} \right) = Q^{O(1)} \min \left(\frac{1}{\|k\omega'(a)\|}, \frac{1}{(k\omega''(a))^{1/2}} \right)$$

where the implied constant is absolute. Notice that $\omega'(a) \asymp q^{A-1}e^{-q} =: w$ and

$$k\omega''(a) \asymp (e^{u-2q}q^{A-1})^{1/2} =: W.$$

Thus, $\text{Err}(k) \ll g_{w,W}(k)Q^{O(1)}$. Using $\text{Err}(k_i) \ll g_{w,W}(k_i)Q^{O(1)}$ for $i < m$ and $\text{Err}(k_m) \ll Q^{O(1)}/W$ in (4.3) produces the estimate

$$I_{\mathbf{u}} \ll \frac{Q^{O(1)}}{W} \sum_{\substack{|k_i| \asymp e^{u_i} \\ i < m}} \left(\prod_{i < m} g_{w,W}(k_i) \right) = \frac{Q^{O(1)}}{W} \left(\sum_{|k| \asymp e^u} g_{w,W}(k) \right)^{m-1}. \quad (4.4)$$

Suppose $W \geq N^{-\varepsilon}$. Then $g_{w,W}(k) \leq N^{\varepsilon}$ and we obtain that

$$I_{\mathbf{u}} \ll Q^{O(1)} N^{\varepsilon m} e^{u_1 + \dots + u_{m-1}} \ll N^{m-1+\varepsilon m} \ll Q^{-C} N^m.$$

Now suppose $W < N^{-\varepsilon} \leq 1/10$. Then Lemma 4.1 is applicable and yields

$$\sum_{|k| \asymp e^u} g_{w,W}(k) \ll (e^u + 1/w) \log(1/W) \ll (e^u + e^q) \log(1/W) \ll NQ.$$

Plugging this into (4.4) and using $1/W \ll e^{q-u/2} q^{(1-A)/2} \ll N e^{-u/2}$ shows that

$$I_{\mathbf{u}} \ll Q^{O(1)} \frac{(NQ)^{m-1}}{W} \ll Q^{O(1)} (NQ)^m e^{-u/2}.$$

Because $u \geq Q^{(A-1)/2}$, we certainly have $e^{-u/2} \ll Q^{-C}$ for any $C > 0$ and thus the proof is complete. \square

4.2 First application of the B -process

First, following the lead set out in [LST24], we apply the B -process in the n -variable. Assume without loss of generality that $k > 0$ (if $k < 0$ we take complex conjugates and the without loss of generality assumption that f is even).

Then, after applying the B -process, the phase function $x \mapsto k\omega(x) - rx$ will be transformed to

$$\phi(k, r) := k\omega(x_{k,r}) - rx_{k,r} \quad \text{where } x_{k,r} := \begin{cases} \tilde{\omega}\left(\frac{r}{k}\right) & \text{if } \frac{r}{k} > e^{A-1}, \\ 3 & \text{otherwise,} \end{cases}$$

and $\tilde{\omega}(x) := (\omega')^{-1}(x)$ is the inverse of the derivative of ω . Importantly, the next proposition states that $\mathcal{E}_{q,u}$ is well approximated by

$$\mathcal{E}_{q,u}^{(B)}(s) := \frac{e(-1/8)}{N} \sum_{k>0} \mathfrak{K}_u(k) \hat{f}\left(\frac{k}{N}\right) e(ks) \sum_{r \geq 0} \frac{\mathfrak{N}_q(x_{k,r})}{\sqrt{|k\omega''(x_{k,r})|}} e(\phi(k, r)). \quad (4.5)$$

We comment on the definition of $x_{k,r}$ and $\mathcal{E}_{q,u}^{(B)}$ in the next remark.

Remark. Recall that we currently work in the non-degenerate regime, $q \geq \delta Q$. Note that $x_{k,r}$ is the stationary point for the map $x \mapsto k\omega(x) - rx$ when $r/k > e^{A-1}$ (i.e. when there is a unique stationary point in the relevant regime). On the other hand, $\mathfrak{N}_q(3) = 0$ since $\inf(\text{supp}(\mathfrak{N}_q)) \rightarrow \infty$ as $Q \rightarrow \infty$. Thus, $\mathfrak{N}_q(x_{k,r})$ is well defined and smooth as a function of $k, r \geq 0$.

PROPOSITION 4.3. *If $u \geq Q^{(A-1)/2}$, then*

$$\|\mathcal{E}_{q,u} - \mathcal{E}_{q,u}^{(B)}\|_{L^m}^m \ll Q^{-100m} \quad (4.6)$$

uniformly for all non-degenerate $(u, q) \in \mathcal{G}(N)$.

Proof. Let $[a, b] := \text{supp}(\mathfrak{N}_q)$, let $\Phi_r(x) := k\omega(x) - rx$, and let $m(r) := \min\{|\Phi'_r(x)| : x \in [a, b]\}$. As usual when applying the B -process we first apply Poisson summation and integration by parts:

$$\sum_{n \in \mathbb{Z}} \mathfrak{N}_q(n) e(k\omega(n)) = \sum_{r \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathfrak{N}_q(x) e(\Phi_r(x)) dx = M(k) + \text{Err}(k),$$

where $M(k)$ gathers the contributions when $r \in \mathbb{Z}$ with $m(r) = 0$ (i.e. with a stationary point) and $\text{Err}(k)$ gathers the contribution of $0 < m(r)$.

In the notation of Lemma 2.2, let $w(x) := \mathfrak{N}_q(x)$, $\Lambda_\psi := \omega(e^q)e^u = q^A e^u$, and $\Omega_\psi = \Omega_w := e^q$. Since (u, q) is non-degenerate we have that $\Lambda_\psi/\Omega_\psi \gg q$, and hence

$$M(k) = e(-1/8) \sum_{r \geq 0} \frac{\mathfrak{N}_q(x_{k,r})}{\sqrt{|k\omega''(x_{k,r})|}} e(\phi(k, r)) + O((q\Lambda_\psi^{1/2+O(\varepsilon)})^{-1}). \quad (4.7)$$

Summing (4.7) against $N^{-1} \mathfrak{K}_u(k) \hat{f}(k/N) e(ks)$ for $k \geq 0$ gives rise to $\mathcal{E}_{q,u}^{(B)}$. The term coming from

$$\text{Err}(k) N^{-1} \mathfrak{K}_u(k) \hat{f}(k/N) e(ks) = \frac{1}{N} \text{Err}_u(s)$$

can be bounded sufficiently by Lemma 4.2 and the triangle inequality. \square

Since $x_{k,r}$ is roughly of size e^q , if we stop here and apply the triangle inequality to (4.5) we would get

$$|\mathcal{E}_{q,u}^{(B)}(s)| \ll \frac{1}{N} \sum_{k>0} \mathfrak{K}_u(k) e^q \frac{1}{\sqrt{k}} \frac{k}{e^q} \ll \frac{1}{N} e^{3u/2} \ll N^{1/2}.$$

Hence, we still need to find a saving of $O(N^{1/2})$. To achieve most of this, we now apply the B -process in the k variable. This will require the following a priori bounds.

4.3 Amplitude bounds

Before proceeding with the second application of the B -process, we require bounds on the amplitude function

$$\Psi_{q,u}(k, r, s) = \Psi_{q,u} := \frac{\mathfrak{N}_q(x_{k,r}) \mathfrak{K}_u(k)}{\sqrt{|k\omega''(x_{k,r})|}} \widehat{f}\left(\frac{k}{N}\right),$$

and its derivatives; for which we have the following lemma.

LEMMA 4.4. *For any pair q, u as above, and any $j \geq 1$, we have the bounds*

$$\|\partial_k^j \Psi_{q,u}(k, r, \cdot)\|_\infty \ll e^{-uj} Q^{O(1)} \|\Psi_{q,u}\|_\infty, \quad (4.8)$$

where the implicit constant in the exponent depends on j , but not q, u . Moreover,

$$\|\Psi_{q,u}\|_\infty \ll e^{q-u/2} q^{-(1/2)(A-1)}.$$

Proof. First, note that since $\Psi_{q,u}$ is a product of functions of k , if we can establish (4.8) for each of these functions, then the overall bound will hold for $\Psi_{q,u}(k, r, s)$ by the product rule. Moreover, the bound is obvious for $\mathfrak{K}_u(k)$, $\widehat{f}(k/N)$, and $k^{-1/2}$.

Thus, consider first $\partial_k \mathfrak{N}_q(x_{k,r}) = \mathfrak{N}'_q(x_{k,r}) \partial_k(x_{k,r})$. By assumption, since $x_{k,r} \asymp e^q$, we have that $\mathfrak{N}'_q(x_{k,r}) \ll e^{-q}$. Again, by repeated application of the product rule, it suffices to show that $\partial_k^j x_{k,r} \ll e^{q-uj} Q^{O(1)}$. To that end, begin with the equation

$$1 = \partial_x(x) = \partial_x(\widetilde{\omega}(\omega'(x))) = \widetilde{\omega}'(\omega'(x)) \omega''(x).$$

Hence, $\widetilde{\omega}'(\omega'(x)) = 1/\omega''(x)$ which we can write as

$$\widetilde{\omega}'(\omega'(x)) = x^2 f_1(\log(x)),$$

where f_1 is a rational function. Now we take $j-1$ derivatives of each side. Inductively, one sees that there exist rational functions f_j such that

$$\widetilde{\omega}^{(j)}(\omega'(x)) = x^{j+1} f_j(\log(x)).$$

Setting $x = x_{k,r} = \widetilde{\omega}(r/k)$ then gives

$$\widetilde{\omega}^{(j)}(r/k) = x_{k,r}^{j+1} f_j(\log(x_{k,r})). \quad (4.9)$$

With (4.9), we can use repeated application of the product rule to bound

$$\begin{aligned}\partial_k^j x_{k,r} &= \partial_k^j \tilde{\omega}(r/k) \\ &= -\partial_k^{j-1} \tilde{\omega}'(r/k) \left(\frac{r}{k^2} \right) \\ &\ll \tilde{\omega}^{(j)}(r/k) \left(\frac{r}{k^2} \right)^j + \tilde{\omega}'(r/k) \left(\frac{r}{k^{1+j}} \right) \\ &\ll x_{k,r}^{j+1} f_j(\log(x_{k,r})) \left(\frac{r}{k^2} \right)^j + x_{k,r}^2 f_1(\log(x_{k,r})) \left(\frac{r}{k^{1+j}} \right).\end{aligned}$$

Now recall that $k \asymp e^u$, $x_{k,r} \asymp e^q$, and $r \asymp e^{u-q} q^{A-1}$, thus

$$\begin{aligned}\partial_k^j x_{k,r} &\ll \left(e^{q(j+1)} \left(\frac{e^{u-q}}{e^{2u}} \right)^j + e^{2q} \left(\frac{e^{u-q}}{e^{(1+j)u}} \right) \right) Q^{O(1)} \\ &\ll e^{q-ju} Q^{O(1)}.\end{aligned}$$

Hence, $\partial_k^{(j)} \mathfrak{N}_q(x_{k,r}) \ll e^{-ju} Q^{O(1)}$.

The same argument suffices to prove that $\partial_k^j (1/\sqrt{|\omega''(x_{k,r})|}) \ll e^{q-ju} Q^{O(1)}$. \square

4.4 Second application of the B -process

We now apply the B -process in the k -variable. At the present stage, the phase function is $\phi(k, r) + ks$. Thus, for $h \in \mathbb{Z}$, let $\mu = \mu_{h,r,s}$ be the unique stationary point of $k \mapsto \phi(k, r) - (h - s)k$. Namely,

$$(\partial_\mu \phi)(\mu, r) = h - s.$$

After the second application of the B -process, the phase will be transformed to

$$\Phi(h, r, s) = \phi(\mu, r) - (h - s)\mu.$$

With that, let (again for $u > 0$)

$$\mathcal{E}_{q,u}^{(\text{BB})}(s) := \frac{1}{N} \sum_{r \geq 0} \sum_{h \geq 0} \hat{f}\left(\frac{\mu}{N}\right) \mathfrak{K}_u(\mu) \mathfrak{N}_q(x_{\mu,r}) \frac{1}{\sqrt{|\mu \omega''(x_{\mu,r}) \cdot (\partial_{\mu\mu} \phi)(\mu, r)|}} e(\Phi(h, r, s)). \quad (4.10)$$

We can now apply the B -process for a second time and conclude the following proposition.

PROPOSITION 4.5. *We have*

$$\|\mathcal{E}_{q,u}^{(\text{BB})} - \mathcal{E}_{q,u}^{(\text{B})}\|_{L^m([0,1])} = O(N^{-1/2m+\varepsilon}) \quad (4.11)$$

uniformly for any non-degenerate $(q, u) \in \mathcal{G}(N)$.

Before we can prove the above proposition, we need some preparations. Note that

$$k\omega'(n) = Ak \frac{(\log n)^{A-1}}{n} \leq 10Ae^{u-q} q^{A-1}.$$

If $u - q + (A - 1) \log q < -10$ then $10Ae^{u-q} q^{A-1} = 10Ae^{-10A} \leq 0.6$. Hence, there is no stationary point in the first application of the B -process. Thus, the contribution from this regime is disposed of by the first B -process. Therefore, from now on we assume that

$$u \geq q - (A - 1) \log q - 10A, \quad \text{in particular, } e^u \gg e^q q^{1-A}. \quad (4.12)$$

4.5 Non-essential regimes

In this section we estimate the contribution from regimes where u is smaller by a power of a logarithm than the top scale Q . We shall see that this regime can be disposed of. More precisely, let

$$\mathcal{T}(N) := \{(q, u) \in \mathcal{G}(N) : u \leq \log N - 10A \log \log N\}.$$

We shall see that contribution $\mathcal{T}(N)$ is negligible by showing that the function

$$T_N(s) := \sum_{(q,u) \in \mathcal{T}(N)} \mathcal{E}_{q,u}^{(B)}(s) \quad (4.13)$$

has a small $\|\cdot\|_\infty$ -norm (in the $s \in [0, 1]$ variable). To prove this, we need to ensure that in

$$\mathcal{E}_{q,u}^{(B)}(s) = \frac{e(-1/8)}{N} \sum_{r \geq 0} \sum_{k \geq 0} \Psi_{q,u}(k, r, s) e(\phi(k, r) - ks)$$

the amplitude function

$$\Psi_{q,u}(k, r, s) := \frac{\mathfrak{N}_q(x_{k,r}) \mathfrak{K}_u(k)}{\sqrt{|k\omega''(x_{k,r})|}} \hat{f}\left(\frac{k}{N}\right)$$

has a suitably good decay in k .

LEMMA 4.6. *If (4.12) holds, then*

$$\|k \mapsto \partial_k \Psi_{q,u}(k, r, s)\|_{L^1(\mathbb{R})} \ll e^{u/2} q^{-(1/2)(A-1)}$$

uniformly for r and s in the prescribed ranges.

Proof. First, use the triangle inequality to bound

$$\begin{aligned} \|k \mapsto \partial_k \Psi_{q,u}(k, r, s)\|_{L^1(\mathbb{R})} &\ll \left\| \partial_k \left\{ \frac{\mathfrak{N}_q(x_{k,r}) \mathfrak{K}_u(k)}{\sqrt{|k\omega''(x_{k,r})|}} \right\} \hat{f}\left(\frac{k}{N}\right) \right\|_{L^1(\mathbb{R})} \\ &\quad + \left\| \frac{\mathfrak{N}_q(x_{k,r}) \mathfrak{K}_u(k)}{\sqrt{|k\omega''(x_{k,r})|}} \partial_k \hat{f}\left(\frac{k}{N}\right) \right\|_{L^1(\mathbb{R})}. \end{aligned}$$

Since \hat{f} has bounded derivative, the term on the right can be bounded by $1/N$ times the sup norm times e^u . Since $\hat{f}(k/N)$ is bounded, and $\mathfrak{N}_q(x_{k,r}) \mathfrak{K}_u(k) / \sqrt{|k\omega''(x_{k,r})|}$ changes sign finitely many times, we can apply the fundamental theorem of calculus and bound the whole by

$$\|k \mapsto \partial_k \Psi_{q,u}(k, r, s)\|_{L^1(\mathbb{R})} \ll \left\| k \mapsto \frac{1}{\sqrt{|k\omega''(x_{k,r})|}} \right\|_{L^\infty(\mathbb{R})}. \quad \square$$

Now we are in the position to prove that the contribution from (4.13) is negligible thanks to a second derivative test. This is one of the places where, in contrast to the monomial case, we only win by a logarithmic factor. Moreover, this logarithmic saving goes to 0 as A approaches 1.

LEMMA 4.7. *The oscillatory integral*

$$I_{q,u}(h, r) := \int_{-\infty}^{\infty} \Psi_{q,u}(t, r, s) e(\phi(t, r) - t(h - s)) dt \quad (4.14)$$

satisfies the bound

$$I_{q,u}(h, r) \ll e^q q^{1-A} \quad (4.15)$$

uniformly in h , and r in ranges prescribed by Ψ .

Proof. We aim to apply van der Corput's lemma (Lemma 2.3) for a second derivative bound. For that, first note that $\partial_t \phi(t, r) = \omega(x_{t,r}) + t \partial_t(\omega(x_{t,r})) - r \partial_t(x_{t,r})$. Now, since

$$\partial_t(\omega(x_{t,r})) = \omega'(x_{t,r}) \partial_t(x_{t,r}) = \frac{r}{t} \partial_t(x_{t,r}), \quad (4.16)$$

it follows that

$$\partial_t \phi(t, r) = \omega(x_{t,r}). \quad (4.17)$$

Now we bound the second derivative of $\phi(t, r) - t(s + h)$. By (4.16) and (4.17), we have

$$\partial_t^2[\phi(t, r)] = \partial_t[\omega(x_{t,r})] = \frac{r}{t} \partial_t[x_{t,r}].$$

Thus,

$$\partial_t^2[\phi(t, r)] = -\frac{1}{\omega''(x_{t,r})} \frac{r^2}{t^3}.$$

Taking $x_{t,r} \asymp e^q$ into account gives

$$\partial_t^2[\phi(t, r)] \asymp \frac{1}{e^{-2q} q^{A-1}} \frac{(e^{u-q} q^{A-1})^2}{e^{3u}} = e^{-u} q^{A-1}. \quad (4.18)$$

The upshot, by van der Corput's lemma (Lemma 2.3), is that

$$I_{q,u}(h, r) \ll \|\Psi\|_\infty (e^{-u} q^{A-1})^{-1/2} \ll e^q q^{1-A}. \quad \square$$

Now we are in the position to prove the following lemma.

LEMMA 4.8. *We have that, as a function of $s \in [0, 1]$, the sup norm $\|T_N\|_\infty \ll (\log N)^{-8A}$.*

Proof. Note that

$$\mathcal{E}_{q,u}^{(B)}(s) \ll \frac{1}{N} \sum_{r \asymp e^{u-q} q^{A-1}} |\Xi(r)|, \quad \text{where } \Xi(r) := \sum_{k>0} \Psi_{q,u}(k, r, s) e(\phi(k, r) - ks). \quad (4.19)$$

By Poisson summation,

$$\Xi(r) = \sum_{h \in \mathbb{Z}} I_{q,u}(h, r).$$

We decompose the right-hand side into the contribution $\Xi_1(r)$ coming from $|h| > (4Q)^A$, and a contribution $\Xi_2(r)$ from the regime $|h| \leq (4Q)^A$. Next, we argue that $\Xi_1(r)$ can be disposed of by partial integration. Because $x_{k,r} \leq 2N$, we have

$$\omega(x_{k,r}) = (\log x_{k,r})^A \leq (3Q)^A.$$

Note for $|h| > (4Q)^A$, by (4.17), that the inequality

$$\partial_k[\phi(k, r) - k(s + h)] \gg h$$

holds true uniformly in r and s . As a result, partial integration yields, for any constant $C > 0$, the bound

$$I_{q,u}(h, r) \ll \|k \mapsto \partial_k \Psi_{q,u}(k, r, s)\|_{L^1(\mathbb{R})} h^{-C}.$$

Therefore,

$$\Xi_1(r) \ll \|k \mapsto \partial_k \Psi_{q,u}(k, r, s)\|_{L^1(\mathbb{R})} \sum_{h \geq (4Q)^A} h^{-C}.$$

Recall that we have $q \geq 1/(m+1)Q$. Thus, taking C to be large and using Lemma 4.6, we deduce that

$$\Xi_1(r) \ll_{C_1} e^{u/2} Q^{-C_1} \quad (4.20)$$

for any constant $C_1 > 0$. All in all, we have shown so far that

$$\Xi(r) \ll \Xi_2(r) + e^{u/2} Q^{-C_1}.$$

In $\Xi_2(r)$ there are $O(Q^A)$ choices of h . By using Lemma 4.7 we conclude

$$\Xi_2(r) \ll e^q q^{1-A} Q^A. \quad (4.21)$$

By combining (4.20) and (4.21), we deduce from (4.19) that

$$\|\mathcal{E}_{q,u}^{(B)}(\cdot)\|_\infty \ll \frac{1}{N} \sum_{r \asymp e^{u-q} Q^{A-1}} e^q q^{1-A} Q^A \ll \frac{1}{N} e^u Q^A.$$

As a result,

$$\begin{aligned} \|T_N(\cdot)\|_\infty &\ll \frac{1}{N} \sum_{(u,q) \in \mathcal{T}(N)} e^u Q^A \ll \frac{1}{N} \sum_{u \leq \log N - 10A \log \log N} e^u Q^{A+1} \\ &\ll \frac{1}{(\log N)^{10A}} (\log N)^{A+1} \ll \frac{1}{(\log N)^{8A}}. \end{aligned} \quad \square$$

4.6 Essential regimes

We are now ready to apply our stationary phase expansion (Proposition 2.1), and thus effectively apply the B -process a second time. Recall that after applying Poisson summation, the phase will be $\psi_{r,h}(t) = \psi(t) := \phi(t, r) - t(h-s)$. Let

$$W_{q,u}(t) := \frac{\mathfrak{N}_q(x_{t,r}) \mathfrak{K}_u(t)}{\sqrt{|t\omega''(x_{t,r})|}} \hat{f}\left(\frac{t}{N}\right) e\left(\psi(t) - \psi(\mu) - \frac{1}{2}(t-\mu)^2 \psi''(\mu)\right).$$

Furthermore, define

$$p_j(\mu) := c_j \left(\frac{1}{|\psi''(\mu)|} \right)^j W_{q,u}^{(2j)}(\mu),$$

where $p_0(\mu) = e(1/8)W_{q,u}(\mu)$. Note that, by (4.18), one can bound

$$p_j(\mu) \ll p_1(\mu) \ll N^\varepsilon \frac{1}{|\psi''(\mu)|} \frac{1}{\mu^{1/2}} \frac{\omega'''(x_{\mu,r})(\partial_t x_{t,r}|_{t=\mu})^2}{|\omega''(x_{\mu,r})|^{3/2}} \ll e^{u/2-q} N^\varepsilon, \quad j \geq 1. \quad (4.22)$$

Hence, let

$$P_{q,u}(h, r, s) := \frac{e(\psi(\mu))}{\sqrt{|\psi''(\mu)|}} (p_0(\mu) + p_1(\mu)),$$

and set

$$E_{q,u}^{(BB)}(s) := \frac{e(-1/8)}{N} \sum_{r \geq 0} \sum_{h \geq 0} P_{q,u}(h, r, s).$$

Before proving Proposition 4.5 we need the following lemma.

LEMMA 4.9. *For any $c \in [0, 1]$ and any $M > 10$, we have the bound*

$$\int_0^1 \min(\|c+s\|^{-1}, M) ds \leq 2 \log M.$$

Proof. Decomposing into intervals where $\|c + s\|^{-1} \leq M$ as well as intervals where $\|c + s\|^{-1} > M$ and then using straightforward estimates implies the claimed bound. \square

We can now prove Proposition 4.5.

Proof of Proposition 4.5. Fix $s \in [0, 1]$ and recall the definition of $I_{q,u}(h, r)$ from (4.14). Then, by Poisson summation,

$$\mathcal{E}_{q,u}^{(B)}(s) = \frac{e(-1/8)}{N} \sum_{r \geq 0} \sum_{h \in \mathbb{Z}} I_{q,u}(h, r).$$

Let $[a, b] := \text{supp}(\mathfrak{K}_u)$, and

$$m_r(h) := \min_{k \in [a, b]} |\psi'_{r,h}(k)|.$$

We decompose the h -summation into three different ranges:

$$\sum_{h \in \mathbb{Z}} I_{q,u}(h, r) = \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3,$$

where the first contribution $\mathfrak{C}_1(r, s)$ is where $m_r(h) = 0$, the second contribution $\mathfrak{C}_2(r, s)$ is where $0 < m_r(k) \leq N^\varepsilon$, and the third contribution $\mathfrak{C}_3(r, s)$ is where $m_r(h) \geq N^\varepsilon$. Integration by parts shows that

$$\mathfrak{C}_3(r, s) \ll N^{-100}.$$

Next, we handle $\mathfrak{C}_1(r, s)$. To this end, we shall apply Proposition 2.1, in the notation of which we have

$$\Omega_w := e^u, \quad \Lambda_w := e^{q^{-u/2+\varepsilon}}, \quad \Lambda_\psi := e^u q^{A-1}, \quad \Omega_\psi := e^u.$$

The decay of the amplitude function was shown in Lemma 4.4; the decay of the phase function follows from a short calculation we omit. Next, since we have disposed of the inessential regimes, we see

$$Z := \Omega_\psi + \Lambda_w + \Lambda_\psi + \Omega_w + 1 \asymp e^u q^{A-1} \asymp N^{1+o(1)}.$$

Furthermore,

$$\frac{\Omega_\psi}{\Lambda_\psi^{1/2}} Z^{\delta/2} = \frac{e^{u/2}}{q^{(1/2)A}} Z^{\delta/2} \asymp \frac{e^{u(1/2+\delta)}}{q^{(1/2)A+(\delta/2)(A-1)}}.$$

Hence, taking $\delta := 1/11$ is compatible with the assumption (2.1). Thus,

$$\mathfrak{C}_1(r, s) = \sum_{h \geq 0} P_{q,u}(h, r, s) + O(N^{-1/11}).$$

Now we bound $\mathfrak{C}_2(r, s)$. First note that $\omega(x_{t,r})$ is monotonic in t . To see this, set the derivative equal to 0:

$$A \frac{\log(x_{t,r})^{A-1}}{x_{t,r}} \partial_t x_{t,r} = A \frac{\log(x_{t,r})^{A-1}}{x_{t,r}} \omega'(r/t) (-r/t^2) = 0.$$

However, since $x_{t,r} \asymp e^q$, this implies $\omega'(r/k) = 0$, which is a contradiction. Thus, by van der Corput's lemma (Lemma 2.3) for the first derivative, and monotonicity, we have

$$\mathfrak{C}_2(r, s) \ll N^{1/2+\varepsilon} \min(\|\omega(x_{a,r}) + s\|^{-1}, N^{1/2+o(1)}),$$

where we used (4.18) and the fact that $\partial_t \phi(t, r) = \omega(x_{t,r})$. Notice that

$$\left\| \frac{1}{N} \sum_{r \geq 0} \mathfrak{C}_2(r, \cdot) \right\|_{L^m}^m \ll \left\| \frac{1}{N} \sum_{r \geq 0} \mathfrak{C}_2(r, \cdot) \right\|_\infty^{m-1} \left\| \frac{1}{N} \sum_{r \geq 0} \mathfrak{C}_2(r, \cdot) \right\|_{L^1}.$$

By (4.15) we see that

$$\left\| \frac{1}{N} \sum_{r \geq 0} \mathfrak{C}_2(r, s) \right\|_{\infty}^{m-1} \ll N^{O(\varepsilon)}.$$

Hence, it remains to estimate

$$N^{O(\varepsilon)} \sum_{r \asymp e^{u-q} q^{A-1}} \frac{1}{\sqrt{N}} \int_0^1 \min(\|\omega(x_{a,r}) + s\|^{-1}, N^{1/2+o(1)}) ds.$$

By exploiting Lemma 4.9 we see that

$$\left\| \frac{1}{N} \sum_{r \geq 0} \mathfrak{C}_2(r, \cdot) \right\|_{L^m}^m \ll \sum_{r \asymp e^{u-q} q^{A-1}} \frac{N^{o(1)}}{\sqrt{N}} \ll N^{o(1)-1/2}$$

which implies the claim.

Finally, it remains to show that

$$\|\mathcal{E}_{q,u}^{(\text{BB})}(\cdot) - E_{q,u}^{(\text{BB})}(\cdot)\|_{L^m}^m = O(N^{-1/2+\varepsilon})$$

from which we can apply the triangle inequality to conclude Proposition 4.5. For this, recall the bounds (4.22). Since $\mathcal{E}_{q,u}^{(\text{BB})}$ is simply the term arising from $p_0(\mu)$, we have that

$$\|\mathcal{E}_{q,u}^{(\text{BB})}(\cdot) - E_{q,u}^{(\text{BB})}(\cdot)\|_{L^m}^m \ll \frac{1}{N} \sum_{r \in \mathbb{Z}} p_1 \ll \frac{e^{u-q/2} N^\varepsilon}{N}.$$

From here the bound follows from the ranges of q and u . \square

Before proceeding, we note that (4.10) can be simplified. In particular, we have the following lemma.

LEMMA 4.10. *Given, h , r , and s as above, we have*

$$\mu_{h,r,s} = \frac{r}{\omega'(\omega^{-1}(h-s))}, \quad \Phi(h, r, s) = -r\omega^{-1}(h-s)$$

and

$$\mu\omega''(x_{\mu,r}) \cdot (\partial_{\mu\mu}\phi)(\mu, r) = -\frac{r^2}{\mu^2}. \quad (4.23)$$

Proof. Recall $x_{k,r} = \tilde{\omega}(r/k)$. Now, to compute μ , we have

$$\begin{aligned} 0 &= \partial_{\mu}(\mu\omega(x_{\mu,r}) - rx_{\mu,r}) - (h-s) \\ &= \omega(x_{\mu,r}) + \mu\omega'(x_{\mu,r})(\partial_{\mu}x_{\mu,r}) - r\partial_{\mu}x_{\mu,r} - (h-s). \end{aligned}$$

Consider first

$$\partial_{\mu}x_{\mu,r} = \partial_{\mu}\left(\tilde{\omega}\left(\frac{r}{\mu}\right)\right) = \tilde{\omega}'\left(\frac{r}{\mu}\right)\left(-\frac{r}{\mu^2}\right).$$

Furthermore, since $\mu = \tilde{\omega}(\omega'(\mu))$, we may differentiate both sides and then change variables to see that

$$\tilde{\omega}'(r/\mu) = \frac{1}{\omega''(\tilde{\omega}(r/\mu))}. \quad (4.24)$$

Hence,

$$\partial_{\mu}(x_{\mu,r}) = -\omega\left(\tilde{\omega}\left(\frac{r}{\mu}\right)\right) \frac{r}{\mu^2\omega''(\tilde{\omega}(r/\mu))}.$$

Hence,

$$\begin{aligned} 0 &= \omega(x_{\mu,r}) - r \left(\omega \left(\tilde{\omega} \left(\frac{r}{\mu} \right) \right) \frac{r}{\mu^2 \omega''(\tilde{\omega}(r/\mu))} \right) + \omega \left(\tilde{\omega} \left(\frac{r}{\mu} \right) \right) \frac{r^2}{\mu^2 \omega''(\tilde{\omega}(r/\mu))} - (h-s) \\ &= \omega(x_{\mu,r}) - (h-s). \end{aligned}$$

Hence, $\omega(\tilde{\omega}(r/\mu)) = h-s$. Solving for μ gives

$$\mu = \frac{r}{\omega'(\omega^{-1}(h-s))}.$$

Moreover, we can simplify the phase as follows:

$$\begin{aligned} \Phi(h, r, s) &= \phi(\mu, r) - (h-s)\mu \\ &= \mu\omega(x_{\mu,r}) - rx_{\mu,r} - (h-s)\mu \\ &= \mu\omega(\tilde{\omega}(r/\mu)) - r\tilde{\omega}(r/\mu) - (h-s)\mu \\ &= \frac{r(h-s)}{\omega'(\omega^{-1}(h-s))} - r\omega^{-1}(h-s) - (h-s)\frac{r}{\omega'(\omega^{-1}(h-s))} \\ &= -r\omega^{-1}(h-s). \end{aligned}$$

Turning now to (4.23), we note that since, by the definition of μ , we have that $\partial_\mu \phi(\mu, r) = h-s$ and $h-s = \omega(\tilde{\omega}(r/\mu))$, we may differentiate both sides of the former to deduce that

$$\begin{aligned} \partial_{\mu\mu} \phi(\mu, r) &= \partial_\mu (\omega(\tilde{\omega}(r/\mu))) \\ &= \omega'(\tilde{\omega}(r/\mu)) \tilde{\omega}'(r/\mu) (-r/\mu^2) \\ &= -(r^2/\mu^3) \tilde{\omega}'(r/\mu). \end{aligned}$$

Now, using (4.24), we conclude that

$$\begin{aligned} \mu\omega''(x_{r,\mu}) \cdot (\partial_{\mu\mu} \phi)(\mu, r) &= -\mu\omega''(\tilde{\omega}(r/\mu)) (r^2/\mu^3) \tilde{\omega}'(r/\mu) \\ &= -(r^2/\mu^2) \omega''(\tilde{\omega}(r/\mu)) \frac{1}{\omega''(\tilde{\omega}(r/\mu))} = -\frac{r^2}{\mu^2}. \end{aligned} \quad \square$$

Applying Lemma 4.10 and inserting some definitions allows us to write

$$\mathcal{E}_{q,u}^{(\text{BB})}(s) = \frac{1}{N} \sum_{r \geq 0} \sum_{h \geq 0} \hat{f}\left(\frac{\mu}{N}\right) \mathfrak{K}_u(\mu) \mathfrak{N}_q(\tilde{\omega}(r/\mu)) \frac{\mu}{r} e(-r\omega^{-1}(h-s)). \quad (4.25)$$

Returning now to the full L^m norm, let $\sigma_i := \sigma(u_i) := u_i/|u_i|$. Applying Propositions 4.3 and 4.5 and expanding the m th power yields

$$\mathcal{F}(N) = \sum_{\sigma_1, \dots, \sigma_m \in \{\pm 1\}} \sum_{\substack{(u_i, q_i) \in \mathcal{G}(N) \\ u_i > 0}} \int_0^1 \prod_{\substack{i \leq m \\ \sigma_i > 0}} \mathcal{E}_{q_i, u_i}^{(\text{BB})}(s) \prod_{\substack{i \leq m \\ \sigma_i < 0}} \overline{\mathcal{E}_{q_i, u_i}^{(\text{BB})}(s)} ds + O(N^{-\varepsilon/2}). \quad (4.26)$$

To simplify this expression, for a fixed \mathbf{u} and \mathbf{q} , and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$, let $\mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}) := \prod_{i \leq m} \mathfrak{K}_{u_i}(\mu_i)$. The functions $\mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s)$ and $\hat{f}(\boldsymbol{\mu}/N)$ are defined similarly. Aside from the error term, the right-hand side of (4.26) splits into a sum over

$$\mathcal{F}_{\mathbf{q}, \mathbf{u}} := \frac{1}{N^m} \sum_{\mathbf{r} \in \mathbb{Z}^m} \frac{1}{r_1 r_2 \cdots r_m} \int_0^1 \sum_{\mathbf{h} \in \mathbb{Z}^m} \mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}) \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s) A_{\mathbf{h}, \mathbf{r}}(s) e(\varphi_{\mathbf{h}, \mathbf{r}}(s)) ds,$$

where the phase function is given by

$$\varphi_{\mathbf{h},\mathbf{r}}(s) := -(r_1\omega^{-1}(h_1 - s) + r_2\omega^{-1}(h_2 - s) + \cdots + r_m\omega^{-1}(h_m - s)),$$

and where

$$A_{\mathbf{h},\mathbf{r}}(s) := \widehat{f}\left(\frac{\boldsymbol{\mu}}{N}\right)\mu_1\mu_2\cdots\mu_m.$$

Now we distinguish between two cases. First, the set of all (\mathbf{r}, \mathbf{h}) where the phase $\varphi_{\mathbf{h},\mathbf{r}}(s)$ vanishes identically, which we call the *diagonal*; and its complement, the *off-diagonal*. Let

$$\mathcal{A} := \{(\mathbf{r}, \mathbf{h}) \in \mathbb{N} \times \mathbb{N} : \varphi_{\mathbf{h},\mathbf{r}}(s) = 0, \forall s \in [0, 1]\},$$

and let

$$\eta(\mathbf{r}, \mathbf{h}) := \begin{cases} 1 & \text{if } (\mathbf{r}, \mathbf{h}) \notin \mathcal{A}, \\ 0 & \text{if } (\mathbf{r}, \mathbf{h}) \in \mathcal{A}. \end{cases}$$

The diagonal, as we show, contributes the main term, while the off-diagonal contribution is negligible (see § 6).

5. Extracting the diagonal

First, we establish an asymptotic for the diagonal. The sums below range over $\mathbf{q} \in [2Q]^m$, $\mathbf{u} \in [-U, U]$, and $\mathbf{r}, \mathbf{h} \in \mathbb{Z}$. Let

$$\mathcal{D}_N = \frac{1}{N^m} \sum_{\mathbf{q}, \mathbf{u}, \mathbf{r}, \mathbf{h}} (1 - \eta(\mathbf{r}, \mathbf{h})) \frac{1}{r_1 r_2 \cdots r_m} \int_0^1 \mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}) \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s) A_{\mathbf{h},\mathbf{r}}(s) ds.$$

With that, the following lemma establishes the main asymptotic needed to prove Lemma 3.1 (and thus Theorem 1.2).

LEMMA 5.1. *We have*

$$\lim_{N \rightarrow \infty} \mathcal{D}_N = \sum_{\mathcal{P} \in \mathcal{P}_m} \mathbf{E}(f^{|P_1|}) \cdots \mathbf{E}(f^{|P_d|}), \quad (5.1)$$

where the sum is over all non-isolating partitions of $[m]$, which we denote by $\mathcal{P} = (P_1, \dots, P_d)$.

Proof. Since the Fourier transform \widehat{f} is assumed to have compact support, we can evaluate the sum over \mathbf{u} and eliminate the factors $\mathfrak{K}_{\mathbf{u}}$. Hence,

$$\mathcal{D}_N = \frac{1}{N^m} \sum_{\mathbf{q}, \mathbf{r}, \mathbf{h}} 1(|\mu_i| > 0) (1 - \eta(\mathbf{r}, \mathbf{h})) \frac{1}{r_1 r_2 \cdots r_m} \int_0^1 \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s) A_{\mathbf{h},\mathbf{r}}(s) ds,$$

where the indicator function takes care of the fact that we extracted the contribution when $k_i = 0$.

The condition that the phase is zero is equivalent to a condition on \mathbf{h} and \mathbf{r} . Specifically, this happens in the following situation. Let \mathcal{P} be a non-isolating partition of $[m]$. We say that a vector (\mathbf{r}, \mathbf{h}) is \mathcal{P} -adjusted if for every $P \in \mathcal{P}$ we have $h_i = h_j$ for all $i, j \in P$, and $\sum_{i \in P} r_i = 0$. The diagonal is restricted to \mathcal{P} -adjusted vectors. Now

$$\chi_{\mathcal{P},1}(\mathbf{r}) := \begin{cases} 1 & \text{if } \sum_{i \in P} r_i = 0 \text{ for each } P \in \mathcal{P}, \\ 0 & \text{otherwise,} \end{cases} \quad \chi_{\mathcal{P},2}(\mathbf{h}) := \begin{cases} 1 & \text{if } h_i = h_j \text{ for all } i, j \in P \in \mathcal{P}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_{\mathcal{P},1}(\mathbf{r})\chi_{\mathcal{P},2}(\mathbf{h})$ encodes the condition that (\mathbf{r}, \mathbf{h}) is \mathcal{P} -adjusted. Thus, we may write

$$\mathcal{D}_N = \frac{1}{N^m} \sum_{\mathcal{P} \in \mathcal{P}_m} \sum_{\mathbf{q}, \mathbf{r}, \mathbf{h}} \chi_{\mathcal{P},1}(\mathbf{r})\chi_{\mathcal{P},2}(\mathbf{h}) \frac{1}{r_1 r_2 \cdots r_m} \left(\int_0^1 \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s) \widehat{f}\left(\frac{\boldsymbol{\mu}}{N}\right) \mu_1 \mu_2 \cdots \mu_m ds \right) + o(1).$$

Inserting the definition of μ_i then gives

$$\mathcal{D}_N = \frac{1}{N^m} \sum_{\mathcal{P} \in \mathcal{P}_m} \sum_{\mathbf{q}, \mathbf{r}, \mathbf{h}} \chi_{\mathcal{P},1}(\mathbf{r})\chi_{\mathcal{P},2}(\mathbf{h}) \int_0^1 \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s) \widehat{f}\left(\frac{\boldsymbol{\mu}}{N}\right) \prod_{i=1}^m \left(\frac{1}{\omega'(\omega^{-1}(h_i - s))} \right) ds + o(1).$$

Now note that the \mathbf{r} variable only appears in $\widehat{f}(\boldsymbol{\mu}/N)$, that is,

$$\begin{aligned} \mathcal{D}_N &= \frac{1}{N^m} \sum_{\mathcal{P} \in \mathcal{P}_m} \sum_{P \in \mathcal{P}} \sum_{\mathbf{q}, h} \int_0^1 \mathfrak{N}_{\mathbf{q}, P}(h) \left(\frac{1}{\omega'(\omega^{-1}(h))} \right)^{|P|} \\ &\quad \times \sum_{\substack{\mathbf{r} \in \mathbb{Z}^{|P|} \\ r_i \neq 0}} \chi(\mathbf{r}) \widehat{f}\left(\frac{1}{N\omega'(\omega^{-1}(h))} \mathbf{r}\right) ds (1 + o(1)), \end{aligned} \quad (5.2)$$

where $\chi(\mathbf{r})$ is 1 if $\sum_{i=1}^{|P|} r_i = 0$ and where $\mathfrak{N}_{\mathbf{q}, P}(h) = \prod_{i \in P} \mathfrak{N}_{q_i}(\mu_i, s)$. We can apply Euler's summation formula ([Apo76, Theorem 3.1]) to conclude that

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^{|P|} \\ r_i \neq 0}} \chi(\mathbf{r}) \widehat{f}\left(\frac{1}{N\omega'(\omega^{-1}(h))} \mathbf{r}\right) = \int_{\mathbb{R}^{|P|}} \chi(\mathbf{x}) \widehat{f}\left(\frac{1}{N\omega'(\omega^{-1}(h))} \mathbf{x}\right) d\mathbf{x} (1 + o(1)).$$

Changing variables then yields

$$\int_{\mathbb{R}^{|P|}} \chi(\mathbf{x}) \widehat{f}\left(\frac{1}{N\omega'(\omega^{-1}(h))} \mathbf{x}\right) d\mathbf{x} = N^{|P|-1} \omega'(\omega^{-1}(h))^{|P|-1} \int_{\mathbb{R}^{|P|}} \chi_{\mathcal{P}}(\mathbf{x}) \widehat{f}(\mathbf{x}) d\mathbf{x} (1 + O(N^{-\theta}));$$

note that $\chi(\mathbf{x})$ fixes $x_{|P|} = -\sum_{i=1}^{|P|-1} x_i$. Plugging this into our (5.2) gives

$$\mathcal{D}_N = \frac{1}{N^d} \sum_{\mathcal{P} \in \mathcal{P}_m} \sum_{P \in \mathcal{P}} \sum_{\mathbf{q}, h} \mathfrak{N}_{\mathbf{q}, P}(h) \omega'(\omega^{-1}(h)) \int_{\mathbb{R}^{|P|-1}} \widehat{f}(x_1, \dots, x_{|P|-1}, -\mathbf{x} \cdot \mathbf{1}) d\mathbf{x} (1 + o(1)).$$

Next, we may apply Euler's summation formula and a change of variables to conclude that

$$\mathcal{D}_N = \sum_{\mathcal{P} \in \mathcal{P}_m} \sum_{P \in \mathcal{P}} \left(\int_{\mathbb{R}^{|P|-1}} \widehat{f}(x_1, \dots, x_{|P|-1}, -\mathbf{x} \cdot \mathbf{1}) d\mathbf{x} \right) (1 + o(1)).$$

From there we apply Fourier analysis as in [LT21, Proof of Lemma 5.1] to conclude (5.1). \square

6. Bounding the off-diagonal

Recall that the off-diagonal contribution is given by

$$\mathcal{O}_N = \sum_{\mathbf{q}, \mathbf{u}} \frac{1}{N^m} \sum_{\mathbf{r}, \mathbf{h}} \eta(\mathbf{r}, \mathbf{h}) \int_0^1 \frac{\mu_1 \mu_2 \cdots \mu_m}{r_1 r_2 \cdots r_m} \widehat{f}\left(\frac{\boldsymbol{\mu}}{N}\right) \mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}) \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s) e(\Phi(\mathbf{h}, \mathbf{r}, s)) ds,$$

where $r_i \asymp e^{u_i - q_i} q_i^{A-1}$ and the variable $h_i \asymp q^A$. Finally, we have the phase function

$$\Phi(\mathbf{h}, \mathbf{r}, s) = -\sum_{i=1}^m r_i \exp((h_i - s)^{1/A}).$$

If we were to bound the oscillatory integral trivially, we would achieve the bound $\mathcal{O}_N \ll (\log N)^{(A+1)m}$. Therefore, all that is needed is a small power saving, for which we can exploit the oscillatory integral

$$I(\mathbf{h}, \mathbf{r}) := \int_0^1 \mathcal{A}_{\mathbf{h}, \mathbf{r}}(s) e(\Phi(\mathbf{h}, \mathbf{r}, s)) ds,$$

where

$$\mathcal{A}_{\mathbf{h}, \mathbf{r}}(s) := \frac{\mu_1 \mu_2 \cdots \mu_m}{r_1 r_2 \cdots r_m} \widehat{f}\left(\frac{\boldsymbol{\mu}}{N}\right) \mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}) \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}, s).$$

While bounding this integral is more involved in the present setting, we can nevertheless use the proof in [LT21, § 6] as a guide. In Proposition 6.1 we achieve a power-saving; for this reason we can ignore the sums over \mathbf{q} and \mathbf{u} which give a logarithmic number of choices.

Since we are working on the off-diagonal we may write the phase as

$$\Phi(\mathbf{h}, \mathbf{r}, s) = \sum_{i=1}^l r_i \exp((h_i - s)^{1/A}) - \sum_{i=l}^L r_i \exp((h_i - s)^{1/A}), \quad (6.1)$$

where we may now assume that $r_i > 0$, the h_i are pairwise distinct, and $L < m$. We restrict attention to the case $L = m$ (this is also the most difficult case and the other cases can be done analogously).

PROPOSITION 6.1. *Let Φ be as above. Then, for any $\varepsilon > 0$, we have*

$$I(\mathbf{h}, \mathbf{r}) \ll N^\varepsilon \mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}_0) \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}_0, 0) \frac{e^{u_1 + \cdots + u_m}}{r_1 \cdots r_m} N^{-1/m+\varepsilon}$$

as $N \rightarrow \infty$, where $\mu_{0,i} = r_i/\omega'(\omega^{-1}(h_i))$. The implied constants are independent of \mathbf{h} and \mathbf{r} provided $\eta_{\mathbf{r}}(\mathbf{h}) \neq 0$.

Proof. As in [LT21], we shall prove Proposition 6.1 by showing that one of the first m derivatives of Φ is small. Then we can apply van der Corput's lemma to the integral and achieve the necessary bound. However, since the phase function is a sum of exponentials (as oppose to a sum of monomials as it was in our previous work), achieving these bounds is significantly more involved than in [LT21].

The j th derivative is given by (we will send $s \mapsto -s$ to avoid having to deal with minus signs at the moment)

$$\begin{aligned} D_j &= \sum_{i=1}^m r_i \exp((h_i + s)^{1/A}) \\ &\quad \times \{A^{-j}(h_i + s)^{j/A-j} + c_{j,1}(h_i + s)^{(j-1)/A-j} + \cdots + c_{j,j-1}(h_i + s)^{(1/A-j)}\} \\ &=: \sum_{i=1}^m b_i P_j(h_i) \end{aligned}$$

where the $c_{j,k}$ depend only on A and j , where $b_i := r_i \exp((h_i + s)^{1/A})$.

In matrix form, let $\mathbf{D} := (D_1, \dots, D_m)$ denote the vector of the first m derivatives, and let $\mathbf{b} := (b_1, \dots, b_m)$. Then

$$\mathbf{D} = \mathbf{b}M, \quad \text{where } (M)_{ij} := P_j(h_i).$$

To prove Proposition 6.1 we will lower-bound the determinant of M . Thus, we will show that M is invertible, and hence we will be able to lower-bound the ℓ^2 -norm of \mathbf{D} . For this purpose,

consider the j th row of M ,

$$(M)_j = (P_j(h_1), \dots, P_j(h_m)).$$

We can write $P_j(h_1) := \sum_{k=0}^{j-1} c_{j,k}(h_i + s)^{t_k/A-j}$, where $t_k := j - k$. Since the determinant is multilinear in the rows, we can decompose the determinant of M as

$$\det(M) = \sum_{\mathbf{t} \in \mathcal{T}} c_{\mathbf{t}} \det(((h_i + s)^{t_j/A-j})_{i,j \leq m}), \quad (6.2)$$

where $c_{\mathbf{t}}$ are constants depending only on \mathbf{t} and the sum over \mathbf{t} ranges over the set

$$\mathcal{T} := \{\mathbf{t} \in \mathbb{N}^m : t_j \leq j, \forall j \in [1, m]\}.$$

Let $X_{\mathbf{t}} := ((h_i + s)^{t_j/A-j})_{i,j \leq m}$. We claim that $\det(M) = c_{\mathbf{t}_M} \det(X_{\mathbf{t}_M})(1 + O(\max_i(h_i^{-1/A})))$ as $N \rightarrow \infty$, where $\mathbf{t}_M := (1, 2, \dots, m)$.

To establish this claim, we appeal to the work of Khare and Tao; see Lemma 2.4. Namely, let $\mathbf{H} := (h_1 + s, \dots, h_m + s)$ with $h_1 > h_2 > \dots$ and let $\mathbf{T}(\mathbf{t}) := (t_1/A - 1, \dots, t_m/A - m)$. Then we can write

$$X_{\mathbf{t}} := \mathbf{H}^{\circ \mathbf{T}(\mathbf{t})}.$$

Now invoking Lemma 2.4, we have

$$\det(X_{\mathbf{t}}) \asymp V(\mathbf{H}) \mathbf{H}^{\mathbf{T}(\mathbf{t}) - \mathbf{n}_{\min}}.$$

Note that we may need to interchange the rows and/or columns of $X_{\mathbf{t}}$ to guarantee that the conditions of Lemma 2.4 are met. However, this will only change the sign of the determinant and thus will not affect the magnitude.

Now, fix $\mathbf{t} \in \mathcal{T}$ such that $\mathbf{t} \neq \mathbf{t}_M$ and compare

$$|\det(X_{\mathbf{t}_M})| - |\det(X_{\mathbf{t}})| \geq |V(\mathbf{H})| (|\mathbf{H}^{\mathbf{T}(\mathbf{t}_M) - \mathbf{n}_{\min}}| - |\mathbf{H}^{\mathbf{T}(\mathbf{t}) - \mathbf{n}_{\min}}|).$$

Since $\mathbf{t}_M \neq \mathbf{t}$ we conclude that all coordinates $t_i \leq (\mathbf{t}_M)_i$ and there exists a k such that $t_k < (\mathbf{t}_M)_k$. Therefore,

$$\begin{aligned} |\det(X_{\mathbf{t}_M})| - |\det(X_{\mathbf{t}})| &= |V(\mathbf{H}) \mathbf{H}^{\mathbf{T}(\mathbf{t}_M) - \mathbf{n}_{\min}}| (1 + O(\max_i(h_i^{-1/A}))) \\ &= |\det(X_{\mathbf{t}_M})| (1 + O(\max_i(h_i^{-1/A}))). \end{aligned}$$

This proves our claim.

Hence,

$$\begin{aligned} |\det(M)| &= |c_{\mathbf{t}_M} \det(X_{\mathbf{t}_M})| (1 + O(\max_i(h_i^{-1/A}))) \\ &= |c_{\mathbf{t}_M} V(\mathbf{H}) \mathbf{H}^{\mathbf{T}(\mathbf{t}_M) - \mathbf{n}_{\min}}| (1 + O(\max_i(h_i^{-1/A}))) \\ &= |c_{\mathbf{t}_M}| \left(\prod_{j=1}^m (h_j + s)^{j/A-2j+1} \right) \prod_{1 \leq i < j \leq m} (h_i - h_j) (1 + O(\max_i(h_i^{-1/A}))) \end{aligned} \quad (6.3)$$

which is clearly larger than 0 (since $h_i - h_j > 1$ and $s > -1$).

Hence, M is invertible, and we conclude that

$$\begin{aligned}\mathbf{D}M^{-1} &= \mathbf{b}, \\ \|\mathbf{D}\|_2 \|M^{-1}\|_{\text{spec}} &\geq \|\mathbf{b}\|_2, \\ \|\mathbf{D}\|_2 &\geq \frac{\|\mathbf{b}\|_2}{\|M^{-1}\|_{\text{spec}}},\end{aligned}$$

where $\|\cdot\|_{\text{spec}}$ denotes the spectral norm with respect to the ℓ^2 vector norm. Recall that $\|M^{-1}\|_{\text{spec}}$ is simply the largest eigenvalue of M^{-1} . Hence, $\det(M^{-1})^{1/m} \leq \|M^{-1}\|_{\text{spec}}$.

We can bound the spectral norm by the maximum norm,

$$\|M^{-1}\|_{\text{spec}} \ll \max_{i,j} (M^{-1})_{i,j}.$$

However, each entry of M^{-1} is equal to $1/\det(M)$ times a cofactor of M , by Cramer's rule. This, together with the size of the h_i , is enough to show that

$$\|\mathbf{D}\|_2 \gg \|\mathbf{b}\|_2 \log(N)^{-1000m}.$$

Now using the bounds on \mathbf{b} (which come from the essential ranges of h_i and r_i), we conclude that

$$\|\mathbf{D}\|_2 \gg N^{1-\varepsilon}.$$

From here we can apply the localized van der Corput's lemma [TY20, Lemma 3.3] as we did in [LT21] to conclude Proposition 6.1. \square

7. Proof of Lemma 3.1

Thanks to the preceding argument, we conclude that

$$\lim_{N \rightarrow \infty} \mathcal{K}_m(N) = \sum_{\mathcal{P} \in \mathcal{P}_m} \mathbf{E}(f^{|P_1|}) \cdots \mathbf{E}(f^{|P_d|}) + \lim_{N \rightarrow \infty} \mathcal{O}_N.$$

Moreover, the off-diagonal term can be bounded using Proposition 6.1 as follows:

$$\begin{aligned}\mathcal{O}_N &= \frac{1}{N^m} \sum_{\mathbf{q}, \mathbf{u}} \sum_{\mathbf{r}, \mathbf{h}} \eta(\mathbf{r}, \mathbf{h}) I(\mathbf{h}, \mathbf{r}) \\ &\ll \frac{1}{N^m} \sum_{\mathbf{q}, \mathbf{u}} \sum_{\mathbf{r}, \mathbf{h}} \mathfrak{K}_{\mathbf{u}}(\boldsymbol{\mu}_0) \mathfrak{N}_{\mathbf{q}}(\boldsymbol{\mu}_0, 0) \frac{e^{u_1 + \cdots + u_m}}{r_1 \cdots r_m} \max_{i \leq m} e^{-u_i/m} N^\varepsilon.\end{aligned}$$

Note that we are summing over reciprocals of r_i and recall that the h_i have size q_i^A , so we may evaluate the sums over \mathbf{h} and \mathbf{r} and gain at most a logarithmic factor (which can be absorbed into the ε). Thus,

$$\mathcal{O}_N \ll \frac{1}{N^m} \sum_{\mathbf{q}, \mathbf{u}} e^{u_1 + \cdots + u_m} \max_{i \leq m} e^{-u_i/m} N^\varepsilon.$$

Likewise there are logarithmically many \mathbf{q} and \mathbf{u} . Thus, maximizing the upper bound, we arrive at

$$\mathcal{O}_N \ll N^{-1/m+\varepsilon}.$$

This concludes our proof of Lemma 3.1. Hence, Theorem 1.3 follows from the argument in §3.

8. Deducing Theorem 1.2 from Theorem 1.3

To see why Theorem 1.3 implies Theorem 1.2, we focus first in §8.1 on the simplest case, $m = 2$. The spirit of approximation argument therein works for any $m \geq 2$. However, because the notation gets more complicated when $m \geq 3$, we present the general argument in §8.2. In any case, the following lemma is at the core of the matter.

LEMMA 8.1. *Let $f \in C_c^\infty(\mathbb{R})$ be positive. For any $\varepsilon > 0$, there exists a pointwise majorant F of f with*

$$\int_{\mathbb{R}} F(x) - f(x) dx < \varepsilon \quad \text{and} \quad \widehat{F} \in C_c^\infty. \quad (8.1)$$

A proof of this can be found in §8.3.

8.1 The case $m = 2$

Suppose we knew that

$$\lim_{N \rightarrow \infty} R^{(2)}(N, f) = \mathbf{E}(f), \quad \text{for any positive } f \in C_c^\infty. \quad (8.2)$$

Then, by the linearity of $f \mapsto R^{(2)}(N, f)$ and $f \mapsto \int_{\mathbb{R}} f(x) dx$, the convergence of $R^{(2)}(N, f) \rightarrow \mathbf{E}(f)$ follows for all $f \in C_c^\infty$. To verify (8.2) we show, for any positive $f \in C_c^\infty$, that

$$\limsup_{N \rightarrow \infty} R^{(2)}(N, f) \leq \mathbf{E}(f) \leq \liminf_{N \rightarrow \infty} R^{(2)}(N, f). \quad (8.3)$$

We first argue the estimate for the limit superior. Given $\varepsilon > 0$, Lemma 8.1 guarantees that there exists a pointwise majorant F of f satisfying (8.1). Observe that $R^{(2)}(N, f) - R^{(2)}(N, F) = R^{(2)}(N, f - F)$ and $R^{(2)}(N, f - F) = -R^{(2)}(N, F - f)$. As a result,

$$|R^{(2)}(N, f) - R^{(2)}(N, F)| \leq R^{(2)}(N, F - f). \quad (8.4)$$

Let $g := F - f$. Combining (8.4), Theorem 1.3, and (8.1), we see that

$$\limsup_{N \rightarrow \infty} R^{(2)}(N, f) \leq \limsup_{N \rightarrow \infty} R^{(2)}(N, F) + \limsup_{N \rightarrow \infty} R^{(2)}(N, g) \leq \mathbf{E}(f) + \varepsilon + \limsup_{N \rightarrow \infty} R^{(2)}(N, g). \quad (8.5)$$

Using Lemma 8.1, we find a pointwise majorant G of g with (8.1).

Applying Theorem 1.3 once more,

$$\limsup_{N \rightarrow \infty} R^{(2)}(N, g) \leq \limsup_{N \rightarrow \infty} R^{(2)}(N, G) = \mathbf{E}(G) \leq \mathbf{E}(g) + \varepsilon \leq 2\varepsilon. \quad (8.6)$$

By inserting this into (8.5) we obtain

$$\limsup_{N \rightarrow \infty} R^{(2)}(N, f) \leq \mathbf{E}(f) + 3\varepsilon, \quad (8.7)$$

for any choice of $\varepsilon > 0$. Hence, $\limsup_{N \rightarrow \infty} R^{(2)}(N, f) \leq \mathbf{E}(f)$. Next, we show the right-hand side of (8.3). We notice that (8.4) and (8.6) yield

$$\liminf_{N \rightarrow \infty} R^{(2)}(N, f) \geq \liminf_{N \rightarrow \infty} R^{(2)}(N, F) - \limsup_{N \rightarrow \infty} R^{(2)}(N, g) \geq \mathbf{E}(f) - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\liminf_{N \rightarrow \infty} R^{(2)}(N, f) \geq \mathbf{E}(f)$ follows.

8.2 linearity-based $m \geq 2$

Let $f \in C_c^\infty(\mathbb{R}^{m-1})$ be positive, which we may assume by a linearity-based argument. Let $\varepsilon > 0$. It suffices to demonstrate that

$$\limsup_{N \rightarrow \infty} R^{(m)}(N, f) \leq \mathbf{E}(f) + 2\varepsilon \quad \text{and} \quad \mathbf{E}(f) - 2\varepsilon \leq \liminf_{N \rightarrow \infty} R^{(m)}(N, f). \quad (8.8)$$

Up to an error of at most ε , we can approximate f with respect to the supremum norm on \mathbb{R}^{m-1} by a finite linear combination of functions \tilde{f} each of which is the form $\tilde{f}(\mathbf{x}) := f_1(x_1) \cdots f_{m-1}(x_{m-1})$, where all $f_i \in C_c^\infty(\mathbb{R})$. Thus, it suffices to establish (8.8) for \tilde{f} in place of f . Given $1 \leq i < m$, Lemma 8.1 implies that there exists a pointwise majorant F_i of f_i satisfying (8.1). As a result, $F(\mathbf{x}) := F_1(x_1) \cdots F_{m-1}(x_{m-1})$ is a pointwise majorant of \tilde{f} . By using the identity $ab - a'b' = a(b - b') + (a - a')b'$ inductively, we conclude that there exists a constant $C_{\tilde{f}} > 0$ so that

$$\int_{\mathbb{R}^{m-1}} F(\mathbf{x}) - \tilde{f}(\mathbf{x}) d\mathbf{x} < C_{\tilde{f}} \cdot \varepsilon$$

no matter what our initial choice of the small parameter ε was. Since $\widehat{F}(\mathbf{x}) = \widehat{F}_1(x_1) \cdots \widehat{F}_{m-1}(x_{m-1})$ is the product of compactly supported functions, $\widehat{F} \in C_c^\infty(\mathbb{R}^{m-1})$. Notice that

$$|R^{(m)}(N, f) - R^{(m)}(N, F)| \leq R^{(m)}(N, F - f).$$

From here we reason exactly as we did in the case $m = 2$ and thus we infer (8.8).

8.3 Proof of Lemma 8.1

The purpose of this subsection is to establish Lemma 8.1. We do so using arguments inspired by Marklof [Mar03, Section 8.6]. First, chose a $\delta > 0$ small enough to ensure that

$$\int_{\mathbb{R}} \frac{\delta}{1+x^2} dx < \frac{\varepsilon}{2}.$$

Next, we perturb f slightly by considering an auxiliary function h defined via its Fourier transform

$$\widehat{h}(x) := f(x) + \frac{\delta}{1+x^2}.$$

The reason for working with this perturbation is to create the leeway needed to ensure that the final approximation F is indeed positive. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be an even, smooth function with $\text{supp}(\chi) \subseteq [-2, 2]$ and $\chi(x) = 1$ for $x \in [-1, 1]$. We consider

$$h_T(x) := h(x)\chi\left(\frac{x}{T}\right),$$

and proceed to argue that there exists a cut-off $T > 0$ so that

$$|\widehat{h_T}(x) - \widehat{h}(x)| < \frac{\delta}{1+x^2} \quad (8.9)$$

for all $x \in \mathbb{R}$. Observe that (8.9) is trivially true at $x = 0$. From now on let $x \neq 0$. By using partial integration twice, we conclude

$$\begin{aligned} \widehat{h_T}(x) - \widehat{h}(x) &= \int_{\mathbb{R}} \left[h(y)\chi\left(\frac{y}{T}\right) - h(y) \right] \left(\frac{d^2}{dy^2} \frac{e(-yx)}{(-2\pi ix)^2} \right) dy \\ &= \frac{1}{(-2\pi ix)^2} \int_{\mathbb{R}} \left(\frac{d^2}{dy^2} \left[h(y)\chi\left(\frac{y}{T}\right) - h(y) \right] \right) e(-yx) dy. \end{aligned}$$

Furthermore,

$$\frac{d^2}{dy^2} \left(h(y) \left(1 - \chi\left(\frac{y}{T}\right) \right) \right) = h''(y) \left(1 - \chi\left(\frac{y}{T}\right) \right) + 2\frac{1}{T} h'(y) \chi'\left(\frac{y}{T}\right) + \frac{1}{T^2} h(y) \chi''\left(\frac{y}{T}\right). \quad (8.10)$$

Let $j \geq 0$ be an integer. To estimate the right-hand side of (8.10) efficiently, our next goal is to control the decay of the j th derivative of h uniformly. To this end, thanks to Fourier inversion,

we see that

$$h^{(j)}(y) = \int_{\mathbb{R}} \widehat{h}(\xi) (2\pi i \xi)^j e(y\xi) d\xi.$$

Let $y \neq 0$. By using partial integration $t \geq j$ many times,

$$\begin{aligned} |h^{(j)}(y)| &= \left| \int_{\mathbb{R}} \widehat{h}(\xi) (2\pi i \xi)^j \frac{d^t}{d\xi^t} \left(\frac{e(y\xi)}{(2\pi i y)^t} \right) d\xi \right| \leq \frac{(2\pi)^j}{|2\pi y|^t} \cdot \int_{\mathbb{R}} \left| \frac{d^t}{d\xi^t} (\xi^j \widehat{h}(\xi)) \right| d\xi \\ &\leq \frac{1}{|y|^t} \int_{\mathbb{R}} \left| \frac{d^t}{d\xi^t} \left(\xi^j f(\xi) + \frac{\delta \xi^j}{1 + \xi^2} \right) \right| d\xi. \end{aligned}$$

By applying Leibniz's rule we conclude that, for any integers $t \geq j \geq 0$, there exists a constant $C(j, t, f) > 0$ so that $|h^{(j)}(y)| \leq C(j, t, f) |y|^{-t}$ for any $y \neq 0$. Taking this information in (8.10) into account yields

$$\frac{d^2}{dy^2} \left(h(y) \left(1 - \chi \left(\frac{y}{T} \right) \right) \right) \ll \frac{1}{y^2} \left[1 - \chi \left(\frac{y}{T} \right) + \frac{1}{T} \chi' \left(\frac{y}{T} \right) + \frac{1}{T^2} \chi'' \left(\frac{y}{T} \right) \right] \ll \frac{1}{y^2}$$

whenever $y \neq 0$. Here the implied constants depend only on f and χ . Because the left-hand side of (8.10) is supported on $|y| \geq T$, it follows that

$$|\widehat{h_T}(x) - \widehat{h}(x)| \ll \frac{1}{|x|^2} \int_{\mathbb{R}} \left| \frac{d^2}{dy^2} \left(h(y) \left(1 - \chi \left(\frac{y}{T} \right) \right) \right) \right| dy \ll \frac{1}{|x|^2} \int_{|y| > T} \frac{1}{y^2} dy \ll \frac{1}{|x|^2 T}.$$

Upon choosing T large enough to overwhelm the implicit constant, we deduce that (8.9) holds true.

Now put $F(x) := \widehat{h_T}(x)$. By the convolution theorem, $F(x) = \int_{\mathbb{R}} \widehat{h}(x - y) \widehat{\chi}(y) dy$. Notice that \widehat{h} and $\widehat{\chi}$ are real-valued (the latter being the Fourier transform of an even, real-valued function). Therefore, F is real-valued. To proceed, we notice that the triangle inequality implies

$$f(x) = \widehat{h}(x) - \frac{\delta}{1 + x^2} < \widehat{h}(x) + (\widehat{h_T}(x) - \widehat{h}(x)) = F(x).$$

Hence, F is a pointwise majorant of f and, in particular, positive. Furthermore,

$$\int_{\mathbb{R}} F(x) - f(x) dx < \int_{\mathbb{R}} |\widehat{h_T}(x) - \widehat{h}(x)| dx + \int_{\mathbb{R}} \frac{\delta}{1 + x^2} dx < \varepsilon.$$

Appendix A. A sublacunary partition of unity

Choose large integers $N, Q \geq 10$, with $e^Q \leq N < e^{Q+1}$. In this section we justify that the functions \mathfrak{N}_q , where $0 \leq q \leq 2Q - 1$, from § 3.2 exist. We assume in the following that Q is an odd integer. The case that Q is even can be handled very similarly. We start from the disjoint decomposition

$$[1, N] = [1, e^{Q-1}) \cup [e^{Q-1}, N].$$

To construct a suitable partition of unity for the first and the second set we require somewhat different treatment. However, the main idea is rather simple, and the mechanics are the same for both. We begin with the former. We cut $[1, e^{Q-1})$ into a union of the dyadic intervals $[e^q, e^{q+1})$. Then we smooth out the corners of their indicator functions in the following way. We smooth every second indicator function (changing their support only marginally), and adjust the remaining ones so that neighboring functions always sum to 1. Furthermore, to decompose $[e^{Q-1}, N]$ in a sublacunary manner, we use a similar strategy. This time we cut the interval into a union

of sublacunary intervals, smoothing the ones with odd indices and then adjusting the corners of the even ones appropriately.

To execute this plan, fix a smooth function $\beta : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ which is supported in a compact interval. For example,

$$\beta(x) := \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{for } 0 \leq |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

is a viable choice. By translating (if need be) and scaling, we modify β to obtain a function $B : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ which is smooth, supported in $[-1/100, 1/100]$, and L^1 -normalized, that is,

$$\int_{\mathbb{R}} B(x) dx = 1.$$

Denote the convolution of functions $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(g_1 * g_2)(x) := \int_{\mathbb{R}} g_1(x-y)g_2(y) dy.$$

Clearly, if g_1 is j times continuously differentiable, then $g_1 * g_2$ is j times continuously differentiable and

$$(g_1 * g_2)^{(j)} = g_1^{(j)} * g_2. \quad (\text{A.1})$$

A.1 The regime $0 \leq q < Q$

Denote the indicator function of an interval I by $\mathbf{1}_I$. The smooth function $B_y(x) := e^{-y}B(e^{-y}x)$ is important in what follows, noting in particular that it is supported in the interval $[-e^y/100, e^y/100]$. For even index $q = 2t \in \{0, \dots, Q-1\}$, we let

$$\mathfrak{N}_{2t}(x) := (\mathbf{1}_{[e^q, e^{q+1}]} * B_q)(x).$$

These functions inherit the smoothness of B . By (A.1), we have

$$\|\mathfrak{N}_{2t}^{(j)}\|_{\infty} \ll e^{-qj} \quad (\text{A.2})$$

for each integer $j \geq 0$. (The implied constant is allowed to depend on the choice of B , which, however, is fixed throughout and therefore this dependency is suppressed.) Next, for every odd index $q = 2t+1 \in \{0, \dots, Q-1\}$ we define

$$\mathfrak{N}_{2t+1}(x) := \begin{cases} 0, & \text{if } x < 0.98 \cdot e^{2t}, \\ 1 - \mathfrak{N}_{2t}(x), & \text{if } 0.98 \cdot e^{2t} \leq x < 1.02 \cdot e^{2t}, \\ 1, & \text{if } 1.02 \cdot e^{2t} \leq x < 0.98 \cdot e^{2t+2}, \\ 1 - \mathfrak{N}_{2t+2}(x), & \text{if } 0.98 \cdot e^{2t+2} \leq x < 1.02 \cdot e^{2t+2}, \\ 0, & \text{if } e^{2t+2} \leq x. \end{cases}$$

Since \mathfrak{N}_{2t+1} is given piecewise by concatenating five smooth functions, we can infer that \mathfrak{N}_{2t+1} is a smooth function as soon as we establish that the four relevant boundary points do not cause issues. A quick inspection reveals that \mathfrak{N}_q , for any $q < Q$, is supported in $[0.98 \cdot e^q, 1.02 \cdot e^{q+1}] \subseteq [e^q/2, 3e^q]$. Hence, \mathfrak{N}_{2t+1} is smooth. Furthermore, \mathfrak{N}_{2t+1} is monotonically increasing (though not always strictly) on the interval $[0.98 \cdot e^q, 1.02 \cdot e^q]$, equal to 1 on $[1.02 \cdot e^q, 0.98 \cdot e^{q+1}]$, and monotonically decreasing (not always strictly) on $[0.98 \cdot e^{q+1}, 1.02 \cdot e^{q+1}]$. The function \mathfrak{N}_{2t} has

analogous monotonicity properties. Hence, \mathfrak{N}'_q also has exactly one sign change. By construction,

$$\mathfrak{N}_q(x) + \mathfrak{N}_{q+1}(x) = 1$$

whenever $e^q \leq x \leq e^{q+1}$, irrespective of the parity of q . Moreover, (A.2) implies that

$$\|\mathfrak{N}_{2t+1}^{(j)}\|_\infty \ll e^{-qj}. \quad (\text{A.3})$$

Consequently, the family of functions $\{\mathfrak{N}_q : 0 \leq q < Q\}$ satisfies (3.7).

A.2 The regime $Q \leq q < 2Q$

Put $\mathcal{Q} := 2.8 \cdot e^Q / Q$. To render the subsequent formulas more transparent, define $a_t := e^Q + (t - 1/2)\mathcal{Q}$. Observe that

$$[e^{Q-1}, N) = [e^{Q-1}, e^Q) \cup [e^Q, N) \subset [e^{Q-1}, e^Q) \cup \bigcup_{0 \leq t \leq T+1} \mathcal{I}_t, \quad \text{where } \mathcal{I}_t := [a_t, a_t + \mathcal{Q}),$$

and where T is the unique integer such that $N \in \mathcal{I}_T$. In fact, due to the fact that $e/2.8 < 0.971$, we see that $0 \leq T \leq 0.98Q$. Informally, our next step is to again smooth out every other partition interval in the aforementioned cover. However, this time we need to be mindful at the end points of the partition which, necessarily, play a distinguished role. The end points will be handled at the end of this paragraph, and we first deal with the bulk of the intervals. Writing any even index $q \in \{Q+1, Q+T+1\}$ in the form $q = Q + 2t + 1$, with $0 \leq t \leq T$, we define

$$\mathfrak{N}_{Q+2t+1}(x) := (\mathbf{1}_{\mathcal{I}_{2t+1}} * B_{\mathcal{Q}})(x).$$

Next, writing any odd index $q \in \{Q+2, Q+T+1\}$ in the form $q = Q + 2t$, with $1 \leq t \leq T$, we put

$$\mathfrak{N}_{Q+2t}(x) := \begin{cases} 0, & \text{if } x < a_{2t-2} - 0.02 \cdot \mathcal{Q}, \\ 1 - \mathfrak{N}_{Q+2t-2}(x), & \text{if } a_{2t-2} - 0.02 \cdot \mathcal{Q} \leq x < a_{2t-2} + 0.02 \cdot \mathcal{Q}, \\ 1, & \text{if } a_{2t-2} + 0.02 \cdot \mathcal{Q} \leq x < a_{2t+2} - 0.02 \cdot \mathcal{Q}, \\ 1 - \mathfrak{N}_{Q+2t+2}(x) & \text{if } a_{2t+2} - 0.02 \cdot \mathcal{Q} \leq x < a_{2t+2} + 0.02 \cdot \mathcal{Q}, \\ 0, & \text{if } a_{2t+2} + 0.02 \cdot \mathcal{Q} \leq x. \end{cases}$$

The function \mathfrak{N}_Q plays a distinguished role as it is the final partition element and of a different scale of unity given provided by the functions $\{\mathfrak{N}_q : 0 \leq q < Q\}$ and the sublacunary one furnished by $\{\mathfrak{N}_q : Q < q < 2Q - 1\}$. To smoothly link these two, we let

$$\mathfrak{N}_Q(x) := \begin{cases} 0, & \text{if } x < 0.98e^Q, \\ 1 - \mathfrak{N}_{Q-1}(x), & \text{if } 0.98e^Q \leq x < 0.98e^Q, \\ 1, & \text{if } a_{2t-1} + 00.2 \cdot \mathcal{Q} \leq x < a_{2t+1} - 00.2 \cdot \mathcal{Q}, \\ 1 - \mathfrak{N}_{Q+1}(x) & \text{if } a_{2t+1} - 00.2 \cdot \mathcal{Q} \leq x < a_{2t+1} + 00.2 \cdot \mathcal{Q}, \\ 0, & \text{if } a_{2t+1} + 00.2 \cdot \mathcal{Q} \leq x. \end{cases}$$

To formally complete the construction, we put

$$\mathfrak{N}_q(x) := 0, \quad \text{for } q \in \{Q+T+2, \dots, 2Q-1\}.$$

By arguing very much as in the regime $0 \leq q < Q$, one can check that $\{\mathfrak{N}_q : Q \leq q < 2Q\}$ has the required properties and, in particular, is such that (3.7) holds true.

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CONFLICTS OF INTEREST

None.

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REFERENCES

- AEM21 C. Aistleitner, D. El-Baz and M. Munsch, *A pair correlation problem, and counting lattice points with the zeta function*, Geom. Funct. Anal. **31** (2021), 483–512.
- Apo76 T. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics (Springer, Berlin, 1976).
- BKY13 V. Blomer, R. Khan and M. Young, *Distribution of mass of holomorphic cusp forms*, Duke Math. J. **162** (2013), 2609–2644.
- EMV15 D. El-Baz, J. Marklof and I. Vinogradov, *The two-point correlation function of the fractional parts of \sqrt{n} is Poisson*, Proc. Amer. Math. Soc. **143** (2015), 2815–2828.
- EM04 N. Elkies and C. McMullen, *Gaps in $\sqrt{n} \bmod 1$ and ergodic theory*, Duke Math. J. **123** (2004), 95–139.
- Hea10 D. R. Heath-Brown, *Pair correlation for fractional parts of αn^2* , Math. Proc. Cambridge Philos. Soc. **148** (2010), 385–407.
- KT21 A. Khare and T. Tao, *On the sign patterns of entrywise positivity preservers in fixed dimension*, Amer. J. Math. **143** (2021), 1863–1929.
- KN74 L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics (John Wiley & Sons, New York–London–Sydney, 1974).
- LST24 C. Lutsko, A. Sourmelidis and N. Technau, *Pair correlation of the fractional parts of αn^θ* , J. Eur. Math. Soc. (JEMS) (2024), doi:10.4171/jems/1449.
- LT21 C. Lutsko and N. Technau, *Correlations of the fractional parts of αn^θ* , Preprint (2021), [arXiv:2112.11524](https://arxiv.org/abs/2112.11524).
- Mar03 J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. of Math. (2) **158** (2003), 419–471.
- MS13 J. Marklof and A. Strömbergsson, *Gaps between logs*, Bull. Lond. Math. Soc. **45** (2013), 1267–1280.
- RS98 Z. Rudnick and P. Sarnak, *The pair correlation function of fractional parts of polynomials*, Comm. Math. Phys. **194** (1998), 61–70.

- RT22 Z. Rudnick and N. Technau, *The metric theory of the pair correlation function for small non-integer powers*, J. Lond. Math. Soc. (2) **106** (2022), 2752–2772.
- Ste93 E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 3 (Princeton University Press, Princeton, NJ, 1993).
- TY20 N. Technau and N. Yesha, *On the correlations of $\alpha n^\theta \bmod 1$* , J. Eur. Math. Soc. (JEMS) **25** (2020), 4123–4154.
- Wey16 H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352.

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