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SOLUTIONALLY COMPLETE VARIETIES

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Abstract

A variety $\mathfrak{A}$ is called solutionally complete if any solvable system of algebraic equations over an algebra A in $\mathfrak{A}$ which has at most one solution in every extension of A in $\mathfrak{A}$ has the solution in A. A necessary and sufficient condition for solutional completeness is given which is a weaker form of the strong amalgamation property.

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Let A be a universal algebra in the variety $\mathfrak{A}$, $X = \{x_1, \ldots, x_n\}$ a finite set of indeterminates, and $W(A, X)$ the word algebra in X over A in the sense of Lausch and Nöbauer (1973). A system of algebraic equations (shortly 'algebraic system') in X over A is a family of pairs of elements of $W(A, X)$. An algebraic system $\langle (p_i(x_1, \ldots, x_n), q_i(x_1, \ldots, x_n)\rangle_{i \in I}$ is 'solvable over $(A, \mathfrak{A})$' if there exist a $\mathfrak{A}$-extension B of A (that is, an algebra $B \in \mathfrak{A}$ which has A as a subalgebra) and elements $b_1, \ldots, b_n \in B$ such that $p_i(b_1, \ldots, b_n) = q_i(b_1, \ldots, b_n)$ for all $i \in I$.

Lausch and Nöbauer (1973) posed the following problem: If S is an algebraic system solvable over $(A, \mathfrak{A})$ which has at most one solution in every $\mathfrak{A}$-extension of A, is then the unique solution of S in A?

Negative answers to this question have been given independently by Taylor (1976) and Hule (1976). In some well-known varieties, however (for instance, the varieties of groups and of lattices), the answer is affirmative. Such varieties are called 'solutionally complete'.

We want to give a necessary and sufficient condition for solutional completeness. For this purpose we consider simple extensions of an algebra A, that is, algebras generated by $A \cup \{b\}$ for some $b \notin A$. Such an algebra will be denoted by $A(b)$. Two simple extensions $A(b)$ and $A(c)$ will be called 'isomorphic over A' if there exists an isomorphism $\tau\colon A(b) \to A(c)$ such that $\tau a = a$ for every $a \in A$ and $\tau b = c$.

THEOREM 1. *A variety $\mathfrak{A}$ is solutionally complete if and only if the following condition* (A) *holds:*

(A) *For any algebra $A \in \mathfrak{A}$ and any two simple $\mathfrak{A}$-extensions $A(b)$, $A(c)$ isomorphic over A there exist a $\mathfrak{A}$-extension D of A and homomorphisms φ: $A(b) \to D$, ψ: $A(c) \to D$ such that φ and ψ fix A and $\varphi b \neq \psi c$.*

PROOF. Suppose that $\mathfrak{A}$ satisfies condition (A). In Hule (1976) it is shown that a variety is solutionally complete if the definition is satisfied for algebraic systems in one indeterminate. So let $S = \langle p_i(x), q_i(x) \rangle_{i \in I}$ be an algebraic system in $\{x\}$ over A which is solvable over $(A, \mathfrak{A})$ and has no solution in A. Then the system has a solution b in some $\mathfrak{A}$-extension B of A. We take the subalgebra $A(b)$ of B and construct an isomorphic copy $A(c)$ of $A(b)$ by renaming the elements of $A(b) - A$ and defining an isomorphism τ such that $\tau b = c$ and $\tau a = a$ for every $a \in A$. Then $A(b)$ and $A(c)$ are simple $\mathfrak{A}$-extensions isomorphic over A. Let D, φ, ψ be as in the theorem. Then $p_i(b) = q_i(b)$ implies $p_i(\varphi b) = q_i(\varphi b)$ and $p_i(\psi c) = q_i(\psi c)$ for every $i \in I$. Hence φb and ψc are two different solutions of S in D. This shows that $\mathfrak{A}$ is solutionally complete.

Now suppose that condition (A) is not satisfied in $\mathfrak{A}$. Then there exist an algebra $A \in \mathfrak{A}$ and a pair of simple $\mathfrak{A}$-extensions $A(b)$, $A(c)$ isomorphic over A such that for any $\mathfrak{A}$-extension D of A and any pair of homomorphisms φ: $A(b) \to D$ and ψ: $A(c) \to D$ which fix A, $\varphi b = \psi c$. By Lemma 4.43, Chap. 1 of Lausch and Nöbauer (1973) there exist a homomorphism λ from the polynomial algebra $A(\{x\}, \mathfrak{A})$ onto $A(b)$ with $\lambda x = b$ and $\lambda a = a$ for every $a \in A$, and a homomorphism μ from $A(\{y\}, \mathfrak{A})$ onto $A(c)$ with $\mu y = c$ and $\mu a = a$ for every $a \in A$ (we assume $x \neq y$, without loss of generality). Let β be the kernel of λ and γ the kernal of μ. Now we define $D = A(\{x, y\}, \mathfrak{A})/\delta$, where δ is the congruence on $A(\{x, y\}, \mathfrak{A})$ generated by $\beta \cup \gamma$. This means that $p(x, y) \equiv q(x, y) \bmod \delta$ holds if and only if there exists a sequence of words $w_0, w_1, \ldots, w_k \in W(A, \{x, y\})$ such that w_0 is a word representation of $p(x, y)$, w_k represents $q(x, y)$, and for any $i \in \{1, \ldots, k\}$ either w_{i-1} and w_i represent the same polynomial or w_i is obtained from w_{i-1} replacing some subword u of w_{i-1} by v where u and v represent polynomials congruent under β or under γ. We first show that δ separates the elements of A. Assume a_1, a_2 to be different elements of A and $w_0, w_1, \ldots, w_k$ a sequence of words as above which establishes $a_1 \equiv a_2 \bmod \delta$. For each i, let $\bar{w}_i$ be the element of $A(b)$ obtained from w_i by substituting each occurrence of x or y by b and performing the operations in $A(b)$. Then $a_1 = \bar{w}_0 = \bar{w}_1 = \ldots = \bar{w}_k = a_2$. Therefore, we can consider D as a $\mathfrak{A}$-extension of A. Since $\beta \subseteq \delta$ and $\gamma \subseteq \delta$, the canonical homomorphism χ from $A(\{x, y\}, \mathfrak{A})$ onto D induces homomorphisms φ: $A(b) \to D$ and ψ: $A(c) \to D$ defined by $\varphi\lambda x = \chi x$, $\psi\mu y = \chi y$, $\varphi a = \psi a = a$ for every $a \in A$. By hypothesis we have $\varphi b = \psi c$ and conclude

$$\chi x = \varphi\lambda x = \varphi b = \psi c = \psi\mu y = \chi y,$$

whence $x \equiv y \bmod \delta$. Now let $\{(p_i, q_i)\ \ i \in I\}$ be a generating set of the congruence β. Then the family $S = \langle (p_i, q_i) \rangle_{i \in I}$ (where we identify the polynomials p_i, q_i with corresponding words) is an algebraic system over A which has the solution b in $A(b)$, not in A. We now show that S has at most one solution in any $\mathfrak{A}$-extension of A, which will prove that $\mathfrak{A}$ is not solutionally complete. Let e and e' be solutions of S in some $\mathfrak{A}$-extension E of A. Then $u(x) \equiv v(x) \bmod \beta$ implies $u(e) = v(e)$ and $u(y) \equiv v(y) \bmod \gamma$ implies $u(e') = v(e')$. Let $w_0, w_1, \ldots, w_k$ be a sequence of words which establishes the relation $x \equiv y \bmod \delta$. This sequence is converted into a chain of equal elements of E if we substitute x by e and y by e'. This completes the proof of the theorem.

As an immediate consequence of Theorem 1 we get a sufficient condition for solutional completeness found previously by Hule (1976).

COROLLARY. *A variety $\mathfrak{A}$ is solutionally complete if it satisfies the following condition* (B):

(B) *If* B *and* C *are $\mathfrak{A}$-extensions of an algebra $A \in \mathfrak{A}$, then there exists an algebra $D \in \mathfrak{A}$ which is a common extension of B and C.*

Condition (B) is usually called the 'strong amalgamation property'. Actually, it suffices that the strong amalgamation property hold for simple $\mathfrak{A}$-extensions of A isomorphic over A. A counterexample in Hule (1978) shows that this condition is not necessary for solutional completeness.

We want to prove a generalization of the preceding result for algebraic systems in arbitrary (not necessarily finite) sets of indeterminates. Thus, we consider an algebraic system $\langle (p_i(X), q_i(X)) \rangle_{i \in I}$ where $X = \{x_j \mid j \in J\}$ is an arbitrary set of indeterminates and $p_i(X)$, $q_i(X)$ are elements of $W(A, X)$. The system is 'solvable over $(A, \mathfrak{A})$' if there exist a $\mathfrak{A}$-extension E of A and a family $\bar{E} = (e_j)_{j \in J}$ of elements of E such that $p_i(\bar{E}) = q_i(\bar{E})$ for all $i \in I$. (For any $p(X) \in W(A, X)$, $p(\bar{E})$ is the element of E which we obtain by substituting each x_j occurring in $p(X)$ by e_j and performing the operations in E.)

A variety is called 'strongly solutionally complete' if the following condition holds: If S is an algebraic system (in an arbitrary set of indeterminates) solvable over $(A, \mathfrak{A})$ which has at most one solution in every $\mathfrak{A}$-extension of A, then the unique solution of S consists of elements of A.

In order to establish a necessary and sufficient condition for strong solutional completeness, we consider extensions $A(B)$ of A where $A(B)$ is generated by $A \cup B$ and $A \cap B = \emptyset$. Two extensions $A(B)$ and $A(C)$ will be called 'isomorphic over A' if there exists an isomorphism $\tau: A(B) \to A(C)$ which takes B onto C and such that $\tau a = a$ for every $a \in A$. When considering extensions $A(B)$ and $A(C)$ isomorphic over A, we shall always assume that an isomorphism τ with the required properties is defined.

THEOREM 2. *A variety* $\mathfrak{A}$ *is strongly solutionally complete if and only if the following condition* (C) *holds:*

(C) *For any algebra* $A \in \mathfrak{A}$ *and any two* $\mathfrak{A}$*-extensions* $A(B)$, $A(C)$ *isomorphic over* A *there exist a* $\mathfrak{A}$*-extension* D *of* A *and homomorphisms* $\varphi: A(B) \to D$, $\psi: A(C) \to D$ *such that* φ *and* ψ *fix* A *and* $\varphi b \neq \psi\tau b$ *for at least one* $b \in B$.

PROOF. Suppose that $\mathfrak{A}$ satisfies condition (C). Let $S = \langle p_i(X), q_i(X)\rangle_{i\in I}$ be an algebraic system in $X = \{x_j \mid j \in J\}$ solvable over $(A, \mathfrak{A})$ which has no solution in A (which means that no family $(a_j)_{j\in J}$ with $a_j \in A$ is a solution). Then S has a solution $\bar{B} = (b_j)_{j\in J}$ in some $\mathfrak{A}$-extension E of A where $b_j \notin A$ for at least one j. Let $B = \{b_j \mid j \in J$ and $b_j \notin A\}$. Then we take the subalgebra $A(B)$ of E and construct an isomorphic copy $A(C)$ of $A(B)$ by renaming the elements of $A(B) - A$ and defining an isomorphism $\tau: A(B) \to A(C)$ such that the two extensions are isomorphic over A. Then we define $\bar{C} = (c_j)_{j\in J}$ where $c_j = \tau b_j$ if $b_j \in B$ and $c_j = b_j$ if $b_j \in A$. Let D, φ, ψ be as in condition (C), $\varphi\bar{B} = (\varphi b_j)_{j\in J}$ and $\psi\bar{C} = (\psi c_j)_{j\in J}$. Then $p_i(\bar{B}) = q_i(\bar{B})$ implies $p_i(\varphi\bar{B}) = q_i(\varphi\bar{B})$ and $p_i(\psi\bar{C}) = q_i(\psi\bar{C})$ for every $i \in I$. Hence $\varphi\bar{B}$ and $\psi\bar{C}$ are two solutions of S in D which are different because $\varphi b_j \neq \psi\tau b_j = \psi c_j$ for at least one $b_j \in B$. This shows that $\mathfrak{A}$ is strongly solutionally complete.

Now suppose that condition (C) is not satisfied in $\mathfrak{A}$. Then there exist an algebra $A \in \mathfrak{A}$ and a pair of $\mathfrak{A}$-extensions $A(B)$, $A(C)$ isomorphic over A such that for any $\mathfrak{A}$-extension D of A and any pair of homomorphisms $\varphi: A(B) \to D$ and $\psi: A(C) \to D$ which fix A, $\varphi b = \psi\tau b$ for every $b \in B$. We index B and C by an appropriate set J such that $B = \{b_j \mid j \in J\}$, $C = \{c_j \mid j \in J\}$ and $c_j = \tau b_j$ for all $j \in J$, then we take disjoint sets of indeterminates $X = \{x_j \mid j \in J\}$ and $Y = \{y_j \mid j \in J\}$. There exist a homomorphism λ from $A(X, \mathfrak{A})$ onto $A(B)$ with $\lambda x_j = b_j$ for every $j \in J$ and $\lambda a = a$ for every $a \in A$, and a homomorphism μ from $A(Y, \mathfrak{A})$ onto $A(C)$ with $\mu y_j = c_j$ for $j \in J$ and $\mu a = a$ for $a \in A$. Let β be the kernel of λ, γ the kernel of μ, and $D = A(X \cup Y, \mathfrak{A})/\delta$, where δ is the congruence on $A(X \cup Y, \mathfrak{A})$ generated by $\beta \cup \gamma$. Like in the proof of Theorem 1, taking for $\bar{w}_i$ the element of $A(B)$ obtained by substituting each x_j or y_j occurring in w_i by b_j, we see that δ separates A, and hence we consider D as an extension of A. Homomorphisms $\varphi: A(B) \to D$ and $\psi: A(C) \to D$ are defined by the conditions $\varphi\lambda x_j = \chi x_j$, $\psi\mu y_j = \chi y_j$ and $\varphi a = \psi a = a$ for every $a \in A$, where χ is the canonical homomorphism from $A(X \cup Y, \mathfrak{A})$ onto D. By hypothesis we have $\varphi b_j = \psi\tau b_j = \psi c_j$ for every j which implies $x_j \equiv y_j \bmod \delta$. Also the rest of the proof is analogous to that of Theorem 1. The algebraic system S constructed as in that proof has the solution $\bar{B} = (b_j)_{j\in J}$ in $A(B)$, not in A, and for two arbitrary solutions of S in some $\mathfrak{A}$-extension of A, $(e_j)_{j\in J}$ and $(e'_j)_{j\in J}$, we deduce $e_j = e'_j$ from $x_j \equiv y_j \bmod \delta$. So $\mathfrak{A}$ is not strongly solutionally complete.

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