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NOTE ON SUMS INVOLVING THE EULER FUNCTION

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Abstract

In this note, we provide refined estimates of two sums involving the Euler totient function,

$$\sum_{n\le x}\phi\Big(\Big[\frac{x}{n}\Big]\Big) \quad\text{and}\quad \sum_{n\le x}\frac{\phi([x/n])}{[x/n]},$$

where $[x]$ denotes the integral part of real x. The above summations were recently considered by Bordellès *et al.* ['On a sum involving the Euler function', Preprint, 2018, arXiv:1808.00188] and Wu ['On a sum involving the Euler totient function', Preprint, 2018, hal-01884018].

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1. Introduction

Let $[x]$ denote the integral part of a real number x. In a recent paper, Bordellès *et al.* [3] studied the asymptotic behaviour of the function

$$S_f := \sum_{n\le x} f\Big(\Big[\frac{x}{n}\Big]\Big).$$

In particular, if $f(n)$ is set to be $\phi(n)$ and $\phi(n)/n$ where $\phi(n)$ is the Euler totient function, Bordellès *et al.* obtained the estimates

$$\sum_{n\le x}\frac{\phi([x/n])}{[x/n]} = x\sum_{n\ge 1}\frac{\phi(n)}{n^2(n+1)} + O(x^{1/2}) \tag{1.1}$$

and

$$\Big(\frac{2629}{4009}\cdot\frac{1}{\zeta(2)} + o(1)\Big)x\log x \le \sum_{n\le x}\phi\Big(\Big[\frac{x}{n}\Big]\Big) \le \Big(\frac{2629}{4009}\cdot\frac{1}{\zeta(2)} + \frac{1380}{4009} + o(1)\Big)x\log x \tag{1.2}$$

for $x\to\infty$.

Subsequently, Wu improved the upper and lower bounds in (1.2) in [7] and the error term in (1.1) in [8]. More precisely, Wu showed that the error term in (1.1) can be sharpened to $O(x^{1/3}\log x)$, while the bounds in (1.2) can be refined as

$$\frac{2}{3}\cdot\frac{1}{\zeta(2)}x\log x+O(x)\le\sum_{n\le x}\phi\Big(\Big[\frac{x}{n}\Big]\Big)\le\Big(\frac{2}{3}\cdot\frac{1}{\zeta(2)}+\frac{1}{3}\Big)x\log x+O(x). \tag{1.3}$$

To bound $\sum_{n\le x}\phi([x/n])$, the main idea in Bordellès *et al.* [3] and Wu [7] relies on an estimate of the summation

$$\mathfrak{S}_\delta(x,N):=\sum_{N<n\le 2N}\phi(n)\psi\Big(\frac{x}{n+\delta}\Big)$$

for $x\ge 2$ and $1\le N\le x$ where $\psi(x)=x-[x]-\frac{1}{2}$ and $\delta\in\{0,1\}$. Such an estimate is built on Vaaler's expansion formula of $\psi(x)$ (see [6] or [2, Theorem 6.1]) and the theory of exponential pairs (see [2, Section 6.6.3]). Further, as Wu has shown in [8], the estimate of a similar summation

$$\mathfrak{S}^*_\delta(x,N):=\sum_{N<n\le 2N}\frac{\phi(n)}{n}\psi\Big(\frac{x}{n+\delta}\Big)$$

is useful to deduce the error term in (1.1).

We observe that, with the aid of an elaborate result due to Huxley (see [5] or [2, Theorem 6.40]), the estimate of $\mathfrak{S}^*_\delta(x,D)$ in [8] can be further sharpened. In fact, Huxley's result is strong enough in the sense that the best known error term up to now for the Dirichlet divisor problem can be deduced from it.

In this note, we shall prove the following results.

THEOREM 1.1. *As $x\to\infty$,*

$$\sum_{n\le x}\frac{\phi([x/n])}{[x/n]}=x\sum_{n\ge1}\frac{\phi(n)}{n^2(n+1)}+O(x^{131/416}(\log x)^{26947/8320}). \tag{1.4}$$

THEOREM 1.2. *As $x\to\infty$,*

$$\begin{aligned}\frac{285}{416}\cdot\frac{1}{\zeta(2)}x\log x&+O(x\log\log x)\\&\le\sum_{n\le x}\phi\Big(\Big[\frac{x}{n}\Big]\Big)\le\Big(\frac{285}{416}\cdot\frac{1}{\zeta(2)}+\frac{131}{416}\Big)x\log x+O(x\log\log x).\end{aligned}$$

We have two remarks to make on these results.

(1) Let $\tau(n)$ denote the number of divisors of n. It is known that the main term of $\sum_{n\le x}\tau(n)$ is $x(\log x+2\gamma-1)$ where γ is the Euler constant. The error term, denoted by $\Delta(x)$, can be trivially bounded to be $O(x^{1/2})$. Hardy [4] also showed that $\Delta(x)$ cannot be $o(x^{1/4})$. The best known bound up to now for $\Delta(x)$ is $O(x^{131/416}(\log x)^{26947/8320})$, which is due to Huxley as we have mentioned above. We can see that the error term in (1.4) can also reach this size.

(2) Numerically,

$$\frac{285}{416}\cdot\frac{1}{\zeta(2)}\approx 0.41649 \quad\text{and}\quad \frac{285}{416}\cdot\frac{1}{\zeta(2)}+\frac{131}{416}\approx 0.73139.$$

This slightly improves the bounds of Wu [7] in (1.3):

$$\frac{2}{3}\cdot\frac{1}{\zeta(2)}\approx 0.40528 \quad\text{and}\quad \frac{2}{3}\cdot\frac{1}{\zeta(2)}+\frac{1}{3}\approx 0.73861.$$

2. An auxiliary estimate

Let $\delta\in\{0,1\}$. We will focus on the following auxiliary function already defined in the introduction:

$$\mathfrak{S}^*_\delta(x,N) := \sum_{N<n\le 2N}\frac{\phi(n)}{n}\psi\Big(\frac{x}{n+\delta}\Big).$$

One has

$$\begin{aligned}\sum_{N<n\le 2N}\frac{\phi(n)}{n}\psi\Big(\frac{x}{n+\delta}\Big) &= \sum_{N<n\le 2N}\frac{1}{n}\psi\Big(\frac{x}{n+\delta}\Big)\sum_{\substack{k,\ell\\ k\ell=n}}\mu(k)\ell\\ &= \sum_{k\le 2N}\frac{\mu(k)}{k}\sum_{N/k<\ell\le 2N/k}\psi\Big(\frac{x}{k\ell+\delta}\Big).\end{aligned}\tag{2.1}$$

Now we will apply the following result due to Huxley [5].

Lemma 2.1 (Huxley, [2, Theorem 6.40]). *Let $r\ge 5$, $M\ge 1$ be integers and suppose $f\in C^r[M,2M]$ is such that there exist real numbers $T\ge 1$ and $1\le c_0\le\cdots\le c_r$ such that, for all $x\in[M,2M]$ and all $j\in\{0,\dots,r\}$,*

$$\frac{T}{M^j}\le |f^{(j)}(x)|\le c_j\frac{T}{M^j}.$$

Then

$$\sum_{M<n\le 2M}\psi(f(n))\ll (MT)^{131/416}(\log MT)^{18627/8320}.$$

Under the setting of Lemma 2.1, let us put $f(z)=x/(kz+\delta)$. It can be easily computed that for $j\ge 1$,

$$f^{(j)}(z)=(-1)^j\frac{j!k^jx}{(kz+\delta)^{j+1}}.$$

Trivially, $[N/k]\asymp N/k$ when $k<N$. It can be shown with almost no effort that T can be chosen to be $\asymp x/N$. In fact, $T=Cx/N$ is admissible where $C=6!/(3\cdot 6^6)$. Now we assume that $N\le Cx$.

For $k<N$,

$$\sum_{N/k<\ell\le 2N/k}\psi\Big(\frac{x}{k\ell+\delta}\Big)=\sum_{[N/k]<\ell\le 2[N/k]}\psi\Big(\frac{x}{k\ell+\delta}\Big)+O(1).$$

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It follows that, for $k < N \le Cx$,

$$\sum_{N/k<\ell\le 2N/k} \psi\Big(\frac{x}{k\ell+\delta}\Big) \ll \Big(\frac{x}{k}\Big)^{131/416}\Big(\log\frac{Cx}{k}\Big)^{18627/8320}.$$

Further, for $N \le k \le 2N$,

$$\sum_{N/k<\ell\le 2N/k} \psi\Big(\frac{x}{k\ell+\delta}\Big) \ll 1.$$

Hence, by (2.1), we conclude that

$$\begin{aligned}\sum_{N<n\le 2N} \frac{\phi(n)}{n}\psi\Big(\frac{x}{n+\delta}\Big) &= \sum_{k\le 2N}\frac{\mu(k)}{k}\sum_{N/k<\ell\le 2N/k}\psi\Big(\frac{x}{k\ell+\delta}\Big)\\ &\ll \sum_{k<N}\frac{1}{k}\Big(\frac{x}{k}\Big)^{131/416}(\log x)^{18627/8320}\\ &\ll x^{131/416}(\log x)^{18627/8320}.\end{aligned}$$

To summarise, we have proved the following result.

Proposition 2.2. *Let* $\delta \in \{0, 1\}$. *Then*

$$\sum_{N<n\le 2N} \frac{\phi(n)}{n}\psi\Big(\frac{x}{n+\delta}\Big) \ll x^{131/416}(\log x)^{18627/8320}$$

uniformly for $1 \le N \le 6!/(3\cdot 6^6)x$.

3. A partial summation

Consider the following partial summation of a general positive-valued function f on $\mathbb{N}$ with the parameter $D \le x$:

$$\sum_{D<n\le x} f\Big(\Big[\frac{x}{n}\Big]\Big).$$

If we assume that $d = [x/n]$ with $D < n \le x$, then $d \le x/D$ and

$$\begin{aligned}\sum_{D<n\le x} f\Big(\Big[\frac{x}{n}\Big]\Big) &= \sum_{d\le x/D} f(d)\cdot \mathrm{card}\Big\{D<n\le x : \Big[\frac{x}{n}\Big]=d\Big\}\\ &= \sum_{d\le x/D} f(d) \sum_{\substack{x/(d+1)<n\le x/d\\ D<n\le x}} 1.\end{aligned}$$

We now observe that when $d \le x/D - 1$, the interval $(x/(d+1), x/d]$ is indeed a subinterval of $(D, x]$ for in this case

$$\frac{x}{d+1} \ge \frac{x}{x/D-1+1} = D$$

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and $x/d \leq x$. It follows that

$$\begin{aligned}\sum_{D<n\leq x} f\Big(\Big[\frac{x}{n}\Big]\Big) &= \sum_{d\leq x/D-1} f(d) \sum_{x/(d+1)<n\leq x/d} 1 + \sum_{x/D-1<d\leq x/D} f(d) \sum_{\substack{x/(d+1)<n\leq x/d\\ D<n\leq x}} 1\\ &= \sum_{d\leq x/D} f(d) \sum_{x/(d+1)<n\leq x/d} 1\\ &\quad - \sum_{x/D-1<d\leq x/D} f(d)\Big(\sum_{x/(d+1)<n\leq x/d} 1 - \sum_{\substack{x/(d+1)<n\leq x/d\\ D<n\leq x}} 1\Big).\end{aligned}$$

On the one hand,

$$\sum_{x/(d+1)<n\leq x/d} 1 - \sum_{\substack{x/(d+1)<n\leq x/d\\ D<n\leq x}} 1 \ll \sum_{x/(d+1)<n\leq x/d} 1 \ll 1 + \frac{x}{d^2}.$$

On the other hand, there is only one d such that $x/D - 1 < d \leq x/D$, which is $[x/D]$. It turns out that

$$\sum_{x/D-1<d\leq x/D} f(d)\Big(\sum_{x/(d+1)<n\leq x/d} 1 - \sum_{\substack{x/(d+1)<n\leq x/d\\ D<n\leq x}} 1\Big) \ll f\Big(\Big[\frac{x}{D}\Big]\Big)\Big(1 + \frac{D^2}{x}\Big).$$

To summarise, we have proved the following proposition.

PROPOSITION 3.1. *Let f be a positive-valued function on $\mathbb{N}$ and D a parameter with $D \leq x$. Then,*

$$\sum_{D<n\leq x} f\Big(\Big[\frac{x}{n}\Big]\Big) = \sum_{d\leq x/D} f(d) \sum_{x/(d+1)<n\leq x/d} 1 + O\Big(f\Big(\Big[\frac{x}{D}\Big]\Big)\Big(1 + \frac{D^2}{x}\Big)\Big).$$

4. Proof of Theorem 1.1

Again, let $C = 6!/(3 \cdot 6^6)$. Then

$$\sum_{n\leq x} \frac{\phi([x/n])}{[x/n]} = \sum_{1/C<n\leq x} \frac{\phi([x/n])}{[x/n]} + O(1).$$

It follows from Proposition 3.1 that

$$\begin{aligned}\sum_{1/C<n\leq x} \frac{\phi([x/n])}{[x/n]} &= \sum_{d\leq Cx} \frac{\phi(d)}{d} \sum_{x/(d+1)<n\leq x/d} 1 + O(1)\\ &= \sum_{d\leq Cx} \frac{\phi(d)}{d}\Big(\frac{x}{d(d+1)} + \psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big) + O(1)\\ &= x\sum_{d\geq 1} \frac{\phi(d)}{d^2(d+1)} + O(1) + \sum_{d\leq Cx} \frac{\phi(d)}{d}\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big).\end{aligned}$$

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Using a dyadic split together with Proposition 2.2, we see that for $\delta \in \{0, 1\}$,

$$\sum_{d \le Cx} \frac{\phi(d)}{d} \psi\Big(\frac{x}{d+\delta}\Big) \ll x^{131/416} (\log x)^{26947/8320}.$$

We therefore arrive at Theorem 1.1.

5. Proof of Theorem 1.2

We first split the sum $\sum_{n \le x} \phi([x/n])$ into two parts:

$$\sum_{n \le x} \phi\Big(\Big[\frac{x}{n}\Big]\Big) = \sum_{n \le D} \phi\Big(\Big[\frac{x}{n}\Big]\Big) + \sum_{D < n \le x} \phi\Big(\Big[\frac{x}{n}\Big]\Big),$$

where the parameter D with $1/C \le D \le x^{1/2}$ is to be determined later.

It follows again from Proposition 3.1 that

$$\begin{aligned}
\sum_{D < n \le x} \phi\Big(\Big[\frac{x}{n}\Big]\Big) &= \sum_{d \le x/D} \phi(d) \sum_{x/(d+1) < n \le x/d} 1 + O\Big(\frac{x}{D} + D\Big) \\
&= x \sum_{d \le x/D} \frac{\phi(d)}{d(d+1)} + \sum_{d \le x/D} \phi(d)\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big) + O\Big(\frac{x}{D} + D\Big) \\
&= \frac{1}{\zeta(2)} x \log \frac{x}{D} + O(x) + \sum_{d \le x/D} \phi(d)\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big). \qquad (5.1)
\end{aligned}$$

In the last identity we use the standard result (see [1, Exercise 3.6]) that

$$\sum_{n \le x} \frac{\phi(n)}{n^2} = \frac{1}{\zeta(2)} \log x + O(1).$$

Thus

$$\sum_{n \le x} \frac{\phi(n)}{n(n+1)} = \sum_{n \le x} \phi(n)\Big(\frac{1}{n^2} + O\Big(\frac{1}{n^3}\Big)\Big) = \frac{1}{\zeta(2)} \log x + O(1).$$

Applying Abel's summation formula to the last part in (5.1) yields

$$\begin{aligned}
\sum_{d \le x/D} \phi(d)\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big) &= \frac{x}{D} \sum_{d \le x/D} \frac{\phi(d)}{d}\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big) \\
&\quad - \int_1^{x/D} \sum_{d \le t} \frac{\phi(d)}{d}\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big)\, dt. \qquad (5.2)
\end{aligned}$$

For $t \in [1, x/D]$ and $\delta \in \{0, 1\}$, by a dyadic split, it follows from Proposition 2.2 that

$$\sum_{d \le t} \frac{\phi(d)}{d} \psi\Big(\frac{x}{d+\delta}\Big) \ll x^{131/416} (\log x)^{26947/8320}.$$

It turns out that by (5.2)

$$\sum_{d \le x/D} \phi(d)\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big) \ll \frac{x}{D} x^{131/416} (\log x)^{26947/8320}. \qquad (5.3)$$

Let us choose

$$D = x^{131/416}(\log x)^{26947/8320}.$$

It follows from (5.1) and (5.3) that

$$\begin{aligned}\sum_{D<n\le x}\phi\Big(\Big[\frac{x}{n}\Big]\Big) &= \frac{1}{\zeta(2)}\Big(1-\frac{131}{416}\Big)x\log x + O(x\log\log x)\\ &= \frac{285}{416}\cdot\frac{1}{\zeta(2)}x\log x + O(x\log\log x).\end{aligned} \tag{5.4}$$

We can also trivially bound

$$0 \le \sum_{n\le D}\phi\Big(\Big[\frac{x}{n}\Big]\Big) \le \sum_{n\le D}\frac{x}{n} = \frac{131}{416}x\log x + O(x\log\log x). \tag{5.5}$$

Theorem 1.2 is a direct combination of (5.4) and (5.5).

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