Adv. Appl. Prob. **23**, 972–974 (1991)
Printed in N. Ireland

LETTERS TO THE EDITOR

RANDOM SET AND COVERAGE MEASURE

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Abstract

It is well known that a random set determines its random coverage measure. The paper gives a necessary and sufficient condition for the reverse implication. An equivalent formulation of the condition constitutes a first step in the search for a way to recognize a random measure as being the random coverage measure of a random set.

RANDOM MEASURE; RANDOM COVERAGE MEASURE

1. Introduction

The interest in random closed sets (Matheron (1975)) has increased during recent years along with the increasing development of their applications. These sets are a cornerstone in the model approach to stereology, as can be seen in Stoyan (1990). Given a random set Φ, its associated random coverage measure is a partial description of it. Recent results in stereology are focused on the estimation of random set characteristics related to this associated coverage measure, in particular its second-moment measure (Cruz-Orive (1989), Jensen et al. (1990)).

Obviously, a random set always determines its coverage measure. But under what conditions does the coverage measure determine the distribution of a random set? The answer to this question can be found as a corollary to another more general question: how can one recognize a random measure μ as being the random coverage measure of a random set with distribution determined by μ?

2. Results

In $\mathbb{R}^k$ with Borel σ-field β^k, a random (d, k)-set Φ is defined as a measurable mapping from a probability space into the measurable space of ν-rectifiable closed sets in $\mathbb{R}^k$, where ν stands for the corresponding d-dimensional Hausdorff measure in $\mathbb{R}^k$ (see Jensen et al. (1990) and Zähle (1982) for more details). This random set determines a unique random coverage measure defined as $\mu_\Phi(B) = \nu(\Phi \cap B)$, $B \in \beta^k$. The following theorem establishes which condition the random closed set must satisfy in order to recover its distribution from the associated coverage measure.

Received 19 March 1991; revision received 16 July 1991.

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This work was supported in part by DGICYT grant no. PB 87-0992.

Theorem 1. Let Φ be a random (d, k)-set and μ_Φ its associated coverage measure. The distribution of Φ is recoverable from μ_Φ if and only if

$$P\{\Phi \cap K \neq \varnothing, \mu_\Phi(K \oplus \varepsilon B) = 0 \quad \text{for some } \varepsilon\} = 0 \tag{2.1}$$

for all compact sets K, where $\oplus$ denotes Minkowski addition and B is the open unit ball in $\mathbb{R}^k$.

Proof. The distribution of Φ is determined by the probabilities $T(K) = P\{\Phi \cap K \neq \varnothing\}$ for every compact K (see Matheron (1975)).

Notice that

$$\mu_\Phi(K \oplus \varepsilon B) > 0, \qquad \forall \varepsilon > 0 \Rightarrow \Phi \cap K \neq \varnothing. \tag{2.2}$$

On the other hand, $\{\Phi \cap K \neq \varnothing\}$ can be written as the disjoint decomposition

$$\begin{aligned} \{\Phi \cap K \neq \varnothing\} = &\{\mu_\Phi(K \oplus \varepsilon B) > 0, \forall \varepsilon > 0\} \\ &\cup \{\Phi \cap K \neq \varnothing, \mu_\Phi(K \oplus \varepsilon B) = 0 \quad \text{for some } \varepsilon\}, \end{aligned} \tag{2.3}$$

and $\{\mu_\Phi(K \oplus \varepsilon B) > 0, \forall \varepsilon > 0\}$ being an event with its probability determined by the distribution of μ_Φ, the sufficiency of (2.1) is established.

To prove necessity, suppose that for some compact K and $\varepsilon > 0$, $P(\Phi \cap K \neq \varnothing, \mu_\Phi(K \oplus \varepsilon B) = 0) > 0$. Set

$$\Phi' = \begin{cases} \Phi - (K \oplus \varepsilon B), & \text{if } \mu_\Phi(K \oplus \varepsilon B) = 0 \\ \Phi, & \text{otherwise.} \end{cases}$$

It follows immediately that $\mu_\Phi = \mu_{\Phi'}$, but the disjoint decomposition

$$\{\Phi' \cap K = \varnothing\} = \{\Phi \cap K = \varnothing\} \cup \{\Phi \cap K \neq \varnothing, \mu_\Phi(K \oplus \varepsilon B) = 0\}$$

implies $P\{\Phi \cap K \neq \varnothing) > P(\Phi' \cap K \neq \varnothing)$.

If follows from the theorem that $P(\Phi \cap K \neq \varnothing) = \lim_{i \to \infty} P\{\mu_\Phi(K \oplus \varepsilon_i B) > 0\}$ for any sequence of ε_i decreasing to 0. In fact, we can associate with each random measure μ a random closed set Φ_μ, satisfying the condition (2.1), as follows.

Definition. Given a random measure μ let Φ_μ be the *random closed support* of μ, defined for any sequence of ε_i decreasing to 0 and for any $\{x_1, x_2, \cdots\}$ dense in $\mathbb{R}^k$ by

$$\Phi_\mu = \textstyle\bigcap_{i=1}^{\infty} \text{closure } \{x_j, \mu(x_j \oplus \varepsilon_i B) > 0\}.$$

Note that this definition is independent of the choice of sequences $\{\varepsilon_i\}$ and $\{x_1, x_2, \cdots\}$ and it allows us the following alternative formulation of Theorem 1.

Theorem 2. The distribution of the random closed set Φ is recoverable from μ_Φ if and only if Φ is distributed as the random closed support of μ_Φ.

Theorem 2 gives a natural answer to the second question in the introduction: a random measure μ will be the random coverage of a random closed set when its closed support Φ_μ has μ as its random coverage measure. But the fundamental question remains: what natural and verifiable conditions must be imposed on μ in order for it to have this property?

Acknowledgement

We are grateful to Professor D. Stoyan for his helpful comments. For the final version of this paper we are much indebted to the interesting and detailed comments of the referee.

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