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GROUPS WHICH SATISFY A WEAK FORM OF POINCARE DUALITY

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Our main result is that a "restricted Poincaré duality" property with respect to finite dimensional coefficient modules over a field holds for a certain class of groups which includes all soluble groups of finite Hirsch length. This relies on a generalisation to the given class of a module construction by Stammbach; an extension of his result on homological dimension to these groups is given. We also generalise the well-known result that torsion-free soluble groups of finite rank are countable.

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0. Introduction

0.1 Notation, definitions and statement of the main theorems

We adopt the conventions of [**2**], except that we use right coefficient modules for cohomology, left for homology. K will be a field, and the dimension of a KG-module will mean its dimension as a K-vector space.

We now define the property with which we are primarily concerned.

Definition. A group G has restricted Poincaré duality of dimension n on a subcategory ${}_1\mathfrak{M}_{KG}$ of the category $\mathfrak{M}_{KG}$ of right KG-modules if there is a left KG-module $\mathscr{D}_G$ (called the *dualizing module*) such that:

(i) $\mathscr{D}_G \cong K$ as a K-module, and $H_n(G,\mathscr{D}_G)\cong K$ as a KG-module;

(ii) $\exists\ [\omega]\in H_n(G,\mathscr{D}_G)$ such that the cap product

$$H^k(G,M)\xrightarrow{\ \cap[\omega]\ } H_{n-k}(G,M\otimes\mathscr{D}_G)$$

is an isomorphism for all $k\in\mathbb{Z}$, and all objects M of ${}_1\mathfrak{M}_{KG}$. (We call $[\omega]$ the *dualizing class*).

Putting ${}_1\mathfrak{M}_{KG}=\mathfrak{M}_{KG}$ here gives Poincaré duality of dimension n over K. $\cap[\omega]$ is a natural transformation between the functors $H^k(G,M)$ and $H_{n-k}(G,M\otimes\mathscr{D}_G)$ from ${}_1\mathfrak{M}_{KG}$ to $\mathfrak{M}_K$ for all $k\in\mathbb{Z}$, by naturality of cap product. (See [**2**, p. 146] and [**3**, p. 109] for elementary properties of this map.)

Define ${}_{fd}\mathfrak{M}_{KG}$ to be the full subcategory of $\mathfrak{M}_{KG}$ whose objects are the finite dimensional (f.d.) KG-modules (the K will be omitted where it is clear which field we mean).

We now define a class of groups including all PD^n-groups, which will later be seen to have restricted Poincaré duality on ${}_{fd}\mathfrak{M}_{KG}$.

Fix $n\in\mathbb{N}$. A group G is *locally orientable Poincaré duality* ($LOPD^n$) over K if each finite subset of G is contained in a subgroup of G which is orientable PD^n over K. If G is countable $LOPD^n$ over K, there is an ascending chain of PD^n subgroups

$$N_0\leqq N_1\leqq \dots \text{ of } G \text{ s.t. } G=\bigcup_{i\in\mathbb{N}} N_i. \tag{1}$$

The $|N_i:N_{i-1}|$ are all finite ([**2**, Proposition 9.22]). G is PD^n if and only if $G=N_k$ for some k; otherwise, G is infinitely generated.

For G countable $LOPD^n$ over K, let X_G be the set of primes p s.t. given any PD^n-subgroup S of G there is a pair of PD^n-subgroups P_1, P_2 of G satisfying $S\leqq P_1<P_2$, $p\,\big|\,|P_1:P_2|$. For any choice of ascending chain (1), X_G is easily seen to be equal to the set of primes p such that $p\,\big|\,|N_{i+1}:N_i|$ for infinitely many $i\in\mathbb{N}$.

Definition. Take $\mathscr{X}_r$ (resp. $\bar{\mathscr{X}}_r$) to be the class of groups G with a series

$$G=G_0\geqq G_1\geqq\cdots\geqq G_r=1 \tag{2}$$

with each G_i normal in G, and each G_i/G_{i+1} is $LOPD^{n(i)}$ (resp. countable $LOPD^{n(i)}$) over K with char K not contained in the union of the $\mathscr{X}_{G_i/G_{i+1}}$, $0\leqq i\leqq r-1$. Let $\bar{\mathscr{X}}:=\bigcup_{r\in\mathbb{N}}\bar{\mathscr{X}}_r$, $\mathscr{X}:=\bigcup_{r\in\mathbb{N}}\mathscr{X}_r$. All soluble groups of finite Hirsch length lie in some $\mathscr{X}_r$.

Henceforth we fix a field K and use the notation of (2); X_G, $\mathscr{X}_r$, $\bar{\mathscr{X}}_r$ will always be defined with respect to this field.

Now we state the main results.

Theorem 1. *If $G\in\mathscr{X}$, then G has restricted Poincaré duality of dimension $\sum_{i=0}^{r-1}n(i)$ on ${}_{fd}\mathfrak{M}_{KG}$.*

We will first prove this where G is countable; that is, $G\in\bar{\mathscr{X}}$. We then look at groups in $\mathscr{X}$, obtaining the following, which enables us to prove Theorem 1 in general.

Theorem 2. *Each $G\in X_r$ has a locally finite characteristic subgroup N such that $G/N\in\bar{\mathscr{X}}_r$.*

Finally, we define a property used in the proof of Theorem 1. We will say that a group G is *finite dimension preserving* if, for M a finite dimensional KG-module and $k\in\mathbb{N}$, $H_k(G,M)$ and $H^k(G,M)$ are finite dimensional.

0.2 Layout of paper

In Section 1 we give some general properties of groups which have restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$ and some other subcategories of $\mathfrak{M}_G$. In Section 2, the countable case of Theorem 1 is proved. Some properties of groups in $\mathscr{X}$ are given in Section 3, and we prove Theorem 2, enabling us to complete the proof of Theorem 1. In Section 4 certain groups in $\mathscr{X}$, including all soluble minimax groups, are shown to have restricted Poincaré duality on other subcategories of $\mathfrak{M}_G$ for some K. In Section 5 we consider the action of $G \in \mathscr{X}$ on the dualizing module.

0.3 Acknowledgements

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1. Some properties of restricted Poincaré duality groups

Here we give several properties of groups with restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$, and some information on whether these hold over other subcategories of $\mathfrak{M}_G$. We also provide a cup product formulation of the duality for finite dimension preserving groups with restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$.

1.1 Uniqueness of the dualizing module

Suppose that G has restricted Poincaré duality on ${}_1\mathfrak{M}_G$, with $\mathscr{D}_G$, $\mathscr{D}'_G$ both satisfying (i) and (ii) (see 0.1). Let $\mathscr{D}'^{op}_G$ be the unique left KG-module such that $\mathscr{D}'^{op}_G \otimes \mathscr{D}'_G \cong K$. Consider $\mathscr{D}_G \otimes \mathscr{D}'^{op}_G$ as a right KG-module via $(n \otimes m)g = g^{-1}n \otimes g^{-1}m$. If K and $\mathscr{D}_G \otimes \mathscr{D}'^{op}_G$ are objects of ${}_1\mathfrak{M}_G$, then $\mathscr{D}_G = \mathscr{D}'_G$.

As left KG-modules, $(\mathscr{D}_G \otimes \mathscr{D}'^{op}_G) \otimes \mathscr{D}'_G \cong \mathscr{D}_G \otimes (\mathscr{D}'^{op}_G \otimes \mathscr{D}'_G)$

$$\text{via } (m \otimes n) \otimes r \longrightarrow m \otimes (n \otimes r).$$

Therefore

$$K \cong H^0(G, K) \overset{\alpha}{\cong} H_n(G, \mathscr{D}_G) \overset{\phi_*}{\cong} H_n(G, (\mathscr{D}_G \otimes \mathscr{D}'^{op}_G) \otimes \mathscr{D}'_G) \overset{\beta}{\cong} H^0(G, \mathscr{D}_G \otimes \mathscr{D}'^{op}_G)$$

where α, β are Poincaré duality isomorphisms. Hence $\mathscr{D}_G \otimes \mathscr{D}'^{op}_G \cong K$, so $\mathscr{D}_G \cong \mathscr{D}'_G$ as required.

The hypothesis clearly holds for ${}_{fd}\mathfrak{M}_G$, hence $\mathscr{D}_G$ is unique for groups with restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$.

1.2 Quotients by locally finite normal subgroups

Let G be a group with restricted Poincaré duality on ${}_1\mathfrak{M}_G$, and T a locally finite normal subgroup of G. Take ${}^T\mathfrak{M}_G$ (${}^T_1\mathfrak{M}_G$) to be the full subcategory of $\mathfrak{M}_G$ (respectively ${}_1\mathfrak{M}_G$) with those objects on which T acts trivially, and let F be the natural functor ${}^T\mathfrak{M}_G \to \mathfrak{M}_{G/T}$. Now define ${}_1\mathfrak{M}_{G/T}$ to be $F({}^T_1\mathfrak{M}_G)$. Then G/T has restricted Poincaré duality on ${}_1\mathfrak{M}_{G/T}$, as we will now show.

$K \cong H_n(G, \mathscr{D}_G)^{**} \cong H^n(G, \mathscr{D}_G^*)^*$ (by a basic adjunction; see 1.6 for details). Hence $H^n(G, \mathscr{D}_G^*) \cong K$. By a spectral sequence corner argument, it follows that $H^n(G/T, H^0(T, \mathscr{D}_G^*)) \cong K$. Therefore T acts trivially on $\mathscr{D}_G^*$, and hence on $\mathscr{D}_G$.

Let M be an KG-module on which T acts trivially. The maps

$$H^k(G/T, M) \xrightarrow{\phi^*} H^k(G, M)$$

$$H_k(G, M) \xrightarrow{\phi_*} H_k(G/T, M)$$

induced from the natural map ϕ: $G \twoheadrightarrow G/T$ are isomorphisms (by the LHS spectral sequence).

By naturality of $\cap$ with respect to change of group, the following diagram commutes.

$$\begin{array}{ccc} H^k(G/T, M) & \xrightarrow{\cap \phi_*[\omega]} & H_{n-k}(G/T, M \otimes \mathscr{D}_G) \\ \downarrow \phi^* & & \uparrow \phi_* \\ H^k(G, M) & \xrightarrow{\cap[\omega]} & H_{n-k}(G, M \otimes \mathscr{D}_G) \end{array}$$

$\cap[\omega]$ is an isomorphism by hypothesis, hence $\cap \phi_*[\omega]$ also is.

Observe that where ${}_1\mathfrak{M}_G = {}_{fd}\mathfrak{M}_G$, we have ${}_1\mathfrak{M}_{G/T} = {}_{fd}\mathfrak{M}_{G/T}$. Hence G/T has restricted Poincaré duality on finite dimensional modules if G does.

1.3 Inverse restricted Poincaré duality

Definition. G has *inverse restricted Poincaré duality* of dimension n on ${}_1\mathfrak{M}^1_G$, a subcategory of the category $\mathfrak{M}^1_G$ of left KG-modules, if there is a left KG-module $\mathscr{D}_G$ satisfying:

(i) $\mathcal{D}_G \cong K$ as a K-module, and $H_n(G, \mathcal{D}_G) \cong K$ as a KG-module;

(ii) $\exists\ [\omega] \in H_n(G, \mathcal{D}_G)$ such that the modified cap product (see [2, p. 147] for definition and properties)

$$H^k(G, \mathrm{Hom}_K(\mathcal{D}_G, M)) \xrightarrow{\psi[\omega]} H_{n-k}(G, M)$$

is an isomorphism for all $k \in \mathbb{Z}$ and all objects M of ${}_1\mathfrak{M}_G^1$.

Putting ${}_1\mathfrak{M}_G^1 = \mathfrak{M}_G^1$ here gives inverse Poincaré duality, which is equivalent to Poincaré duality (in the sense that a group with one has the other, with the same dualizing module), since the dualizing module is K-projective (see [2, Theorem 9.4]). As in restricted Poincaré duality, $\psi[\omega]$ is a natural transformation between the functors $H^k(G, \mathrm{Hom}_K(\mathcal{D}_G, \_))$ and $H_{n-k}(G, \_)$ from ${}_1\mathfrak{M}_G^1$ to $\mathfrak{M}^1$.

ψ is defined by

$$\begin{array}{ccc} H^r(G, \mathrm{Hom}_K(A, B)) & \xrightarrow{\ \cap[\omega]\ } & H_{n-r}(G, \mathrm{Hom}_K(A, B) \otimes A) \\ & \searrow^{\psi[\omega]} & \big\downarrow (\mathrm{ev})_* \\ & & H_{n-r}(G, B) \end{array} \qquad (3)$$

where ev: $\mathrm{Hom}_K(A, B) \otimes A \to B$ is given by

$$f \otimes a \to f(a)$$

and $[\omega] \in H_n(G, A)$.

Where $A = \mathcal{D}_G$, it is easy to see that ev is an isomorphism.

Lemma 1. (a) *Suppose G has restricted Poincaré duality on ${}_1\mathfrak{M}_G$, with dualizing module $\mathcal{D}_G$. Then G has inverse restricted duality with dualizing module $\mathcal{D}_G$ on any subcategory ${}_2\mathfrak{M}_G^1$ of $\mathfrak{M}_G^1$ in which each object M satisfies $\mathrm{Hom}_K(\mathcal{D}_G, M)$ an object of ${}_1\mathfrak{M}_G$.*

b) *Suppose G has inverse restricted Poincaré duality on ${}_1\mathfrak{M}_G^1$, with dualizing module $\mathcal{D}_G$. Then G has restricted Poincaré duality with dualizing module $\mathcal{D}_G$ on any subcategory ${}_2\mathfrak{M}_G$ of $\mathfrak{M}_G$ in which each object M satisfies $\mathrm{Hom}_K(\mathcal{D}_G^{op}, M)$ an object of ${}_1\mathfrak{M}_G^1$.*

Proof. (a) Trivial.

(b) Substitute $\mathrm{Hom}_K(\mathscr{D}_G^{op}, M)$ for B in (3). There is a natural isomorphism $\mathrm{Hom}_K(\mathscr{D}_G, \mathrm{Hom}_K(\mathscr{D}_G^{op}, M)) \to \mathrm{Hom}_K(\mathscr{D}_G \otimes \mathscr{D}_G^{op}, M) \cong M$ given by a basic adjunction. Applying this, we obtain the result.

Where M is a f.d. left KG-module, $\mathrm{Hom}_K(\mathscr{D}_G, M)$ is a f.d. right KG-module, and where M is a f.d. right KG-module, $\mathrm{Hom}_K(\mathscr{D}_G^{op}, M)$ is a f.d. left KG-module. Hence G has inverse restricted Poincaré duality on f.d. modules if and only if G has restricted Poincaré duality on f.d. modules, and, if both hold, the dualizing modules are the same.

1.4 Duality groups with restricted Poincaré duality

All soluble groups s.t. $\mathrm{cd}_{\mathbb{Z}} G = \mathrm{hd}_{\mathbb{Z}} G < \infty$ are duality groups over $\mathbb{Z}$ and hence over $\mathbb{Q}$ by [8]. We will see that these groups are also restricted Poincaré duality on finite dimensional modules. Many of them, $\mathbb{Z}[1/2] \wr \langle x2 \rangle$ for instance, are not Poincaré duality groups. However, there is a connection between the duality and the restricted Poincaré duality on a subcategory of $\mathfrak{M}_G$ for some groups with both properties.

Lemma 2. *Let G be a duality group with dualizing module $\mathscr{D}_G$. The following are equivalent:*

(i) *G has restricted Poincaré duality of dimension n on ${}_1\mathfrak{M}_G$ with dualizing module $\mathscr{D}_G$ such that $\mathscr{D}_G^{op}$ is an object of ${}_1\mathfrak{M}_G$.*

(ii) *There is a KG-module $\mathscr{D}_G \overset{K}{\cong} K$ s.t. $\mathscr{D}_G^{op}$ is an object of ${}_1\mathfrak{M}_G$, and a KG-module homomorphism $\phi: D_G \twoheadrightarrow \mathscr{D}_G$ which induces isomorphisms $\phi_*: H_r(G, M \otimes D_G) \to H_r(G, M \otimes \mathscr{D}_G)$ for all $r \in \mathbb{N}$ and all objects M of ${}_1\mathfrak{M}_G$.*

Proof. (ii)$\Rightarrow$(i): Follows easily from naturality of cap product with respect to coefficient homomorphisms.

(i)$\Rightarrow$(ii): Define ρ: $G \to K$ by $\rho(g) = g.1$ in $\mathscr{D}_G$. Let J be the two-sided ideal of KG generated by the $\rho(g) - g, g \in G$. Then

$$\begin{array}{ccc} D_G = H^n(G, KG) & \xrightarrow{\sigma} & H^n(G, KG/J) \\ & & \Big\downarrow \cap[\omega] \\ & & \mathscr{D}_G \cong H_0(G, KG/J \otimes \mathscr{D}_G) \end{array}$$

where $[\omega]$ is the dualizing class of $H_n(G, \mathscr{D}_G)$ (see 0.1).

σ is onto since $cd_K G = n$. KG/J regarded as a right module is $\mathscr{D}_G^{op}$, so the vertical map

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is an isomorphism. So we have $\phi\colon D_G \twoheadrightarrow \mathscr{D}_G$. By naturality of $\cap$, the following diagram commutes.

$$\begin{array}{ccc}
H^r(G,M)\otimes H_n(G,D_G) & \xrightarrow{1\otimes\phi_*} & H^r(G,M)\otimes H_n(G,\mathscr{D}_G) \\
\Big\downarrow \cap & & \Big\downarrow \cap \\
H_{n-r}(G,M\otimes D_G) & \xrightarrow{(1\otimes\phi)_*} & H_{n-r}(G,M\otimes\mathscr{D}_G)
\end{array}$$

Where M is an object of ${}_1\mathfrak{M}_G$, the $\cap$ maps are both isomorphisms, so it suffices to prove that $\phi_*\colon H_n(G,D_G)\to H_n(G,\mathscr{D}_G)$ is an isomorphism. By [2, Lemma 9.1] the following diagram commutes and the vertical maps are isomorphisms.

$$\begin{array}{ccc}
H_n(G,D_G) & \xrightarrow{\phi_*} & H_n(G,\mathscr{D}_G) \\
\Big\downarrow \psi & & \Big\downarrow \psi \\
\operatorname{Hom}_{KG}(D_G,D_G) & \xrightarrow{\phi_*} & \operatorname{Hom}_{KG}(D_G,\mathscr{D}_G)
\end{array}$$

The bottom ϕ_* is nonzero, hence the top ϕ_* is an isomorphism. □

Note that an analogous result holds for restricted inverse Poincaré duality, where G is an inverse duality group.

Observe that not all duality groups have restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$; the free group on two generators is a duality group, but does not have restricted Poincaré duality on this category.

1.5 The cap product formulation

Corollary. *Suppose G has restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$ and G is finite dimension preserving. Then for all finite dimensional KG-modules M, the map*

$$H^r(G,M)\otimes H^{n-r}(G,\operatorname{Hom}_K(M,\mathscr{D}_G))\xrightarrow{(\mathrm{ev})_*\circ\cup} H^n(G,\mathscr{D}_G)$$

is an exact pairing, where ev *is the map*

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$$M \otimes \mathrm{Hom}_K(M, \mathcal{D}_G^*) \to \mathcal{D}_G^*$$

given by $m \otimes \phi \to m\phi$.

Proof. Take L any group, V any KL-module. Let

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \twoheadrightarrow K$$

be a projective resolution for K over KL.

Applying $\mathrm{Hom}_{KL}(\_, \mathrm{Hom}_K(V, K))$ to this gives a cochain complex whose cohomology is $H^*(L, V^*)$. Using $\mathrm{Hom}_K(\_ \otimes_{KL} V, K)$ instead gives a cochain complex with cohomology $H_*(L, V)^*$. The basic adjunction

$$\mathrm{Hom}_{KL}(\_, \mathrm{Hom}_K(V, K)) \to \mathrm{Hom}_K(\_ \otimes_{KL} V, K)$$

gives an isomorphism between the cochain complexes which induces the isomorphism

$$H^*(L, V^*) \xrightarrow{\alpha} H_*(L, V)^*.$$

Now suppose G satisfies restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$ and is finite dimension preserving. Let M be a finite dimensional module. Consider the diagram

$$\begin{array}{ccc}
H^r(G, M) \otimes H_n(G, \mathcal{D}_G) & \xrightarrow[\cong]{\cap} & H_{n-r}(G, M \otimes \mathcal{D}_G) \\
\downarrow 1 \otimes ** & & \downarrow ** \\
H^r(G, M) \otimes H_n(G, \mathcal{D}_G)^{**} & & H_{n-r}(G, M \otimes \mathcal{D}_G)^{**} \\
\downarrow 1 \otimes \alpha^* & & \downarrow \alpha^* \\
H^r(G, M) \otimes H^n(G, \mathcal{D}_G^*)^* & & H^{n-r}(G, (M \otimes \mathcal{D}_G)^*)^* \\
& \searrow \psi & \downarrow (\text{basic adj.})_* \\
& & H^{n-r}(G, \mathrm{Hom}_K(M, D_G^*))^*
\end{array}$$

As all the other maps are isomorphisms, ϕ may be defined by this diagram; it too is clearly an isomorphism. Let θ be defined by

$$\begin{array}{ccc} H^{n-r}(G, \mathrm{Hom}_K(M, \mathcal{D}_G^*))^{**} & \xrightarrow{\psi^*} & (H^r(G, M) \otimes H^n(G, \mathcal{D}_G^*)^*)^* \\ & \overset{\theta}{\searrow} & \Big\downarrow \text{basic adj.} \\ & & \mathrm{Hom}_K(H^r(G, M), H^n(G, \mathcal{D}_G^*)^{**}) \end{array}$$

Define β: $H^r(G, M) \otimes H^{n-r}(G, \mathrm{Hom}_K(M, \mathcal{D}_G^*))^{**} \to H^n(G, \mathcal{D}_G^*)^{**}$ by

$$\beta(v \otimes w) = v(w\theta).$$

Taking bases of the K-vector spaces involved, then performing a routine calculation, it can be seen that the following diagram commutes.

$$\begin{array}{ccc} H^r(G, M) \otimes H^{n-r}(G, \mathrm{Hom}_K(M, \mathcal{D}_G^*)) & \xrightarrow{(\mathrm{ev})_* \circ \cap} & H^n(G, \mathcal{D}_G^*) \\ \Big\downarrow ** & & \Big\downarrow ** \\ H^r(G, M) \otimes H^{n-r}(G, \mathrm{Hom}_K(M, \mathcal{D}_G^*))^{**} & \xrightarrow{\beta} & H^n(G, \mathcal{D}_G^*)^{**} \end{array}$$

The result follows. □

2. Proof of the countable case of Theorem 1

We prove by induction on r that if $G \in \bar{X}$, then G is finite dimension preserving and has restricted Poincaré duality on finite dimensional modules.

2.1 Proof for $r = 1$

A PD^n group (over K) plainly has restricted Poincaré duality on ${}_{fd}\mathfrak{M}_G$, and is finite dimension preserving because it is of type (FP). Throughout this section, M will be a f.d. KG-module.

Suppose

$$G = \bigcup_{i=0}^{\infty} N_i, \quad N_0 \leqq N_1 \leqq \cdots$$

with the N_i all orientable PD^n. There is a short exact sequence of KG-modules

$$\underleftarrow{\lim}{}^{(1)} H^{k-1}(N_i, M) \to H^k(G, M) \to \underleftarrow{\lim} H^k(N_i, M)$$

(see [4]) where $\underleftarrow{\lim}$, $\underleftarrow{\lim}{}^{(1)}$ have been taken over

$$\left(\cdots \xrightarrow{\text{Res}} H^s(N_i, M) \xrightarrow{\text{Res}} H^s(N_{i-1}, M) \xrightarrow{\text{Res}} \cdots \xrightarrow{\text{Res}} H^s(N_0, M)\right) \tag{4}$$

for $s=k$, $s=k-1$ respectively.

There is also an isomorphism

$$\underrightarrow{\lim} H_k(N_i, M) \to H_k(G, M)$$

where $\underrightarrow{\lim}$ is taken over the system

$$\left(H_k(N_0, M) \xrightarrow{\text{Cor}} H_k(N_1, M) \xrightarrow{\text{Cor}} \cdots \xrightarrow{\text{Cor}} H_k(N_i, M) \xrightarrow{\text{Cor}} \cdots\right). \tag{5}$$

As automorphisms of $H_n(N_i, M)$ or $H^n(N_i, M)$, it is well known that

$$\text{Cor} \circ \text{Res} = x|N_i : N_{i-1}| \tag{6}$$

char $K \notin X_G$, so $x|N_i : N_{i-1}|$ is an isomorphism, for sufficiently large i. $H_n(N_i, K) \overset{KN_i}{\cong} K$ for all $i \in \mathbb{N}$, since the N_i are orientable PD^n. The Cor maps in (5) are onto by (6), hence $H_n(G, K) \overset{KG}{\cong} K$.

The duality isomorphism in the orientable PD^n groups is given by a cap product, which may be regarded [2] as a map

$$H_n(N_i, K) \to \text{Hom}_{KN_i}(H^k(N_i, M), H_{n-k}(N_i, M))$$

which is natural with respect to changes of group and module. Choose an element $[\omega]$ of $H_n(G, K)$ and let $[\omega_i]$ be its image under restriction to $H_n(N_i, K)$. The following commutes, since $\cap$ is natural in the group.

$$\begin{array}{ccccc} \cdots \xrightarrow{\text{Res}} H^k(N_{i+1},M) & \xrightarrow{\text{Res}} & H^k(N_i,M) \xrightarrow{\text{Res}} \cdots \\ \downarrow \cap[\omega_{i+1}] & & \downarrow \cap[\omega_i] \\ \cdots \xleftarrow{\text{Cor}} H_{n-k}(N_{i+1},M) & \xleftarrow{\text{Cor}} & H_{n-k}(N_i,M) \xleftarrow{\text{Cor}} \cdots \end{array} \tag{7}$$

By (6), the Res maps are $(1-1)$ and the Cor maps are onto. The N_i are finite dimension preserving, so all the modules in (7) are finite dimensional. Hence $\exists\ n_0 \in \mathbb{N}$ such that

$$\text{Res}: H^k(N_{l+1},M) \to H^k(N_l,M) \text{ and Cor}: H_{n-k}(N_l,M) \to H_{n-k}(N_{l+1},M) \text{ are isomorphisms for } l > n_0. \tag{8}$$

The vertical maps therefore induce an isomorphism

$$\psi: \varprojlim H^k(N_i,M) \to \varinjlim H_{n-k}(N_i,M)$$

which is the unique such KG-module isomorphism satisfying $\psi|_{N_i} = \cap[\omega_i]$. $\varprojlim^{(1)} H^{k-1}(N_i,M) = 0$ by the Mittag–Leffler condition (see [1]), satisfied since all the $H^{k-1}(N_i,M)$ are finite dimensional. It follows that α is an isomorphism, and ψ is the unique isomorphism $H^k(G,M) \to H_{n-k}(G,M)$ such that $\psi|_{N_i} = \cap[\omega_i]$. But $\cap[\omega]|_{N_i} = \cap[\omega_i]$, hence $\psi = \cap[\omega]$. Therefore $\cap[\omega]$ is an isomorphism, so G has restricted Poincaré duality on finite dimensional modules, with dualizing module K.

By (8), $\dim_K(\varprojlim H^k(N_i,M)) = \dim_K(H^k(N_l,M))$ and $\dim_K(\varinjlim H_{n-k}(N_i,M)) = \dim_K(H^k(N_l,M))$. Hence G is finite dimension preserving.

2.2 The induction step

Throughout this part, $G \in \bar{X}_r$, the notation of (2) is used, and $N := G_1$. Assume result true for $G \in \bar{X}_{r-1}$ (hence for N).

2.2.1 G is finite dimension preserving

For the extension $N \twoheadrightarrow G \twoheadrightarrow G/N$, there is a Lyndon–Hochschild–Serre spectral sequence

$$H^{k-l}(G/N, H^l(N,M)) \Rightarrow H^k(G,M). \text{ (see [5], [9].)}$$

So $H^k(G,M)=S_k \geqq S_{k-1} \geqq \cdots \geqq S_0 = 1$ where S_i/S_{i-1} is a section of $H^{k-i}(G/N, H^i(N,M))$.

The $H^{k-i}(G/N, H^i(N,M))$ are finite dimensional, since N and G/N are finite dimension preserving by the induction hypothesis. Therefore $H^k(G,M)$ is finitely generated. A similar argument may be used on homology.

2.2.2 Construction of the dualizing module in the general case

Let $L_i := H_{n(i)}(G_i/G_{i+1}, K)$. G acts on L_i via conjugation on G_i/G_{i+1}. Set $\mathscr{D}_G := L_0^{op} \otimes \cdots \otimes L_{r-1}^{op}$. From (1.1) we know that $\mathscr{D}_G \cong K$ as a K-module. By the method of [2, Lemma 7.13], it can be shown that $H_n(G, \mathscr{D}_G) \cong K$ as a KG-module.

2.2.3 The induction argument

Set $l := n(1)+n(2)+\cdots+n(r-1)$.

Lemma 3.

$$H^k(N,M) \xrightarrow{\cap[\omega]} H_{l-k}(N, M \otimes D_G)$$

is a KG-module isomorphism, where $[\omega] \in H_l(N, D_G)$.

Proof. $\mathrm{Res}_N^G(\mathscr{D}_G) = \mathscr{D}_N$, so the given map is a KN-module isomorphism by the induction hypothesis. It now suffices to show that it is also a KG-module homomorphism. Since $\cap$ is natural with respect to group homomorphisms, the diagram below commutes,

$$\begin{array}{ccc} H^k(N,M) & \xrightarrow{\cap[\omega]} & H_{l-k}(N, M\otimes \mathscr{D}_G) \\ \downarrow \nu_g & & \downarrow \nu_g \\ H^k(N,M) & \xrightarrow{\cap([\omega]\nu_g)} & H_{l-k}(N, M\otimes \mathscr{D}_G) \end{array}$$

where ν_g are the natural maps in homology given by the action of $g \in G$.

$$H_l(N, \mathscr{D}_G) \overset{KN}{\cong} H_l(N, \mathscr{D}_N) \cong K$$

$K \cong H_n(G, \mathscr{D}_G) \cong H_{n-l}(G/N, H_l(N, \mathscr{D}_G)) \cong H_l(N, \mathscr{D}_G)^{G/N}$ (by result for $r=1$). Hence G acts trivially on $H_l(N, \mathscr{D}_G)$ so $[\omega]\nu_g = [\omega]$. Therefore $\cap[\omega]$ is a KG-module isomorphism as required.

$$\text{Let } \cdots \xrightarrow{\partial_1} P_3 \xrightarrow{\partial_1} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_1} P_0 \twoheadrightarrow K$$

be a projective resolution for K over KG, and let

$$\cdots \xrightarrow{\partial_2} Q_3 \xrightarrow{\partial_2} Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_2} Q_0 \twoheadrightarrow K$$

be a projective resolution for K over KG/N.

Using these, we form two double cochain complexes of the form given below. In (I), $X_{i,j} = \mathrm{Hom}_{G/N}(Q_i, \mathrm{Hom}_N(P_j, M))$ and the differentials are $\partial' = \partial_2^*$, $\partial'' = (-1)^i \partial_1^{*}{}_*$. In (II), $X_{i,j} = Q_{n-l-i} \otimes_{G/N} P_{l-j} \otimes_N (M \otimes \mathscr{D}_G)$ and the differentials are $\partial' = \dot{\partial}_{2*}$, $\partial'' = (-1)^i \partial_{1**}$. It is well known that I and II give the Lyndon–Hochschild–Serre spectral sequences for cohomology and homology respectively. (See [5,9] for details.)

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
X_{0,2} & \xrightarrow{\partial'} & X_{1,2} & \xrightarrow{\partial'} & X_{2,2} & \xrightarrow{\partial'} & \cdots \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
X_{0,1} & \xrightarrow{\partial'} & X_{1,1} & \xrightarrow{\partial'} & X_{2,1} & \xrightarrow{\partial'} & \cdots \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
X_{0,0} & \xrightarrow{\partial'} & X_{0,1} & \xrightarrow{\partial'} & X_{0,2} & \xrightarrow{\partial'} & \cdots
\end{array}$$

Let ω be a representative cycle of a nonzero element of $H_n(G, \mathscr{D}_G)$ and take $\alpha\omega$ to be a representative cycle of the corresponding element of $H_{n-l}(G/N, H_l(N, \mathscr{D}_G))$ under the spectral sequence corner isomorphism.

On the chain level, $\cap$ is natural with respect to module homomorphisms; by this and Lemma 3 we may define a map from double complex (I) to double complex (II) by $\cap * \circ \cap(\alpha\omega)$.

Consider the filtration

$$_{\mathrm{I}}F^p((\mathrm{Tot}\, X)^n) = \bigoplus_{\substack{r+s=n \\ r \geq p}} X_{r,s}.$$

The spectral sequences associated with this for (I) and (II) respectively satisfy

$$ {}^{(\mathrm{I})}_{1}E_2^{p,q} = H^p(G/N, H^q(N,M)) $$

$$ {}^{(\mathrm{II})}_{1}E_2^{p,q} = H_{n-l-p}(G/N, H_{l-q}(N, M\otimes\mathscr{D}_G)). $$

The map between these induced from $\cap * \circ \cap(\alpha\omega)$ is $\cap * \circ \cap[\alpha\omega]$, which is an isomorphism where M is finite dimensional, by the induction hypothesis. By the mapping theorem for spectral sequences, this induces an isomorphism between the graded objects associated with the $H^n(\mathrm{Tot}\, X)$ suitably filtered for the two spectral sequences.

Now consider the filtration

$$ {}_2F^q((\mathrm{Tot}\, X)^m) = \bigoplus_{\substack{r+s=m \\ s\geq q}} X_{r,s}. $$

$$ {}^{(\mathrm{I})}_{2}E_1^{p,q} = \begin{cases} 0 & \text{for } p\neq 0; \\ H^m(G,M) & \text{for } p=0. \end{cases} $$

$$ {}^{(\mathrm{II})}_{2}E_2^{p,q} = \begin{cases} 0 & \text{for } p\neq n-l; \\ H_m(G, M\otimes\mathscr{D}_G) & \text{for } p=n-l. \end{cases} $$

The E_∞-page is clearly the E_2-page, in both cases. Hence $\cap * 0 \cap(\alpha\omega)$ induces an isomorphism

$$ \theta\colon H^p(G,M) \to H_{n-p}(G, M\otimes\mathscr{D}_G). $$

It remains only to check that $\theta = \cap[\omega]$. As both maps are natural maps from a universal cohomology theory for G to a cohomology theory for G, we need only ensure that they coincide for $p=0$; that is, that the following diagram commutes.

$$ \begin{array}{ccc} M^G\otimes H_n(G,\mathscr{D}_G) & \xrightarrow{\ \cap\ } & H_n(G, M\otimes\mathscr{D}_G) \\ \downarrow & & \downarrow \\ (M^N)^{G/N}\otimes H_{n-l}(G/N, H_l(N,\mathscr{D}_G)) & \xrightarrow{\ \cap * \circ \cap\ } & H_{n-l}(G/N, H_l(N, M\otimes\mathscr{D}_G)) \end{array} $$

where the vertical maps are spectral sequence corner isomorphisms.

It is easily seen that

$$M^G \otimes H_n(G, \mathscr{D}_G) \to H_n(G, M \otimes \mathscr{D}_G)$$

is given by $m \cap s = f_m(s)$, $(s \in H_n(G, \mathscr{D}_G))$ where $f_m: H_n(G, \mathscr{D}_G) \to H_n(G, M \otimes \mathscr{D}_G)$ is induced from the module homomorphism $v_m: \mathscr{D}_G \to M \otimes \mathscr{D}_G$ given by

$$e \to m \otimes e.$$

Similarly, $(M^N)^{G/N} \otimes H_{n-l}(G/N, H_l(N, \mathscr{D}_G)) \xrightarrow{\cap * \circ \cap} H_{n-l}(G/N, H_l(N, M \otimes \mathscr{D}_G))$ is given by $m(\cap * \circ \cap)s = f_m(s)$ where

$$f_m: H_{n-l}(G/N, H_l(N, \mathscr{D}_G)) \to H_{n-l}(G/N, H_l(N, M \otimes \mathscr{D}_G))$$

is induced from v_m.

It now only remains to check that the following diagram commutes.

$$\begin{array}{ccc} H_n(G, \mathscr{D}_G) & \xrightarrow{\quad f_m \quad} & H_n(G, M \otimes \mathscr{D}_G) \\ \downarrow & & \downarrow \\ H_{n-l}(G/N, H_l(N, \mathscr{D}_G)) & \xrightarrow{\quad f_m \quad} & H_{n-l}(G/N, H_l(N, M \otimes \mathscr{D}_G)) \end{array}$$

where the vertical maps are spectral sequence corner isomorphisms. But these commute with module homomorphisms, and the result follows.

3. Some properties of groups in $\mathscr{X}$, and the end of the proof of Theorem 1

3.1 Proof of Theorem 2

Definition. We will say that a group G is LPD^n if each finite subset of G is contained in a PD^n-subgroup of G.

Some observations on the PD^n subgroups of an $LOPD^n$ group G

Write $\mathscr{P}$ for the set of all PD^n subgroups of G. Where $P \in \mathscr{P}$, P is generated by a finite subset X of G. If Y is another finite subset of G, $X \cup Y$ lies in some $P_1 \in \mathscr{P}$ containing P. It follows from the remark after (1), that

(i) Given $g \in G$ $\exists$ $k \in \mathbb{N}$ s.t. $g^k \in P$.

(ii) $N_G(P)/P$ is locally finite.

If X, Y are finite generating sets for P_1, $P_2 \in \mathscr{P}$, $\langle P_1, P_2 \rangle$ is generated by $X \cup Y$, so lies in some $P_3 \in \mathscr{P}$. Hence $|\langle P_1, P_2 \rangle : P_1|$, $|\langle P_1, P_2 \rangle : P_2|$ are finite, giving $|P_1 : P_1 \cap P_2|$ finite. By [2, Theorem 9.9], $P_1 \cap P_2$ is PD^n. Hence

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(iii) $P_1, P_2, \ldots, P_n \in \mathcal{P} \Rightarrow P_1 \cap P_2 \cap \cdots \cap P_n \in \mathcal{P}$.

$C_G(P)/(C_G(P) \cap P) \cong C_G(P) \cdot P/P \leqq N_G(P)/P$ and $C_G(P) \cap P \leqq \zeta(C_G(P))$ hence $C_G(P)/\zeta(C_G(P))$ is locally finite. By Schur's Theorem,

(iv) $(C_G(P))'$ is locally finite.

Lemma 4. *$S := \bigcup_{P \in \mathcal{P}} C_G(P)$ is a characteristic subgroup of G, $L := \bigcup_{P \in \mathcal{P}} (C_G(P))'$ is a locally finite characteristic subgroup of G and $S' = L$.*

Proof of lemma. Where $P_1, P_2, \ldots, P_n$ are PD^n subgroups of G, $\langle C_G(P_1), \ldots, C_G(P_n) \rangle \subseteq C_G(P_1 \cap \cdots \cap P_n)$, hence $S < G$. Plainly S is characteristic in G. $\langle (C_G(P_1))', \ldots, (C_G(P_n))' \rangle \subseteq \langle (C_G(P_1), \ldots, C_G(P_n) \rangle' \subseteq (C_G(P_1 \cap \cdots \cap P_n))'$ hence $L < G$; clearly L is characteristic in G. Any subset of L lies in some $(C_G(P))'$, hence L is locally finite. From $\langle C_G(P_1), C_G(P_2) \rangle \subseteq (C_G(P_1 \cap P_2))'$, it follows that $S' = L$.

Proof of Theorem 2 for $r=1$. S/L is abelian. The torsion elements of S/L therefore generate a locally finite subgroup W/L. W is a locally finite subgroup of S, and S/W is torsion free abelian.

We will now show that G/W is countable. G/W will then be in $\bar{\mathfrak{X}}_1$ by 1.2.

Choose a PD^n subgroup N of G. Given $g \in G$, let $N_g := N \cap N^g \cap \ldots \cap N^{g^{k-1}}$, where $g^k \in N$. N_g is PD^n by (iv), hence finitely generated. N is countable, therefore so is $\{N_g; g \in G\}$. As $g \in N_G(N_g)$, $G = \bigcup_{g \in G} N_G(N_g)$, a countable union. It now suffices to show that, for $P \in \mathcal{P}$, $N_G(P)/(N_G(P) \cap W)$ is countable. $N_G(P)/C_G(P)$ is isomorphic to a subgroup of Aut P, so is countable. So it only remains to prove that $C_G(P)/(C_G(P) \cap W)$ is countable. Suppose this is not true. Write V for $C_G(P) \cap W$. Plainly $C_G(P)/V$ is abelian. For $h \in \zeta(P)$, $k \in \mathbb{N}$, let $P_{k,h} := \{gV \in C_G(P)/V : g^k V = hV\}$. As $\zeta(P)$ is countable, there are only countably many $P_{k,h}$, so by the Dirichlet pigeonhole principle, some $P_{k,h}$ is uncountable. Hence $\exists$ distinct $g_1 V, g_2 V \in C_G(P)/V$ s.t. $g_1^k V = g_2^k V$. Since $C_G(P)/V$ is abelian, $(g_1^{-1} g_2)^k \in V$. By definition of V, $g_1^{-1} g_2 \in V$, hence $g_1 V = g_2 V$, so we have a contradiction.

Induction step. We use the notation of (1). Suppose the theorem holds for smaller r. Let H/G_1 be the maximal locally finite subgroup of the union of the centralizers of the PD subgroups of G_0/G_1. $G_0/H \in \bar{\mathfrak{X}}_1$ by result for $r=1$. We will prove that H/G_2 is (locally finite)-by-(countable $LOPD^n$)-by-(countable locally finite), so $J \in X_{r-1}$, where J/G_2 is the given locally finite normal subgroup.

It follows from the inductive hypothesis that J has a characteristic locally finite subgroup S such that J/S is countable.

$$J = J_0 \geqq J_1 \geqq \cdots \geqq J_{r-1} = 1$$

with the J_i all normal in G and the J_i/J_{i+1} all $LOPD$, since $J \in Y_{r-1}$.

$$\frac{J}{S} = \frac{J_0}{S} \geqq \frac{J_1 S}{S} \cdots \geqq \frac{J_{r-1} S}{S} = 1$$

is a series in which each J_iS is normal in G and each

$$\frac{J_iS}{J_{i+1}S}$$

is $LOPD^n$. If follows that $G/S \in Y_r$ as required.
Now it suffices to prove the following.

Lemma 5. *H/G_2 is (locally finite)-by-(countable $LOPD^n$)-by-(countable locally finite).*

Proof. Here we may assume that G_2 is trivial. First, observe that where $\{g_1,\ldots,g_k\}$ is a finite subset of H, $\langle g_1,\ldots,g_k\rangle \cap G_1$ is finitely generated. This follows from the fact that $\langle g_1,\ldots,g_k\rangle \cap G_1$ is of finite index in $\langle g_1,\ldots,g_k\rangle$ which is finitely generated.

Take an arbitrary finite subset $\{h_1,\ldots,h_k\}$ of H. It follows from the argument above that $\langle h_1,\ldots,h_k\rangle \cap G_1$ is finitely generated, so lies in an OPD^n subgroup P of G_1. Let $x_1,\ldots,x_l$ be generators of P. Then

$$P_2 = \langle h_1,\ldots,h_k, x_1,\ldots,x_l\rangle \cap G_1$$

is finitely generated, so lies inside an OPD^n group P_1 in G_1. However, it also contains P. It follows from [2, Proposition 9.22] that $|P_1 : P|$ is finite, therefore $|P_1 : P_2|$ is finite and P_2 is an OPD^n group. The second isomorphism theorem tells us that

$$\frac{\langle h_1,\ldots,h_k, x_1,\ldots,x_l\rangle}{P_2} \cong \frac{\langle h_1,\ldots,h_k\rangle G_1}{G_1}$$

therefore $\langle h_1,\ldots,h_k, x_1,\ldots,x_l\rangle$ is a finite extension of P_2, so is OPD^n-by-finite. Furthermore, this subgroups is PD^n, because the definition of the class X ensures that the order of

$$\frac{\langle h_1,\ldots,h_k\rangle G_1}{G_1}$$

is not divisible by $char\,K$. The proof of Theorem 2 for $r=1$ does not use orientability, hence H has a locally finite normal subgroup J such that H/J is countable locally (OPD^n-by-finite). H/J is the union of an ascending chain of OPD^n-by-finite groups.

If a PD^n group S has a subgroup T of finite index not divisible by $char\,K$, then the dualizing module E_T of T is the restriction to T of the dualising module E_S of S; considering the corestriction map

$$H_n(T, Res_T(E_S)) \xrightarrow{\mathrm{Cor}} H_n(S, E_S).$$

This is onto (see 4.1), therefore $H_n(T, Res_T(E_S))$ is nonzero. The only one-dimensional module M for which $H_n(T, M)$ can be nonzero is E_T. Hence $Res_T(E_S) = E_T$.

It now follows that every OPD^n-by-finite PD^n group has a unique maximal OPD^n subgroup. The unique maximal OPD^n subgroups of the OPD^n-by-finite groups in the given ascending chain form an ascending chain; they generate an $LOPD^n$ normal subgroup W/J of H/J. It is easy to see that H/W is countable locally finite.

Hence H is (locally finite)-by-(countable $LOPD^n$)-by-(countable locally finite), as required.

3.2 Proof of Theorem 1 for G uncountable

By Theorem 2, it suffices to prove that if G is locally finite with char $K \notin \mathscr{X}_G$ and M is a finite dimensional KG-module, then $H^i(G,M)=0$ for $i>0$ and

$$H_0(G,M) \xrightarrow{\cap[\omega]} H_0(G,M)$$

is an isomorphism for $0 \neq [\omega] \in H_0(G,K)$.

By a routine calculation, the above cap product is an isomorphism if and only if

$$M = M^G \oplus M\mathfrak{g} \tag{9}$$

(Gothic letters represent appropriate augmentation ideals).

Let N be the kernel of the action of G on M. G/N is then a locally finite subgroup of $GL_s(K)$ where $s = \dim_K M$.

By (9), to show that the given cap product is an isomorphism, it is enough to prove that $M = M^L \oplus M\mathfrak{l}$.

Choose a maximal linearly independent subset $\{g_1, \ldots, g_r\}$ of L. This generates a finite group $L_1 < L$. $M^{L_1} \supseteq M^L$, $M\mathfrak{l}_1 \subseteq M\mathfrak{l}$.

Suppose $\exists\ g \in L$ s.t. $g = \sum_{i=1}^r \lambda_i g_i$ with $\sum_{i=1}^r \lambda_i = k+1$, $k \neq 0$. Then for all $m \in M^{L_1}$, $m(g-1) = km$. Thus $M^{L_1} \subseteq M\mathfrak{l}$, hence $M = M\mathfrak{l}$. $M^L = 0$ since $mg = (k+1)m \neq m$.

Suppose there is no such g. Then for all $g \in G$, $g = \sum_{i=1}^r \lambda_i g_i$ with $\sum_{i=1}^r \lambda_i = 1$. Therefore $M^L = M^{L_1}$, $M\mathfrak{l} = M\mathfrak{l}_1$. Hence the given cap product is an isomorphism.

$H^i(N,M)=0$ for $i=0$ by the Universal Coefficients Theorem. By a spectral sequence corner argument, it now suffices to prove that $H^i(G/N,M)=0$ for $i=0$. By a result of Schur (see [**12**, Corollary 9.4]), $L := G/N$ is abelian-by-finite. Let A be an abelian normal subgroup of L s.t. L/A is finite. $H^i(L,M) \cong H^i(A,M)^{L/A}$. M may be expressed as a direct sum of finitely many simple KA-modules by [**12**, Corollary 1.6], hence as their direct product. Therefore $H^i(A,M) = \prod_{\lambda \in \Lambda} H^i(A,M_\lambda)$ where the M_λ are simple. Since A acts nontrivially on M_λ, $H^i(A,M_\lambda)=0$ for $i>0$ as required.

3.3 Homological and cohomological dimensions of groups in $\bar{\mathscr{X}}$

Our construction of $\mathscr{D}_G$ was based on that of the module A used in Stammbach's proof [**11**] that the homological dimension of a group G in the class C is equal to the Hirsch length hG, where C is composed of the groups whose factors are locally finite or abelian. $C \subseteq \mathfrak{X}$, and for $G \in C$, $A = \mathscr{D}_G$ and $n_G = hG$ where $n_G := \sum n(i)$ in the notation of (2). Hence $n_G = \mathrm{hd}_K(G)$. This generalizes as follows.

Lemma 7. $n_G = \mathrm{hd}_K(G)$ *for* $G \in \mathfrak{X}$.

Proof. Clearly $n_G \leqq \mathrm{hd}_K(G)$. For G with $r=1$, $\mathrm{hd}_K(G) = n_G$. Suppose result holds for $r-1$, and consider $G_1 \rightarrowtail G \twoheadrightarrow G/G_1$. $\mathrm{hd}_K(G/G_1) + \mathrm{hd}_K(G_1) \geqq \mathrm{hd}_K(G)$ and $n_{G/G_1} + n_{G_1} = n_G$. Since $\mathrm{hd}_K(G/G_1) = n_{G/G_1}$, $\mathrm{hd}_K(G_1) = n_{G_1}$ by the induction hypothesis, the result follows.

By the method used in the proof of [4, Theorem A], it is easily shown that $\mathrm{cd}_K(G/G_{r-1}) \leqq \mathrm{cd}_K(G) - n_{G_{r-1}}$. From this, $\mathrm{cd}_K(G) \geqq \mathrm{cd}_K(G/G_1) + n_{G_1}$, by induction on $r-1$.

3.4 A property of infinitely generated $LOPD^n$ groups

Where P is a PD^n subgroup of an infinitely generated $LOPD^n$ group G, P has Euler characteristic $\chi(G) = 0$.

To show this, assume G is countable. In the notation of (1), $\chi(N_i) = |N_{i+1} : N_i| \chi(N_{i+1})$ and $\chi(N_j) \in \mathbb{Z}$.

Suppose $\chi(N_i)$ is nonzero for some i. Then it is nonzero for all $i \in \mathbb{N}$, and $|\chi(N_0)| > |N_i : N_0|$ for all i, giving a contradiction.

4. An extension of Theorem 1 for certain groups

Let $G \in \mathscr{X}$ over $\mathbb{Q}$, $\mathbb{Q}_p$ or any other field K such that every locally finite subgroup of $GL_n(K)$ is finite ($\mathbb{Q}, \mathbb{Q}_p$ have this property; see [12, Theorem 9.33]). Also let G be poly (locally finite without subgroups of finite index, or orientable PD^n over K). Note that soluble minimax groups satisfy these conditions. Then G has restricted Poincaré duality on the following full subcategories of $\mathfrak{M}_{KG}$:

${}_{cfd}\mathfrak{M}_G$ whose objects are the $\varinjlim(M_j)$, $(M_j)_{j \in I}$ a direct limit system of finite dimensional KG-modules;

${}_{lfd}\mathfrak{M}_G$ with objects the $\varprojlim(M_j)$, $(M_j)_{j \in I}$ an inverse limit system of finite dimensional KG-modules.

We now prove this. We use the following from [7, Section 2] and [6, Prop. 4]. For $(M_j)_{j \in I}$ a direct limit system of finite dimensional KG-modules, the natural map

$$\varinjlim H^n(G, M_j) \to H^n(G, \varinjlim M_j) \tag{10}$$

is an isomorphism.

For $(M_j)_{j \in I}$ an inverse limit system of finite dimensional KG-modules, the natural maps

$$H^n\left(G, \varprojlim M_j\right) \to \varprojlim H^n(G, M_j)$$

$$H_n\left(G, \varprojlim M_j\right) \to \varprojlim H_n(G, M_j)$$

are isomorphisms.

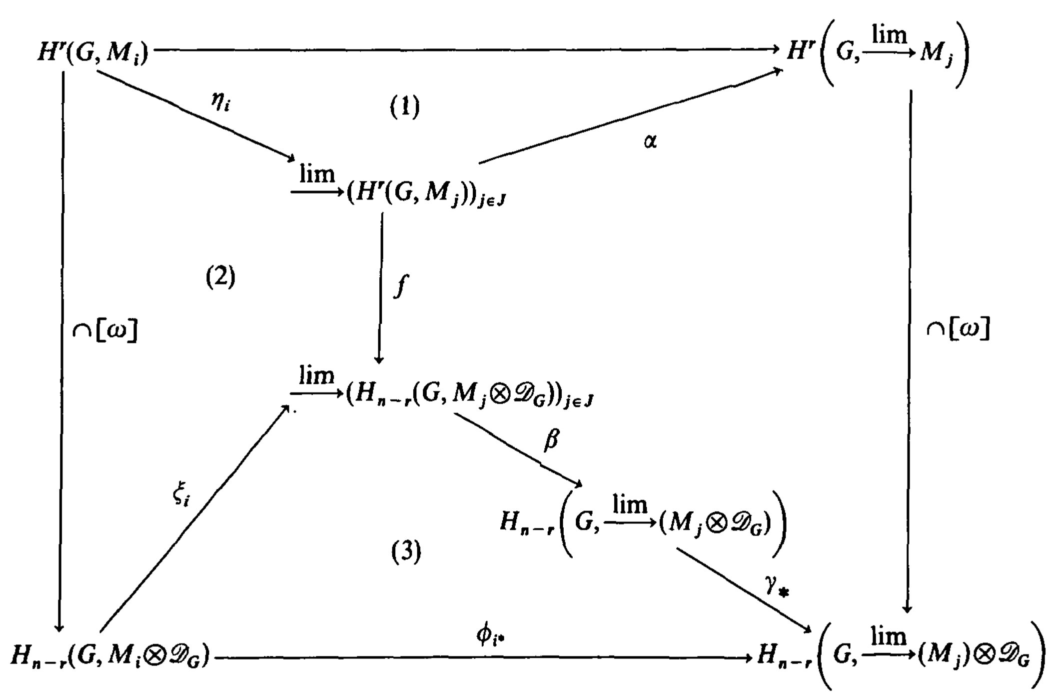
We will prove the result for ${}_{cfd}\mathfrak{M}_G$; that for ${}_{lfd}\mathfrak{M}_G$ is done by a similar method. Consider the diagram on the following page, where $[\omega]$ is a nonzero element of

$$H_n(G, \mathscr{D}_G),\ \phi\colon M_i \to \varinjlim (M_j)_{j\in I},$$

η_i and ξ_i are projections to direct limits, γ_* is induced from the isomorphism

$$\gamma\colon \varinjlim (M_j \otimes \mathscr{D}_G) \to \varinjlim (M_j) \otimes \mathscr{D}_G.$$

α, β, f are the unique maps given by the universal property of direct limit s.t. (1), (2), (3) respectively commute.



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Since (1) commutes, $\cap[\omega]\circ\alpha$ is the unique map given by the universal property such that $\cap[\omega]\circ\alpha\circ\eta_i=\cap[\omega]\circ\phi_{i*}$ for all i. Since (2) and (3) commute, $\gamma_*\circ\beta\circ f$ is the unique map such that $\gamma_*\circ\beta\circ f\circ\eta_i=\phi_{i*}\circ\cap[\omega]$ for all i. The outer square commutes for all $i\in J$, since $\cap[\omega]$ is natural with respect to coefficient homomorphisms. Hence $\gamma_*\circ\beta\circ f=\cap[\omega]\circ\alpha$.

$$H^r(G,M_i)\xrightarrow{\cap[\omega]}H_{n-r}(G,M_i\otimes\mathscr{D}_G)$$

is an isomorphism for all r, i by Theorem 1, hence f is an isomorphism. β is an isomorphism since $H_s(G,\_)$ commutes with exact colimits; α is an isomorphism by (10). Hence

$$H^r\left(G,\xrightarrow{\lim}M_i\otimes\mathscr{D}_G\right)\xrightarrow{\cap[\omega]}H_{n-r}\left(G,\xrightarrow{\lim}(M_i)\otimes\mathscr{D}_G\right)$$

is an isomorphism as required.

5. Action of $G\in\mathscr{X}$ on its dualizing module over $\mathbb{Q}$

All known PD^n groups over $\mathbb{Q}$ have all $g\in G$ acting on the dualizing module by multiplication by ± 1. However, some groups in $\mathscr{X}$ have different actions on $\mathscr{D}_G$. For example, for $G=\mathbb{Z}[1/2]\wr\langle t\rangle$, where t acts on $\mathbb{Z}[1/2]$ by multiplication by 2, t acts on $\mathscr{D}_G$ by multiplication by 2.

By the construction of $\mathscr{D}_G$ in (2.2.2), it suffices to examine actions of Aut N on $H_n(N,\mathbb{Q})$ where N is a $LOPD^n$ group.

It is well known that

$$\left.\begin{matrix}H_n(N,\mathbb{Q})\\ \text{over }\mathbb{Q}\end{matrix}\right\}\cong\left\{\begin{matrix}H_n(N,\mathbb{Q})\\ \text{over }\mathbb{Z}.\end{matrix}\right.$$

Over $\mathbb{Z}$, the Universal Coefficients Theorem gives $H_n(N,\mathbb{Q})\cong H_n(N,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$. Now we examine $H_n(N,\mathbb{Z})$ for N an $LOPD^n$ group. Let $\mathscr{P}$ be the set of all orientable PD^n subgroups of N. The $H_n(P,\mathbb{Z})$ for $P\in\mathscr{P}$ and the corestriction maps between them form a direct limit system, with direct limit $H_n(G,\mathbb{Z})$. We will now see what the Cor maps are. For $P\in\mathscr{P}$, $H_n(P,\mathbb{Z})\cong\mathbb{Z}$.

Lemma 8. *Let P_i, $P_j\in\mathscr{P}$ with $P_i<P_j$. Let $H_n(P_i,\mathbb{Z})=\langle a\rangle$, $H_n(P_j,\mathbb{Z})=\langle b\rangle$. Then* Cor: *$H_n(P_i,\mathbb{Z})\to H_n(P_j,\mathbb{Z})$ is given by*

$$a\to\pm|P_j:P_i|b.$$

Proof. First we will show that the following diagram commutes for all $k \in \mathbb{N}$ and any KP_j module M, where Sh is the map in Shapiro's Lemma, and ϕ is the map

$$\mathrm{Ind}_{P_i}^{P_j}(M) \twoheadrightarrow M \text{ given by}$$

$$r \otimes_{KP_i} m \to rm.$$

$$\begin{array}{ccc} H_k(P_i, M) & \xrightarrow{\mathrm{Cor}} & H_k(P_i, M) \\ \Big\downarrow \mathrm{Sh} & \nearrow \phi_* & \\ H_k(P_j, \mathrm{Ind}_{P_i}^{P_j}(M)) & & \end{array} \qquad (11)$$

Both $\phi \circ \mathrm{Sh}$ and Cor are natural maps from the homology theory $H_*(P_i, \_)$ for P_j to the universal homology theory $H_*(P_j, \_)$ for P_j, so it suffices to show that (11) commutes for $k = 0$. This is easily checked, so (11) commutes.

Now consider

$$\begin{array}{ccccc} H_n(P_i, \mathbb{Z}) & \xrightarrow{\mathrm{Cor}} & H_n(P_j, \mathbb{Z}) & \xleftarrow{\cap[\omega]} & \mathbb{Z} \\ \Big\downarrow \mathrm{Sh} & \nearrow \phi_* & & & \nearrow \phi_* \\ H_n(P_j, \mathrm{Ind}_{P_i}^{P_j}(\mathbb{Z})) & & \xleftarrow{\cap[\omega]} & & (\mathrm{Ind}_{P_i}^{P_j}(\mathbb{Z}))^{P_j} \end{array}$$

where $\cap[\omega]$ is the Poincaré duality isomorphism. By naturality of $\cap$, the square commutes; the triangle commutes by (11).

Take a transversal $1, g_1, g_2, \ldots, g_n$ to P_i in P_j. A general element of $(\mathrm{Ind}_{P_i}^{P_j}(\mathbb{Z}))^{P_j}$ is of the form $1 \otimes l + g_1 \otimes l + \cdots + g_n \otimes l$ where $l \in \mathbb{Z}$. Hence $\mathrm{Im}\, \phi_* = \{|P_j : P_i| b\}$, and the result follows.

By this lemma, $H_n(N, \mathbb{Z}) \cong \mathbb{Z}[1/p_0, \ldots, 1/p_n, \ldots]$ where $X_N = \{p_i\}_{i \in \mathbb{N}}$.

$H_n(N, \mathbb{Z})$ is preserved setwise by the outer automorphisms of $\mathbb{N}$. Any outer automorphism of N must therefore act on $H_n(N, \mathbb{Q})$ by multiplication by s/t, where s, t are products of elements of X_N.

Now we examine $\mathscr{P}$ more closely, to see which actions of this form may occur. We may label the $P \in \mathscr{P}$ by elements of $\mathbb{Q}$ as follows. Choose an arbitrary $P \in \mathscr{P}$, and label it as 1. Label $P_i \in \mathscr{P}$ by

$$\frac{|\langle P, P_1 \rangle : P|}{|\langle P, P_1 \rangle : P_1|}.$$

Where ϕ is an outer automorphism of G, it is easy to see that the action of ϕ on $H_n(G, \mathbb{Z})$ is multiplication by plus or minus the label of $\phi(P)$; furthermore, this multiplication must preserve the labels.

This tells us that no locally finite group N has an outer automorphism which acts on $\mathscr{D}_G$ by multiplication by $r \neq \pm 1$, for a finite subgroup of lowest order (not necessarily unique) has a label smaller than all the others. By Section 3.1, if N is an uncountable $LOPD^n$ group such that $g \in \operatorname{Aut} N$ acts by multiplication by $r \neq \pm 1$ on $H_n(N, \mathbb{Q})$, there is a countable quotient N/S s.t. g acts by multiplication by $r \neq 1$ on $H_n(N/S, \mathbb{Q})$.

We may also deduce that no PD^n subgroup P_1 of an infinitely generated $LOPD^n$ group N admitting an outer automorphism which induces multiplication by $r \neq 1$ on $H_n(G, \mathbb{Z})$ may be a hyperbolic manifold group. Take $P_2 := P_1 \cap \phi^{-1}(P_1)$, $P_3 := \phi(P_2) \subseteq P_1$, where ϕ is an outer automorphism inducing multiplication by $r \neq \pm 1$ on $H_n(G, \mathbb{Z})$. P_2 and P_3 will be hyperbolic manifold groups with different labels, hence distinct hyperbolic volumes. By rigidity (see [**10**]), these cannot be isomorphic, so we have a contradiction.

Lemma 9. *For G an arbitrary group, let X be a one-dimensional KG-module on which $g \in G$ acts by multiplication by $\rho_g \in K$. Let σ be a field automorphism of K. Then $H_r(G, X) \cong H_r(G, {}_\sigma X)$ where ${}_\sigma X$ is the one-dimensional KG-module on which g acts by multiplication by $\sigma^{-1}(\rho_g)$.*

Proof. Let α: $(KG)_1 \to (KG)_2$, $(KG)_1$ and $(KG)_2$ both copies of KG, be given by $kg \to \sigma(k)g$. It is well known (see [**2**, p. 2]) that α induces an isomorphism

$$\operatorname{Tor}_n^{(KG)_1}({}^\alpha X, K) \cong \operatorname{Tor}_n^{(KG)_2}(X, (KG)_2 \otimes_{(KG)_1} K) \tag{12}$$

where ${}^\alpha X$ is X viewed as a $(KG)_1$ module via α in the usual way.

We now examine $(KG)_2 \otimes_{(KG)_1} K$ and ${}^\alpha X$.

The action of $l \in K$ on $(KG)_2 \otimes_{(KG)_1} K$ is given by $l \otimes_{(KG)_1} k \to l \otimes_{(KG)_1} k = 1 \otimes_{(KG)_1} \sigma^{-1}(l)k$. We now write $\sigma(k)_s$ for $1 \otimes_{(KG)_1} k$. $k_s + l_s = (k+l)_s$, and the induced K-action on k_s is $k_s \to (lk)_s$. Hence $(KG)_2 \otimes_{(KG)_1} K \cong K$.

Take an isomorphism of K-modules ϕ: $K \to X$. The action of g on X is given by $g \cdot \phi(k) = \rho_g \phi(k) = \phi(\rho_g \cdot k)$. $l \in K$ acts on ${}^\alpha X$ by $l \cdot {}^\alpha\phi(k) = {}^\alpha\phi(\sigma(l) \cdot k)$, and $g \in G$ acts by $g \cdot {}^\alpha\phi(k) = {}^\alpha\phi(\rho_g \cdot k)$. Now write k_t for $\alpha_\phi(\sigma(k))$. $k_t + l_t = (k+l)_t$, the induced action of $l \in K$ is $l \cdot k_t = (lk)_t$, and the action of g is $k_t \to (\sigma^{-1}(\rho_g) \cdot k)_t$. Hence ${}^\alpha X = {}_\sigma X$, and the result follows from (12).

Corollary. *Let G be a group with restricted Poincaré duality of dimension n on ${}_{fd}\mathfrak{M}_{KG}$.*

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Then the action of $g \in G$ on the dualizing module $\mathscr{D}_G$ is multiplication by an element of the subfield of K which is fixed by all field automorphisms of K.

Proof. There is a unique one-dimensional KG-module $\mathscr{D}_G$ such that $H_n(G, \mathscr{D}_G) \cong K$. Hence, for all field automorphisms σ, ${}_{\sigma^{-1}}\mathscr{D}_G \cong \mathscr{D}_G$ as KG-modules, hence $\sigma^{-1}(\rho_g) = \rho_g$ as required.

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