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A REMARKABLE CONTINUED FRACTION

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We study a particular oscillating continued fraction, and find its two limit points.

Let $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ be sequences of positive integers, and consider the continued fraction

$$\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots$$

The *convergents* p_k/q_k to this continued fraction are defined recursively,

$$p_k = a_k p_{k-1} + b_k p_{k-2}, \quad q_k = a_k q_{k-1} + b_k q_{k-2},$$

with the initial conditions $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$. It is then easy to show that

$$\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots\frac{b_k}{a_k} = \frac{p_k}{q_k},$$

that q_k increases without limit as $k \to \infty$, and that

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^k b_k b_{k-1}\cdots b_1}{q_k q_{k-1}}. \tag{2}$$

Since a_k and b_k are positive, the convergent

$$\frac{p_k}{q_k} = \frac{a_k p_{k-1} + b_k p_{k-2}}{a_k q_{k-1} + b_k q_{k-2}}$$

lies strictly between p_{k-1}/q_{k-1} and p_{k-2}/q_{k-2}; it is easy to check that the second convergent is less than the first, and so

$$\frac{p_2}{q_2} < \frac{p_4}{q_4} < \frac{p_6}{q_6} < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

It follows immediately that

$$\lim_{k\to\infty}\frac{p_{2k}}{q_{2k}} \quad\text{and}\quad \lim_{k\to\infty}\frac{p_{2k-1}}{q_{2k-1}} \tag{3}$$

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both exist. If every b_k is 1, then from (2) we have

$$\lim_{k\to\infty}\left(\frac{p_k}{q_k}-\frac{p_{k-1}}{q_{k-1}}\right)=0$$

and so p_{2k}/q_{2k} and p_{2k-1}/q_{2k-1} approach a common limit. In the general case, however, this need not be true. The continued fraction (1) is said to *converge* if the two limits (3) are the same, and to *oscillate* if not. A simple sufficient condition for convergence is that $a_k \geqslant b_k$ for all large k, and a necessary and sufficient condition may be found in [**1**]: the continued fraction (1) converges if and only if at least one of the series

$$\frac{a_1}{b_1}+\frac{a_3b_2}{b_3b_1}+\frac{a_5b_4b_2}{b_5b_3b_1}+\frac{a_7b_6b_4b_2}{b_7b_5b_3b_1}+\cdots$$

and

$$\frac{a_2b_1}{b_2}+\frac{a_4b_3b_1}{b_4b_2}+\frac{a_6b_5b_3b_1}{b_6b_4b_2}+\cdots$$

diverges. From this result it is easy to see that the continued fraction given by $a_k=1$, $b_k=2^k$

$$\frac{2}{1+}\frac{4}{1+}\frac{8}{1+}\frac{16}{1}+\cdots$$

oscillates, but it seems difficult to evaluate the two limit points of p_k/q_k. We made the surprising discovery that if we tweak the b_k just a little, we obtain a tractable problem. Indeed we prove

THEOREM. *For the continued fraction given by* $a_k=1$, $b_k=2^k+2$,

$$\frac{4}{1+}\frac{6}{1+}\frac{10}{1+}\frac{18}{1+}\cdots\frac{2^k+2}{1}+\cdots$$

we have

$$\lim_{k\to\infty}\frac{p_{2k}}{q_{2k}}=1 \quad\text{and}\quad \lim_{k\to\infty}\frac{p_{2k-1}}{q_{2k-1}}=2.$$

PROOF: We have

(4)
$$\begin{aligned} &p_0=0,\quad p_1=4,\quad p_k=p_{k-1}+\left(2^k+2\right)p_{k-2},\\ &q_0=1,\quad q_1=1,\quad q_k=q_{k-1}+\left(2^k+2\right)q_{k-2}. \end{aligned}$$

The first few p_k, q_k are given by the table

k	0	1	2	3	4	5	$\cdots$
p_k	0	4	4	44	116	1612	$\cdots$
q_k	1	1	7	17	143	721	$\cdots$

It is not hard to show from (4) that

$$p_k = (2^k + 2^{k-1} + 5)p_{k-2} - (2^{2k-3} + 2^k + 2^{k-1} + 4)p_{k-4}, \tag{5}$$

and, of course, the same recurrence holds for the $\{q_k\}$. Thus,

$$\begin{aligned} p_k &= p_{k-1} + (2^k + 2)p_{k-2} \\ &= p_{k-2} + (2^{k-1} + 2)p_{k-3} + (2^k + 2)p_{k-2} \\ &= (2^k + 3)p_{k-2} + (2^{k-1} + 2)p_{k-3} \\ &= (2^k + 3)p_{k-2} + (2^{k-1} + 2)\left(p_{k-2} - (2^{k-2} + 2)p_{k-4}\right) \\ &= (2^k + 2^{k-1} + 5)p_{k-2} - (2^{2k-3} + 2^k + 2^{k-1} + 4)p_{k-4}. \end{aligned}$$

So we have

$$p_{2k} = (2^{2k} + 2^{2k-1} + 5)p_{2k-2} - (2^{4k-3} + 2^{2k} + 2^{2k-1} + 4)p_{2k-4} \tag{6}$$

and

$$p_{2k+1} = (2^{2k+1} + 2^{2k} + 5)p_{2k-1} - (2^{4k-1} + 2^{2k+1} + 2^{2k} + 4)p_{2k-3} \tag{7}$$

and the same recurrences hold for q_{2k}, q_{2k+1} respectively.

We now define

$$P_e(x) = \sum_{k\geqslant 0} p_{2k}x^k, \quad P_o(x) = \sum_{k\geqslant 0} p_{2k+1}x^k,$$
$$Q_e(x) = \sum_{k\geqslant 0} q_{2k}x^k, \quad Q_o(x) = \sum_{k\geqslant 0} q_{2k+1}x^k.$$

From (6) it follows that

$$\begin{aligned} &P_e(x) - 5xP_e(x) + 4x^2P_e(x) - 6xP_e(4x) + 24x^2P_e(4x) + 32x^2P_e(16x) \\ &\qquad = p_0 + (p_2 - 11p_0)x = 4x, \end{aligned} \tag{8}$$

while from (7)

$$\begin{aligned} &P_o(x) - 5xP_o(x) + 4x^2P_o(x) - 12xP_o(4x) + 48x^2P_o(4x) + 128x^2P_o(16x) \\ &\qquad = p_1 + (p_3 - 17p_1)x = 4 - 24x. \end{aligned} \tag{9}$$

Similar relations hold for $Q_e(x)$, $Q_o(x)$ respectively.

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(8) and (9) can be written

$$(10)\qquad P_e(x)=\frac{4x}{(1-x)(1-4x)}+\frac{6x}{(1-x)}P_e(4x)-\frac{32x^2}{(1-x)(1-4x)}P_e(16x),$$

$$(11)\qquad P_o(x)=\frac{4-24x}{(1-x)(1-4x)}+\frac{12x}{(1-x)}P_o(4x)-\frac{128x^2}{(1-x)(1-4x)}P_o(16x).$$

Iteration of (10) leads to to the following, which we prove by induction.

$$(12)\qquad \begin{aligned} P_e(x)=&\sum_{k=0}^{n}\frac{\left(2^{k^2+3k+3}-2^{k^2+2k+2}\right)x^{k+1}}{(1-x)(1-4x)\cdots(1-4^{k+1}x)}\\ &+\frac{\left(2^{n^2+3n+3}-2^{n^2+2n+1}\right)x^{n+1}}{(1-x)\cdots(1-4^nx)}P_e\left(4^{n+1}x\right)\\ &-\frac{\left(2^{n^2+5n+6}-2^{n^2+4n+5}\right)x^{n+2}}{(1-x)(1-4x)\cdots(1-4^{n+1}x)}P_e\left(4^{n+2}x\right).\end{aligned}$$

First, (12) is true for $n=0$ by (10). Also, if we put $4^{n+1}x$ for x in (10), we obtain

$$(13)\qquad \begin{aligned} P_e\left(4^{n+1}x\right)=&\frac{4^{n+2}x}{(1-4^{n+1}x)(1-4^{n+2}x)}+\frac{6\times 4^{n+1}x}{(1-4^{n+1}x)}P_e\left(4^{n+2}x\right)\\ &-\frac{32\times 4^{2n+2}x^2}{(1-4^{n+1}x)(1-4^{n+2}x)}P_e\left(4^{n+3}x\right).\end{aligned}$$

If we suppose (12) true for some $n\geqslant 0$, and we substitute (13) into (12), we obtain

$$(14)\qquad \begin{aligned} P_e(x)=&\sum_{k=0}^{n}\frac{\left(2^{k^2+3k+3}-2^{k^2+2k+2}\right)x^{k+1}}{(1-x)(1-4x)\cdots(1-4^{k+1}x)}\\ &+\frac{\left(2^{n^2+3n+3}-2^{n^2+2n+1}\right)x^{n+1}}{(1-x)\cdots(1-4^nx)}\\ &\times\left\{\frac{4^{n+2}x}{(1-4^{n+1}x)(1-4^{n+2}x)}+\frac{6\times 4^{n+1}x}{(1-4^{n+1}x)}P_e\left(4^{n+2}x\right)\right.\\ &\left.-\frac{32\times 4^{2n+2}x^2}{(1-4^{n+1}x)(1-4^{n+2}x)}P_e\left(4^{n+3}x\right)\right\}\\ &-\frac{\left(2^{n^2+5n+6}-2^{n^2+4n+5}\right)x^{n+2}}{(1-x)(1-4x)\cdots(1-4^{n+1}x)}P_e\left(4^{n+2}x\right)\\ =&\sum_{k=0}^{n+1}\frac{\left(2^{k^2+3k+3}-2^{k^2+2k+2}\right)x^{k+1}}{(1-x)(1-4x)\cdots(1-4^{k+1}x)}\end{aligned}$$

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$$+\frac{\left(2^{(n+1)^2+3(n+1)+3}-2^{(n+1)^2+2(n+1)+1}\right)x^{n+2}}{(1-x)(1-4x)\cdots(1-4^{n+1}x)}P_e\left(4^{n+2}x\right)$$
$$-\frac{\left(2^{(n+1)^2+5(n+1)+6}-2^{(n+1)^2+4(n+1)+5}\right)x^{n+3}}{(1-x)(1-4x)\cdots(1-4^{n+2})x}P_e\left(4^{n+3}x\right).$$

Here we have used the facts that

$$4^{n+2}\left(2^{n^2+3n+3}-2^{n^2+2n+1}\right)=2^{(n+1)^2+3(n+1)+3}-2^{(n+1)^2+2(n+1)+2},$$
$$6\times 4^{n+1}\left(2^{n^2+3n+3}-2^{n^2+2n+1}\right)-\left(2^{n^2+5n+6}-2^{n^2+4n+5}\right)$$
$$=2^{(n+1)^2+3(n+1)+3}-2^{(n+1)^2+2(n+1)+1},$$
$$32\times 4^{2n+2}\left(2^{n^2+3n+3}-2^{n^2+2n+1}\right)=2^{(n+1)^2+5(n+1)+6}-2^{(n+1)^2+4(n+1)+5}.$$

That is, (12) is true for $n+1$. So (12) is true for $n\geq 0$ by induction. If we let $n\to\infty$ we find

(15)
$$P_e(x)=\sum_{k\geqslant 0}\frac{\left(2^{k^2+k+1}-2^{k^2+1}\right)x^k}{(1-x)\cdots(1-4^kx)}.$$

In the same way, we can show that

$$P_o(x)=\sum_{k\geqslant 0}\frac{\left(2^{k^2+2k+1}-2^{k^2+k}\right)(4-24\times 4^kx)x^k}{(1-x)(1-4x)\cdots(1-4^{k+1}x)},$$
$$Q_e(x)=\sum_{k\geqslant 0}\frac{\left(2^{k^2+k+1}-2^{k^2}\right)x^k}{(1-x)\cdots(1-4^kx)},$$
$$Q_o(x)=\sum_{k\geqslant 0}\frac{\left(2^{k^2+2k+1}-2^{k^2+k}\right)x^k}{(1-x)(1-4x)\cdots(1-4^{k+1}x)}.$$

It is not hard to show that

(16)
$$P_o(x)=\sum_{k\geqslant 0}\frac{\left(3\times 2^{k^2+2k+1}-2\times 2^{k^2+k}\right)x^k}{(1-x)\cdots(1-4^kx)},$$
$$Q_o(x)=\sum_{k\geqslant 0}\frac{\left(3\times 2^{k^2+2k}-2\times 2^{k^2+k}\right)x^k}{(1-x)\cdots(1-4^kx)}.$$

For instance,

$$\begin{aligned}
P_o(x) &= \sum_{k\geqslant 0} \frac{\left(2^{k^2+2k+1} - 2^{k^2+k}\right)\left(6\left(1-4^{k+1}x\right)-2\right)x^k}{(1-x)(1-4x)\cdots(1-4^{k+1}x)} \\
&= \sum_{k\geqslant 0} \frac{6\left(2^{k^2+2k+1} - 2^{k^2+k}\right)x^k}{(1-x)\cdots(1-4^k x)} - \sum_{k\geqslant 0} \frac{2\left(2^{k^2+2k+1} - 2^{k^2+k}\right)x^k}{(1-x)\cdots(1-4^k x)} \cdot \left(1 + \frac{4^{k+1}x}{1-4^{k+1}x}\right) \\
&= \sum_{k\geqslant 0} \frac{4\left(2^{k^2+2k+1} - 2^{k^2+k}\right)x^k}{(1-x)\cdots(1-4^k x)} - \sum_{k\geqslant 0} \frac{\left(2^{k^2+4k+4} - 2^{k^2+3k+3}\right)x^{k+1}}{(1-x)\cdots(1-4^{k+1}x)} \\
&= \sum_{k\geqslant 0} \frac{4\left(2^{k^2+2k+1} - 2^{k^2+k}\right)x^k}{(1-x)\cdots(1-4^k x)} - \sum_{k\geqslant 1} \frac{\left(2^{k^2+2k+1} - 2^{k^2+k+1}\right)x^k}{(1-x)\cdots(1-4^k x)} \\
&= \sum_{k\geqslant 0} \frac{\left(3\times 2^{k^2+2k+1} - 2^{k^2+k+1}\right)x^k}{(1-x)\cdots(1-4^k x)}.
\end{aligned}$$

Now let

$$\begin{aligned}
A(x) &= \sum_{k\geqslant 0} \frac{2^{k^2+k}x^k}{(1-x)\cdots(1-4^k x)} = \sum_{k\geqslant 0} a_k x^k, \\
B(x) &= \sum_{k\geqslant 0} \frac{2^{k^2}x^k}{(1-x)\cdots(1-4^k x)} = \sum_{k\geqslant 0} b_k x^k, \qquad (17)\\
C(x) &= \sum_{k\geqslant 0} \frac{3\times 2^{k^2+2k}x^k}{(1-x)\cdots(1-4^k x)} = \sum_{k\geqslant 0} c_k x^k.
\end{aligned}$$

Then

$$\begin{aligned}
&P_e(x) = 2A(x) - 2B(x), \quad P_o(x) = 2C(x) - 2A(x), \\
&Q_e(x) = 2A(x) - B(x), \quad Q_o(x) = C(x) - 2A(x), \qquad (18)
\end{aligned}$$

from which it follows that

$$p_{2k} = 2a_k - 2b_k, \quad p_{2k+1} = 2c_k - 2a_k, \quad q_{2k} = 2a_k - b_k, \quad q_{2k+1} = c_k - 2a_k. \qquad (19)$$

We have

$$\begin{aligned}
&A(x) = \frac{1}{1-x} + \frac{4x}{1-x}A(4x), \quad B(x) = \frac{1}{1-x} + \frac{2x}{1-x}B(4x), \\
&C(x) = \frac{3}{1-x} + \frac{8x}{1-x}C(4x). \qquad (20)
\end{aligned}$$

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or,

$$(21)\qquad \begin{aligned} A(x) &= 1 + xA(x) + 4xA(4x), \\ B(x) &= 1 + xB(x) + 2xB(4x), \\ C(x) &= 3 + xC(x) + 8xC(4x). \end{aligned}$$

It follows that

$$(21)\qquad \begin{aligned} &a_0 = 1, \quad a_k = a_{k-1} + 4^k a_{k-1} = \big(4^k + 1\big)a_{k-1}, \\ &b_0 = 1, \quad b_{k-1} + 2 \times 4^{k-1} b_{k-1} = \big(2 \times 4^{k-1} + 1\big)b_{k-1}, \\ &c_0 = 3, \quad c_k = c_{k-1} + 8 \times 4^{k-1} c_{k-1} = \big(2 \times 4^k + 1\big)c_{k-1}. \end{aligned}$$

It follows that

$$(22)\qquad \begin{aligned} a_k &= (4+1)\big(4^2+1\big) \ \cdots \ \big(4^k + 1\big), \\ b_k &= (2+1)(2 \times 4 + 1) \ \cdots \ \big(2 \times 4^{k-1} + 1\big), \\ c_k &= 3(2 \times 4 + 1)\big(2 \times 4^2 + 1\big) \ \cdots \ \big(2 \times 4^k + 1\big). \end{aligned}$$

Note that $c_k = b_{k+1}$, so (19) becomes

$$(23)\qquad p_{2k} = 2a_k - 2b_k, \quad p_{2k+1} = 2b_{k+1} - 2a_k, \quad q_{2k} = 2a_k - b_k, \quad q_{2k+1} = b_{k+1} - 2a_k.$$

Observe the table

k	0	1	2	3	4	5	$\cdots$
p_k	0	4	4	44	116	1612	$\cdots$
q_k	1	1	7	17	143	721	$\cdots$
a_k	1	5	85	5525	1419925	1455423125	$\cdots$
b_k	1	3	27	891	114939	58963707	$\cdots$

Now,

$$(24)\qquad \frac{b_k}{a_k} = \frac{(2+1)}{(4+1)} \cdots \frac{\big(2 \times 4^{k-1} + 1\big)}{\big(4^k + 1\big)} \leqslant \left(\frac{3}{5}\right)^k$$

and

$$(25)\qquad \frac{a_k}{b_{k+1}} = \frac{1}{(2+1)} \frac{(4+1)}{(2 \times 4 + 1)} \cdots \frac{\big(4^k + 1\big)}{\big(2 \times 4^k + 1\big)} \leqslant \frac{1}{3}\left(\frac{5}{9}\right)^k.$$

It follows that

$$(26)\qquad 1 > \frac{p_{2k}}{q_{2k}} = 1 - \frac{b_k/a_k}{2 - b_k/a_k} \geqslant 1 - \frac{(3/5)^k}{2 - (3/5)^k} \geqslant 1 - \left(\frac{3}{5}\right)^k$$

and

$$(27)\qquad 2 < \frac{p_{2k+1}}{q_{2k+1}} = 2 + \frac{2a_k/b_{k+1}}{1 - 2a_k/b_{k+1}} \leqslant 2 + \frac{(2/3)(5/9)^k}{1 - (2/3)(5/9)^k} \leqslant 2 + 2\left(\frac{5}{9}\right)^k.$$

The result follows.

References

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