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APPROXIMATION BY FOURIER STIELTJES SERIES

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In this paper certain estimates of the rate of convergence of triangular matrix means of the Fourier Stieltjes series and its conjugate series are obtained.

1. INTRODUCTION

Let $F \in BV[0;2\pi]$. Then the Fourier Stieltjes series of F or the Fourier series of dF is defined as

$$dF(x) \sim \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu x}, \tag{1.1}$$

where $c_\nu = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\nu t} dF(t)$, $\nu = 0, \pm 1, \pm 2, \ldots$

The series conjugate to (1.1) is given by

$$-i \sum_{\nu=-\infty}^{\infty} (\text{sign } \nu) c_\nu e^{i\nu x}. \tag{1.2}$$

We denote (1.1) by $S[DF]$ and (1.2) by $\tilde{S}[DF]$.

It is convenient to define $F(x)$ for all values of x by $F(x+2\pi) - F(x) = F(2\pi) - F(0)$. This enables us to integrate, in the formula for c_ν, over any interval of length 2π.

We write

$$F_x(t) = F(x+t) - F(x-t) - 2tF'(x),$$
$$G_x(t) = F(x+t) + F(x-t) - 2F(x),$$

and denote the total variation of $f(t)$ over the interval $[0,t]$ by $V_0^t(f)$.

Let $\Lambda = (\lambda_{n,k})$, $n = 0, 1, 2, \ldots$, $k = 0, 1, 2, \ldots, n$ be a triangular matrix and let

$$\sigma_n = \sum_{k=0}^{n} \lambda_{n,k} s_k,$$

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where $\{s_k\}$ is a given sequence of numbers. σ_n is called the nth Λ-mean of $\{s_k\}$. We suppose that $\{\lambda_{n,k}\}$ is non-negative with $\sum_{k=0}^{n} \lambda_{n,k} = 1$, $n = 0, 1, 2, \ldots$. For $\lambda_{n,k} = \frac{p_{n-k}}{P_n}$, $P_n = p_0 + p_1 + \cdots + p_n$ the Λ-means reduce to Nörlund means (N, p_n). Similarly for $\lambda_{n,k} = \frac{p_k}{P_n}$ we get $(\overline{N}, p_n)$ means.

In what follows we assume that C is a positive constant not necessarily the same at each occurrence.

We prove the following theorem.

THEOREM. *Let $\{\lambda_{n,k}\}$ be non-decreasing with respect to k and let $t_n(x)$ and $\tilde{t}_n(x)$ denote respectively the Λ-means of the series $S[dF]$ and $\tilde{S}[dF]$. Then*

$$|t_n(x) - F'(x)| \leqslant C \sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x) \sum_{\nu=0}^{k} \lambda_{n,n-\nu}, \tag{1.3}$$

$$\left| \tilde{t}_n(x) - \left\{ -\frac{1}{\pi} \int_{\pi/n+1}^{\pi} \frac{G_x(t)dt}{(2\sin t/2)^2} \right\} \right| \leqslant C \sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x) \sum_{\nu=0}^{k} \lambda_{n,n-\nu}. \tag{1.4}$$

2. Proof of the theorems

PROOF OF (1.3): Writing $K_n(t) = \sum_{\nu=0}^{n} \lambda_{n,k} D_k(t)$, with $D_k(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin t/2}$ and denoting by $s_n(x)$ the n-th partial sum of (1.1) we have

$$\begin{aligned} t_n(x) &= \sum_{k=0}^{n} \lambda_{n,k} s_k(x) \\ &= \sum_{k=0}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{-\pi}^{\pi} D_k(x-t)dF(t) \\ &= \frac{1}{\pi} \int_0^{\pi} \sum_{k=0}^{n} \lambda_{n,k} D_k(t) d[F(x+t) - F(x-t)] \\ &= \frac{1}{\pi} \int_0^{\pi} K_n(t) d[F(x+t) - F(x-t)] \end{aligned}$$

and hence

$$\begin{aligned} t_n(x) - F'(x) &= \frac{1}{\pi} \int_0^{\pi} K_n(t) d[F(x+t) - F(x-t) - 2tF'(x)] \\ &= \frac{1}{\pi} \int_0^{\pi} K_n(t) dF_x(t) \\ &= \frac{1}{\pi} \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right) K_n(t) dF_x(t) \\ &= I_1 + I_2, \text{ say}. \end{aligned}$$

Since $|K_n(t)| \leqslant 2n$ uniformly in t, we have

$$|I_1| \leqslant \frac{1}{\pi}\int_0^{\pi/n+1} 2n\,|dF_x(t)| = \frac{2n}{\pi} V_0^{\pi/n+1}(F_x) \leqslant C\sum_{k=0}^{n} V_0^{\pi/k+1}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}, \tag{2.1}$$

in view of the fact that $\left(\sum_{\nu=0}^{k}\lambda_{n,n-\nu}\right)/k+1$ is non-increasing. Let $\gamma_n(t)$ be a linear function on $[k,k+1]$ such that $\gamma_n(k)=\lambda_{n,n-k}$, $k=0,1,2,\ldots,n$ and let

$$\Gamma_n(t)=\int_0^t \gamma_n(u)du, \quad t\geqslant 0.$$

Then

$$\Gamma_n(k)=\sum_{\nu=0}^{k-1}\frac{\gamma_n(\nu+1)+\gamma_n(\nu)}{2}=\sum_{\nu=0}^{k-1}\frac{\lambda_{n,n-\nu-1}+\lambda_{n,n-\nu}}{2} \leqslant \sum_{\nu=0}^{k}\lambda_{n,n-\nu}\leqslant 2\Gamma_n(k).$$

Using the well-known estimate of McFadden [**2**]

$$\left|\sum_{k=a}^{b}\lambda_{n,n-k}e^{i(n-k)t}\right| \leqslant 2(2\pi+1)\Gamma_n(\pi/t), \tag{2.2}$$

where $0\leqslant a\leqslant b\leqslant\infty$, $0<t\leqslant\pi$ and n is any non-negative integer, we have

(2.3)

$$\begin{aligned}|I_2| &\leqslant \frac{1}{\pi}\int_{\pi/n+1}^{\pi}|K_n(t)|\,|dF_x(t)| \leqslant C\int_{\pi/n+1}^{\pi}|dF_x(t)|\,\frac{\Gamma_n(\pi/t)}{t}\\ &= C\int_{\pi/n+1}^{\pi}\frac{\Gamma_n(\pi/t)}{t}dV_0^t(F_x)\\ &= C\{[\frac{\Gamma_n(\pi/t)}{t}V_0^t(F_x)]_{\pi/n+1}^{\pi} + \int_{\pi/n+1}^{\pi}V_0^t(F_x)\frac{\Gamma_n(\pi/t)}{t^2}dt\\ &+ \int_{\pi/n+1}^{\pi}\pi V_0^t(F_x)\frac{\gamma_n(\pi/t)}{t^3}dt\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{\pi}\Gamma_n(1)V_0^{\pi}(F_x) - \frac{(n+1)C}{\pi}\Gamma_n(n+1)V_0^{\frac{\pi}{n+1}}(F_x) \\
&+ \frac{C}{\pi}\int_1^{n+1} V_0^{\pi/t}(F_x)\Gamma_n(t)dt + \frac{C}{\pi}\int_1^{n+1} tV_0^{\pi/t}(F_x)\gamma_n(t)dt \\
&\leqslant C\lambda_{n,n}V_0^{\pi}(F_x) + C(n+1)V_0^{\frac{\pi}{n+1}}(F_x) \\
&+ C\sum_{k=1}^{n}\int_k^{k+1} V_0^{\pi/t}(F_x)\Gamma_n(t)dt + C\sum_{k=1}^{n}\int_k^{k+1} V_0^{\pi/t}(F_x)t\gamma_n(t)dt \\
&\leqslant C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}, \text{ as shown in (2.1),} \\
&+ C\sum_{k=1}^{n} V_0^{\pi/k}(F_x)\Gamma_n(k+1) + C\sum_{k=1}^{n} V_0^{\pi/k}(F_x)(k+1)\left(\frac{\gamma_n(k)+\gamma_n(k+1)}{2}\right) \\
&\leqslant C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu} + C\sum_{k=1}^{n} V_0^{\pi/k}(F_x)(k+1)\lambda_{n,n-k} \\
&\leqslant C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}.
\end{aligned}
$$

Thus from (2.1) and (2.3) we get the required result. ∎

PROOF OF (1.4): We have

$$\widetilde{t}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\widetilde{K}_n(t)dF(x+t),$$

where

$$\widetilde{K}_n(t) = \sum_{\nu=0}^{n}\lambda_{n,k}\widetilde{D}_k(t)$$

with

$$\widetilde{D}_k(t) = \sum_{\nu=1}^{k}\sin\nu t = \frac{\cos t/2 - \cos\left(k+\frac{1}{2}\right)t}{2\sin t/2}.$$

Now

$$
\begin{aligned}
\widetilde{t}_n(x) &= -\frac{1}{\pi}\int_0^{\pi}\widetilde{K}_n(t)d[F(x+t)+F(x-t)] \\
&= -\frac{1}{\pi}\int_0^{\pi}\widetilde{K}_n(t)dG_x(t),
\end{aligned}
$$

so that

$$\tilde{t}_n(x) - \left(-\frac{1}{\pi}\int_{\pi/n+1}^{\pi} \frac{G_x(t)}{(2\sin t/2)^2}dt\right)$$
$$= -\frac{1}{\pi}\int_0^{\pi/n+1} \tilde{K}_n(t)dG_x(t)$$
$$+ \frac{1}{\pi}[-\frac{G_x(t)}{2\tan t/2}]_{\frac{\pi}{n+1}}^{\pi} + \frac{1}{\pi}\int_{\pi/n+1}^{\pi}\{\frac{1}{2\tan t/2} - \tilde{K}_n(t)\}dG_x(t)$$
$$= L_1 + L_2 + L_3, \text{ say .}$$

Since $|\tilde{K}_n(t)| \leqslant n$, as shown in (2.1) we have

(2.4)
$$|L_1| \leqslant \frac{n}{\pi}\int_0^{\frac{\pi}{n+1}} |dG_x(t)|$$
$$\leqslant C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}.$$

Also

(2.5)
$$|L_2| = \frac{1}{\pi}\left|G_x\left(\frac{\pi}{n+1}\right) - G_x(0)\right| \frac{1}{2\tan\frac{\pi}{2(n+1)}}$$
$$\leqslant \frac{(n+1)}{\pi^2}V_0^{\frac{\pi}{n+1}}(G_x)$$
$$\leqslant C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}.$$

Using the estimate (2.2) we observe that

$$\left|\frac{1}{2\tan t/2} - \tilde{K}_n(t)\right| = \left|\sum_{k=0}^{n}\lambda_{n,k}\{\frac{1}{2\tan t/2} - \frac{\cos t/2 - \cos\left(k+\frac{1}{2}\right)t}{2\sin t/2}\}\right|$$
$$= \left|\sum_{k=0}^{n}\lambda_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{2\sin t/2}\right|$$
$$\leqslant \frac{C}{t}\Gamma_n\left(\frac{\pi}{t}\right),$$

and hence

(2.6)
$$|L_3| \leqslant C\int_{\frac{\pi}{n+1}}^{\pi} \frac{1}{t}\Gamma_n\left(\frac{\pi}{t}\right)|dG_x(t)|$$
$$\leqslant C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}$$

as shown in I_2.

The proof now follows from (2.4)-(2.6).

3. Estimates of means

Taking $\lambda_{n,k} = \frac{p_{n-k}}{P_n}$, $P_n = p_0 + p_1 + \cdots + p_n$, we get the following estimates for Nörlund means $N_n(x)$ of $S[dF]$ and $\widetilde{N}_n(x)$ of $\widetilde{S}[dF]$. [1].

Corollary 1. *If $\{p_k\}$ is non-increasing sequence of positive numbers, then*

$$|N_n(x) - F'(x)| \leqslant \frac{C}{P_n} \sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x) P_k,$$

and

$$\left| \widetilde{N}_n(x) - \{-\frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \frac{G_x(t)}{(2\sin t/2)^2} dt\} \right| \leqslant \frac{C}{P_n} \sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x) P_k.$$

We can similarly obtain estimates for $(\overline{N}, p_n)$ means by taking $\lambda_{n,k} = \frac{p_k}{P_n}$, where $\{p_k\}$ is a non-decreasing sequence of positive numbers.

As a special case for $p_k = A_k^{\alpha-1}$, $0 < \alpha \leqslant 1$, we get the following estimates for (C, α) means.

Corollary 1. *Let $\sigma_n^\alpha(x)$ and $\widetilde{\sigma}_n^\alpha(x)$ be the (C, α) means of $S[dF]$ and $\widetilde{S}[dF]$ respectively. If $0 < \alpha \leqslant 1$, then*

$$|\sigma_n^\alpha(x) - F'(x)| \leqslant C n^{-\alpha} \sum_{k=0}^{n} (k+1)^\alpha V_0^{\frac{\pi}{k+1}}(F_x),$$

and

$$|\widetilde{\sigma}_n^\alpha(x) - \{-\frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \frac{G_x(t)}{(2\sin t/2)^2} dt\}| \leqslant C n^{-\alpha} \sum_{k=0}^{n} (k+1)^\alpha V_0^{\frac{\pi}{k+1}}(G_x).$$

In view of known results: $F \in BV[0, 2\pi] \Rightarrow V_0^t(F_x) = o(t)$ and $V_0^t(G_x) = o(t)$ for almost all x ([3],p.105) we deduce the following result of Zygmund [3]:

$$\sigma_n^\alpha(x) \to F'(x),\ n \to \infty \text{ and } \widetilde{\sigma}_n^\alpha(x) + \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \frac{G_x(t)}{(2\sin t/2)^2} dt \to 0, n \to \infty$$

for almost all x.

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