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A FAMILY OF HURWITZ GROUPS WITH NON-TRIVIAL CENTRES

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In this paper a new family of quotients of the triangle group $< x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 >$ is obtained. It is shown that for every positive integer m divisible by 3 there is a Hurwitz group of order $504m^6$ having a centre of size 3, and as a consequence there is a Riemann surface of genus $6m^6 + 1$ with the maximum possible number of automorphisms.

The (2,3,7) triangle group Δ is the abstract group with presentation $\Delta = < x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 >$. A theorem of Hurwitz [4] states that a compact Riemann surface with genus $g > 1$ has at most $84(g - 1)$ conformal automorphisms, that is, homeomorphisms of the surface onto itself which preserve the local structure. Any surface with the maximum possible number of automorphisms must be uniformized by a normal subgroup N of the triangle group Δ, for the latter is the Fuchsian group with fundamental region of smallest hyperbolic area. Moreover, the conformal automorphism group is then isomorphic to the quotient group Δ/N. Conversely, any finite non-trivial quotient G of Δ gives rise to a compact Riemann surface (of genus $\frac{1}{84}|G| + 1$) with the maximum possible number of automorphisms, and G as its automorphism group. For these reasons, any finite non-trivial quotient of Δ is

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called a *Hurwitz* group.

A good number of Hurwitz groups are known, particularly amongst the finite simple groups (see [2], [3], [7], [8] for example).

In a paper on certain normal subgroups of Δ, Leech [5] raised the question of whether there exists a Hurwitz group with non-trivial centre. He answered that question in the affirmative (in a note added in proof) and later [6] produced two infinite families of Hurwitz groups, the groups in these families having centres of size 2 and 4 respectively. The groups themselves had orders $504p^7$, with p running through the positive integers congruent to 2 modulo 4, and $1008p^7$, with p running through the positive integers divisible by 4. Every group in either family was obtained as an extension of a 7-generator group by the simple group $PSL(2,8)$.

In this paper we use similar methods to produce a family of Hurwitz groups each having a centre of size 3. Specifically, for every positive integer m divisible by 3, there is a Hurwitz group G which is an extension by the simple group $PSL(2,7)$ of a 6-generator group K of order $3m^6$, such that the centre $Z(G)$ of G is cyclic of order 3 and coincides with the commutator subgroup K' of K. As a consequence, for each such m there must be a compact Riemann surface of genus $6m^6+1$ with the maximum possible number of conformal automorphisms. (Actually this surface admits a covering projection onto Klein's quartic, with K being the group of covering transformations, but we do not pursue this matter here.)

The construction of our family proceeds as follows:

Let x,y and z be the usual generators of the group Δ, so that $x^2 = y^3 = (xy)^7 = 1$ and $\langle x,y\rangle = \Delta$. Now put $A = y^{-1}xyx$ and $B = (xy)^3$, so that the defining relations for Δ become $B^7 = (AB)^2 = (A^{-1}B)^3 = 1$, and our notation is made consistent with that used by Leech in [5].

Next define the elements a_n $(0 \le n \le 6)$ by $a_0 = A^4$ and $a_n = B^{-n}a_0B^n$ (for $1 \le n \le 6$). According to Leech [5] these elements

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a_n generate a normal subgroup, say Γ, of Δ, with factor group $PSL(2,7)$, the simple group of order 168. Moreover, the generators of Γ satisfy the relations $a_6a_5a_4a_3a_2a_1a_0 = 1$ and $a_6a_3a_0a_4a_1a_5a_2 = 1$, and, eliminating the redundant generator a_6 from these, we obtain the additional relation

$$a_2^{-1}a_5^{-1}a_1^{-1}a_4^{-1}a_0^{-1}a_3^{-1}a_5a_4a_3a_2a_1a_0 = 1.$$

Also it is known that the element A acts by conjugation on the elements a_n as follows:

$$A^{-1}a_0A = a_0$$

$$A^{-1}a_1A = a_6$$

$$A^{-1}a_2A = a_5^{-1}a_6^{-1}$$

$$A^{-1}a_3A = a_2^{-1}$$

$$A^{-1}a_4A = a_0^{-1}a_3^{-1}a_6^{-1}$$

$$A^{-1}a_5A = a_4^{-1}$$

$$A^{-1}a_6A = a_0^{-1}a_1^{-1}$$

(and a convenient check on this list of conjugates is that each of the words $a_6a_5a_4a_3a_2a_1a_0$ and $a_6a_3a_0a_4a_1a_5a_2$ is conjugated by A to a conjugate of the inverse of the other).

Now let $\Sigma = [\Delta, \Gamma']$. This is the normal subgroup of Δ generated by all conjugates and inverses of elements of the form $[a, [b, c]]$ where $a \in \Delta$ and $b, c \in \Gamma$. (Here the notation $[\alpha, \beta]$ stands as usual for the commutator $\alpha^{-1}\beta^{-1}\alpha\beta$ of the elements α, β, so that, for instance, $\Gamma' = [\Gamma, \Gamma]$ is the normal subgroup of Γ generated by commutators of elements $\alpha, \beta \in \Gamma$.)

We choose Σ specifically so that any quotient G of Δ/Σ has a normal subgroup K (namely the image of Γ) with the property that

$G/K \cong \Delta/\Gamma \cong PSL(2,7)$, and also so that $[G,K'] = 1$, in other words $K' \subseteq Z(G)$. In particular we have $K' \subseteq Z(K)$, which means in group-theoretical language that K is nilpotent of class 1 or 2.

Well, let us now suppose that G is any such group. For notational convenience, let A and B and a_n (for $0 \leq n \leq 6$) now stand for the images in G of the corresponding elements of Δ. If K is the image of Γ, then since $K' \subseteq Z(K)$ it is easy to see that

$$[ab,c] = [a,c][b,c] \quad \text{and}$$
$$[a,bc] = [a,b][a,c] \quad \text{for all } a,b,c \in K,$$

and in particular (taking $c = b^{-1}$) also

$$[a,b^{-1}] = [a,b]^{-1} = [b,a] \quad \text{for all } a,b \in K.$$

Thus commutators of elements of K behave very nicely. Indeed, it follows that every element of K' can be expressed as a product of commutators of the form $[a_i,a_j]$, even a product of powers of the elements u_i, v_i, w_i defined for $0 \leq i \leq 6$ as follows:

$$u_i = [a_i,a_{i+1}]\ ,\ v_i = [a_i,a_{i+2}]\ ,\ w_i = [a_i,a_{i+3}]$$

(with subscripts treated modulo 7). Note for example that $[a_i,a_{i+4}] = [a_{i+4},a_i]^{-1} = w_{i+4}{}^{-1}$ since $(i+4) + 3 \equiv i$ modulo 7.

At this point we consider the conjugation action of the elements A and B on the normal subgroup K'. First $B^{-1}u_iB = [B^{-1}a_iB,B^{-1}a_{i+1}B]$ $= [a_{i+1},a_{i+2}] = u_{i+1}$, and similarly $B^{-1}v_iB = v_{i+1}$ and $B^{-1}w_iB = w_{i+1}$ for $0 \leq i \leq 6$. On the other hand,

$$A^{-1}u_2A = [A^{-1}a_2A,A^{-1}a_3A] = [a_5{}^{-1}a_6{}^{-1},a_2{}^{-1}] = [a_2,a_5{}^{-1}a_6{}^{-1}]$$
$$= [a_2,a_5{}^{-1}][a_2,a_6{}^{-1}] = [a_2,a_5]^{-1}[a_6,a_2] = w_2{}^{-1}w_6$$

and $$A^{-1}v_1A = [A^{-1}a_1A,A^{-1}a_3A] = [a_6,a_2{}^{-1}] = [a_6,a_2]^{-1} = w_6{}^{-1}$$

(and in fact the A-conjugates of the other generators of K' are also easy to determine). But we know that $K' \subseteq Z(G)$, hence B and A actually centralize every u_i, v_i and w_i, so that

$$u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 \quad \text{and}$$

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$$v_0 = v_1 = v_2 = v_3 = v_4 = v_5 = v_6 \text{ and}$$

$$w_0 = w_1 = w_2 = w_3 = w_4 = w_5 = w_6 \text{ and}$$

$$u_2 = w_2^{-1}w_6 \text{ and } v_1 = w_6^{-1} .$$

From these we deduce that $u_0 = u_2 = w_2^{-1}w_6 = w_0^{-1}w_0 = 1$ and $v_0 = v_1 = w_6^{-1} = w_0^{-1}$, and as a consequence it is now clear that K' is cyclic, being generated by the element w_0.

We leave it as an exercise for the reader to verify that the calculation of the conjugates of the other u_i, v_i and w_i leads to no further restrictions on K'. This could be interpreted as meaning that the factor group Γ'/Σ (that is, $\Gamma'/[\Delta,\Gamma']$) is cyclic of infinite order - however we find this is not the case, by considering the additional relation $a_2^{-1}a_5^{-1}a_1^{-1}a_4^{-1}a_0^{-1}a_3^{-1}a_5a_4a_3a_2a_1a_0 = 1$. Repeated application of the identity $ab = ba[a,b]$ gives $a_5a_4a_3a_2a_1a_0 =$ $a_3a_0a_4a_1a_5a_2u_3^{-1}v_3^{-1}u_0^{-1}v_0^{-1}w_4v_5u_4^{-1}u_1^{-1}w_5 = a_3a_0a_4a_1a_5a_2w_0^3$ (since $u_i = 1$ and $v_i = w_0^{-1}$ and $w_i = w_0$ for all i), so the additional relation simplifies to $w_0^3 = 1$. Thus K', and as a special case even Γ'/Σ itself, is cyclic of order 3 or possibly trivial.

Also the above observations give us information about the centre $Z(K)$ of K. Since K' has order 1 or 3 it follows that $[a_i^3, a_j] = [a_i, a_j]^3 = 1$ for all i,j, and therefore a_i^3 commutes with each of the generators a_j of K, that is $a_i^3 \in Z(K)$ for all i.

Now for the moment let us see what happens when G is a non-trivial finite group. The generators a_i of K, being conjugate under the element B, must all have the same (finite) order, say m. If m is not divisible by 3, then we can replace each generator a_i by its cube a_i^3, and find $K = \langle a_i \mid 0 \le i \le 6 \rangle = \langle a_i^3 \mid 0 \le i \le 6 \rangle \subseteq Z(K)$, so K

is Abelian. It then follows that G is an extension by $PSL(2,7)$ of an Abelian group of rank 3 or 6 (and exponent m), as considered by Cohen in [2], and in particular the centre $Z(G)$ of G is easily found to be trivial in that case.

On the other hand, suppose m is a positive integer divisible by 3. In this case let us take G to be the infinite group Δ/Σ, and consider the subgroup K^m generated by the elements $a_i^{\,m}$ (for $0 \le i \le 6$). First as $a_i^{\,3} \in Z(K)$ for all i, we find that K^m is central in K. Next it is easy to see that $B^{-1}a_i^{\,m}B = (B^{-1}a_iB)^m = a_{i+1}^{\,m}$ for all i (modulo 7), so that B normalizes K^m. Just as easily we obtain $A^{-1}a_0^{\,m}A = a_0^{\,m}$ and $A^{-1}a_1^{\,m}A = a_6^{\,m}$ and $A^{-1}a_3^{\,m}A = (a_2^{\,m})^{-1}$ and $A^{-1}a_5^{\,m}A = (a_4^{\,m})^{-1}$, from the known action of conjugation by A on the generators of K. But further, whenever $a,b \in K$ we find $(ab)^m = a^mb^m[b,a]^{\frac{1}{2}m(m-1)}$ using the fact that $K' \subseteq Z(K)$, and indeed $(ab)^m = a^mb^m$ since $[b,a]$ must have order 1 or 3. Application of this identity gives:

$$A^{-1}a_2^{\,m}A = (a_5^{-1}a_6^{-1})^m = (a_5^{-1})^m(a_6^{-1})^m = (a_5^{\,m})^{-1}(a_6^{\,m})^{-1}, \text{ and}$$

$$A^{-1}a\,^{m}A=(a_0^{-1}a_3^{-1}a_6^{-1})^m=(a_0^{-1})^m(a_3^{-1})^m(a_6^{-1})^m=(a_0^{\,m})^{-1}(a_3^{\,m})^{-1}(a_6^{\,m})^{-1}, \text{ and}$$

$$A^{-1}a_6^{\,m}A = (a_0^{-1}a_1^{-1})^m = (a_0^{-1})^m(a_1^{-1})^m = (a_0^{\,m})^{-1}(a_1^{\,m})^{-1}.$$

Hence also conjugation by A preserves the subgroup K^m, which is therefore normal in G. In particular, $K^m = \Omega_m/\Sigma$ for some normal subgroup Ω_m of Δ (and indeed Ω_m is the normal subgroup of Δ generated by Σ together with the mth powers of the original elements a_n for $0 \le n \le 6$).

Now define G_m to be the quotient Δ/Ω_m. Obviously G_m is a Hurwitz group, but of course also G_m is an extension by $PSL(2,7)$ of a 6-generator nilpotent group, say K_m, of class 1 or 2.

We claim that in fact K_m has class 2, and moreover that K_m' coincides with the centre $Z(G_m)$ which must have size 3.

Well, from the construction it is clear that $K_m' \subseteq Z(G_m)$. On the other hand, the quotient G_m/K_m' is also a Hurwitz group, indeed it is an extension by $PSL(2,7)$ of an Abelian group of rank $6\cdot$ and order m^6, again as considered by Cohen in [2] . Now using the known conjugation action of A and B on the elements a_i $(0 \le i \le 6)$, a routine calculation shows that no non-trivial element of the factor group K_m/K_m' can be centralized by both generators A and B of G_m; hence G_m/K_m' has trivial centre. (Alternatively, this can be seen from Cohen's calculations in [2] .) Consequently $K_m' \cap Z(G_m) \subseteq K_m'$, in other words $K_m' = Z(G_m)$.

Next consider the special case where $m = 3$. The group G_3 has obviously the presentation

$$< A,B,a_0,w_0 \mid B^7 = (AB)^2 = (A^{-1}B)^3 = a_0^{-1}A^4 = a_0^{\,3} = [a_0,B^{-1}a_0B] = w_0[a_0,B^{-2}a_0B^2]$$

$$= w_0^{-1}[a_0,B^{-3}a_0B^{3}] = [A,w_0] = [B,w_0] = 1 > ,$$

amongst others of course. Now to this presentation we may apply the Todd-Coxeter algorithm, for example to determine the index of the subgroup $< B >$ in G_3.

I have implemented a lookahead version of the Todd-Coxeter algorithm, as described in [1], on an IBM 4341 using the language PASCAL. Approximately 15 Minutes are required by this program to find that in fact $< B >$ has index 52488 in G_3. (Those readers who have access to the CAYLEY group system may like to confirm this result themselves.) It follows that G_3 has order 367416, and then since $|G_3| = |G_3/K_3|\,|K_3/K_3'|\,|K_3'| = |PSL(2,7)|\,3^6|K_3'|$, we deduce that K_3' has order 3. Hence in particular, our claim is true in the case $m = 3$.

But now for any m (divisible by 3) it is obvious that $\Omega_m \subseteq \Omega_3$; indeed as we know that Ω_m/Σ , being generated by the mth powers of the

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generators of Γ/Σ , is a subgroup of index $(\frac{m}{3})^6$ in the central subgroup Ω_3/Σ of Γ/Σ , it is clear that Ω_m has index $(\frac{m}{3})^6$ in Ω_3. Consequently

$$|G_m| = |\Delta/\Omega_m| = |\Delta/\Omega_3||\Omega_3/\Omega_m| = |G_3|(\tfrac{m}{3})^6 = 367416(\tfrac{m}{3})^6 = 504\ m^6 .$$

In particular, this means K_m' must have order 3, so our task is completed.

Actually when m is of the form $3k$ where k is coprime to 6, the group G_m is easily found to be a split extension (that is, a semi-direct product) of an Abelian group of rank 6 and order k^6 by the group G_3, and, as such, G_m can indeed be constructed in this way. We leave the verification of this claim to the interested reader.

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