Tensor Integrals

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1. *Introduction.*

A form of integration of tensors will be introduced here, which will preserve the character of a tensor when so integrated.

Let $T^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ be the components of a given tensor, defined in the co-ordinate system x^r at all points of a curve l, in a space V_n of affine connection[1] with connection coefficients L^i_{jk}. We consider the n^{p+q} differential equations

$$T^{u_1 \ldots u_p}_{v_1 \ldots v_q} = \frac{\delta}{\delta t}\left(X^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right),$$

where the quantities on the right are the intrinsic derivatives[2] of the components $X^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ of a tensor with respect to a parameter t which defines l.

Definition. If these differential equations have a solution

$$X^{u_1 \ldots u_p}_{v_1 \ldots v_q} = Z^{u_1 \ldots u_p}_{v_1 \ldots v_q}$$

(*the indices taking the values* 1 *to* n), *then* $Z^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ *will be called a tensor integral of* $T^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ *along* l *with respect to* t, *and will be written* $\int T^{u_1 \ldots u_p}_{v_1 \ldots v_q}\, \delta t$.

2. *Properties of tensor integrals.*

The quantities $Z^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ transform as components of a tensor and

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[1] Eisenhart, *Non-Riemannian Geometry* (1927), chapter I.

[2] These intrinsic derivatives must not be confused with ordinary differential coefficients. For definitions see Eisenhart, *op. cit.*, chapter I. All quantities used in this paper are real.

represent therefore a tensor which has the same transformation coefficients as the tensor $T^{u_1 \dots u_p}_{v_1 \dots v_q}$.

The tensor integral $\int T^{u_1 \dots u_p}_{v_1 \dots v_q} \delta t$, *if it exists, represents therefore the components of a tensor of the same type and order as the tensor* $T^{u_1 \dots u_p}_{v_1 \dots v_q}$.

From theorems on differential equations[1] we can deduce immediately the following two theorems:—

THEOREM 1. *If the quantities* dx^r/dt, *the coefficients of the connection of the space and the components* $T^{u_1 \dots u_p}_{v_1 \dots v_q}$ *of a tensor in the co-ordinate system* x^r *are continuous functions of the parameter t which defines a curve l, then a tensor integral* $\int T^{u_1 \dots u_p}_{v_1 \dots v_q} \delta t$ *exists, representing components of a tensor which are continuous along l.*

THEOREM 2. *If the co-ordinates* x^r, *the coefficients of the connection of the space and the components* $T^{u_1 \dots u_p}_{v_1 \dots v_q}$ *of a tensor in this co-ordinate system are analytic functions of the parameter t which defines a curve l, then a tensor integral* $\int T^{u_1 \dots u_p}_{v_1 \dots v_q} \delta t$ *exists as a set of analytic functions of t along l.*

3. *Types of tensor integrals.*

Definition. Quantities $P^{u_1 \dots u_p}_{v_1 \dots v_q}$ satisfying the equations

$$\frac{\delta}{\delta t}\left(X^{u_1 \dots u_p}_{v_1 \dots v_q}\right) = 0 \tag{1}$$

will be said to be the components of tensors *parallel with respect to l such that any one of these tensors may be obtained from any other by parallel displacement along this curve.*

Hence, if $X^{u_1 \dots u_p}_{v \dots v_q} = A^{u_1 \dots u_p}_{v_1 \dots v_q}$ be a *particular* solution of the equations

$$T^{u_1 \dots u_p}_{v_1 \dots v_q} = \frac{\delta}{\delta t}\left(X^{u_1 \dots u_p}_{v_1 \dots v_q}\right), \tag{2}$$

[1] Goursat, *Mathematical Analysis*, translated by Hedrick, vol. II (1916).

then, by (1), the *complete* solution of equations (2) is given by

$$X^{u_1 \ldots u_p}_{v_1 \ldots v_q} = A^{u_1 \ldots u_p}_{v_1 \ldots v_q} - P^{u_1 \ldots u_p}_{v_1 \ldots v_q}.$$

(a) Thus we obtain the theorem:—

THEOREM 3. *Under the conditions of either Theorem* 1 *or Theorem* 2, *all tensor integrals of* $T^{u_1 \ldots u_v}_{v_1 \ldots v_q}$ *with respect to t are given by*

$$\int T^{u_1 \ldots u_p}_{v_1 \ldots v_q} \delta t = A^{u_1 \ldots u_p}_{v_1 \ldots v_q} - P^{u_1 \ldots u_p}_{v_1 \ldots v_q},$$

where $A^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ *form a particular tensor integral of the given tensor with respect to t, and* $P^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ *are the components of tensors obtained by parallel displacement along l of an arbitrary initial tensor.*

(b) Under the conditions of Theorem 3, we define the tensor integral of $T^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ along l between the limits t_0 and t_1 by

$$\int_{t_0}^{t_1} T^{u_1 \ldots u_p}_{v_1 \ldots v_q} \delta t = \left(A^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_{t_1} - \left(P^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_{t_1},$$

where $\left(A^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_{t_1}$ are the components of the tensor $A^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ at a point t_1 on l, and $\left(P^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_{t_1}$ are the components of a tensor at the same point t_1 obtained by parallel displacement of the tensor $\left(A^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_{t_0}$ along l from the point t_0 to t_1.

(c) The tensor integral of a tensor of order zero is the ordinary integral of the function representing this tensor.

4. *Tensor integration by parts.*

From the definition and properties of tensor integrals (stated in sections 1 and 2) it follows at once that *tensor integration with respect to a parameter defining a curve and intrinsic differentiation with respect to this parameter are inverse operations.*

THEOREM 4. *Let the quantities dx^r/dt and the coefficients of the connection of the space be continuous along a curve l defined by the parameter t.*

Let the components U of a tensor in the co-ordinate system x^r have continuous intrinsic derivatives $\delta U/\delta t$ along l, and let the components W of another tensor in this co-ordinate system be continuous along l.

Then

$$\int_{t_0}^{t_1} UW\,\delta t = \left(U\int W\delta t\right)_{t_1} - \left(P\right)_{t_1} - \int_{t_0}^{t_1}\left[\frac{\delta U}{\delta t}\int W\delta t\right]\delta t,$$

where $(P)_{t_1}$ are the components of a tensor at the point t_1 obtained by parallel displacement of the tensor $\left(U\int W\delta t\right)_{t_0}$ along l from the point t_0 to t_1.

Proof. Put $\int W\delta t = M$.

We have

$$\frac{\delta}{\delta t}\left(UM\right) = U\frac{\delta M}{\delta t} + M\frac{\delta U}{\delta t}$$

Hence
$$\int_{t_0}^{t_1}\frac{\delta}{\delta t}\left(UM\right)\delta t = \int_{t_0}^{t_1} U\frac{\delta M}{\delta t}\,\delta t + \int_{t_0}^{t_1} M\frac{\delta U}{\delta t}\,\delta t,$$

that is,

$$\left(UM\right)_{t_1} - \left(P\right)_{t_1} = \int_{t_0}^{t_1} U\frac{\delta M}{\delta t}\,\delta t + \int_{t_0}^{t_1} M\frac{\delta U}{\delta t}\,\delta t.$$

The conclusion follows.

5. *The m^{th} tensor integral of a tensor.*

THEOREM 5. *Let the quantities dx^r/dt and the coefficients of the connection of the space be continuous along a curve l defined by the parameter t.*

If the components W of a tensor in the co-ordinate system x^r are continuous along l, then, for all positive integers m, the m^{th} tensor integral of W with respect to t between t_0 and z is given by

$$\frac{1}{(m-1)!}\int_{t_0}^{z}(z-t)^{m-1}\,W\delta t.$$

Proof. $(z-t)^{m-1}$ is a tensor of order zero. Applying Theorem 4, we obtain

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$$\frac{1}{(m-1)!}\int_{t_0}^{z}(z-1)^{m-1}\,W\delta t$$

$$=\left(\frac{(z-t)^{m-1}}{(m-1)!}\int_{t_0}^{t}W\delta t\right)_z-(P)_z+\int_{t_0}^{z}\left[\frac{(z-t)^{m-2}}{(m-2)!}\int_{t_0}^{t}W\delta t\right]\delta t$$

$$=\frac{1}{(m-2)!}\int_{t_0}^{z}\left[(z-t)^{m-2}\int_{t_0}^{t}W\delta t\right]\delta t,$$

since the components $(P)_z$ belong to a tensor obtained by parallel displacement of a zero tensor and are therefore zero for all values of the indices.

Continuing this process of tensor integration by parts, we obtain the conclusion.

6. *Fractional tensor integrals and intrinsic derivatives.*[1]

Now let m have any real value, but let the conditions of Theorem 5 be otherwise satisfied. Then we define the m^{th} tensor integral, or the $(-m)^{\text{th}}$ intrinsic derivative, of W with respect to z by [2]

$$\left(\frac{\delta}{\delta z}\right)^{-m}W=\frac{1}{\Gamma(m+c)}\left(\frac{\delta}{\delta z}\right)^{c}\int_{t_0}^{z}(z-t)^{m+c-1}\,W\delta t,$$

where c is the least integer greater than or equal to zero such that $m+c\geqslant 1$.

Since tensor integration and intrinsic differentiation do not alter the type and order of a tensor, and $(z-t)^{m+c-1}$ is a tensor of order zero, it follows that, *for any real value of m, the m^{th} tensor integral $\left(\frac{\delta}{\delta z}\right)^{-m}W$ is a tensor of the same type and order as the tensor W.*

7. *Tensor expansions.*[3]

THEOREM 6. *Let the quantities dx^r/dt and the coefficients of the connection of the space be continuous along a curve l defined by the parameter t. Let the derivatives $\delta^m T^{u_1\ldots u_p}_{v_1\ldots v_q}/\delta t^m$ of the components $T^{u_1\ldots u_p}_{v_1\ldots v_q}$ of*

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[1] See my paper in *Phil. Mag.* (7), XX (1935), 781-789.

[2] The gamma function $\Gamma(m+c)$ is not to be confused with the Christoffel symbols.

[3] These expansions correspond to the Taylor series for ordinary functions.

a tensor in the co-ordinate system x^r be continuous along l for all positive and zero integers m. Then, in the interval of convergence,

$$\left(T^{u_1 \dots u_p}_{v_1 \dots v_q}\right)_z = \sum_{m=0}^{\infty} \frac{(z-t_0)^m}{m!} \left({}^{(m)} U^{u_1 \dots u_p}_{v_1 \dots v_q}\right)_z ,$$

where $\left({}^{(m)} U^{u_1 \dots u_p}_{v_1 \dots v_q}\right)_z$ *are the components of a tensor obtained by parallel displacement of the tensor* $\left(\dfrac{\delta^m T^{u_1 \dots u_p}_{v_1 \dots v_q}}{\delta t^m}\right)_{t_0}$ *along l from t_0 to z.*

Proof. Write T for $T^{u_1 \dots u_p}_{v_1 \dots v_q}$, and apply the process of tensor integration by parts of Theorem 4 to the tensor integral $\int_{t_0}^{z} 1 T \delta t$. Since 1 is a tensor of order zero (so that its tensor integral is its ordinary integral), we have, by Theorem 4,

$$\begin{aligned} \int_{t_0}^{z} 1 T \delta t &= (-(z-t)\,T)_z - (P)_z + \int_{t_0}^{z} (z-t) \frac{\delta T}{\delta t} \delta t \\ &= -(P)_z + \int_{t_0}^{z} (z-t) \frac{\delta T}{\delta t} \delta t, \end{aligned} \tag{1}$$

where $(P)_z$ is the tensor at the point z obtained by parallel displacement of the tensor $(-(z-t)\,T)_{t_0}$ along l from t_0 to z.

We have also $\dfrac{\delta P}{\delta t} = 0$

and $\dfrac{\delta}{\delta t}({}^{(0)}U) = 0,$ (2)

where ${}^{(0)}U$ stands for ${}^{(0)}U^{u_1 \dots u_p}_{v_1 \dots v_q}$.

At the point $t = t_0$,

$$P = (-(z-t)\,T)_{t_0} = -(z-t_0)\,({}^{(0)}U)_{t_0}. \tag{3}$$

By (2) and (3), $P = -(z-t_0)({}^{(0)}U)$ is a solution of $\delta P/\delta t = 0$.

Hence by (1) we have

$$\int_{t_0}^{z} T \delta t = (z-t_0)\,({}^{(0)}U)_z + \int_{t_0}^{z} (z-t) \frac{\delta T}{\delta t} \delta t$$

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$$= (z - t_0)\,({}^{(0)}U)_z + \ldots + \frac{(z-t_0)^m}{m!}\,({}^{(m-1)}U)_z + \ldots$$

$$\ldots + \int_{t_0}^{z} \frac{(z-t)^m}{m!}\,\frac{\delta^m T}{\delta t^m}\,\delta t$$

on continuing this tensor integration by parts, where ${}^{(m-1)}U$ stands for ${}^{(m-1)}U^{u_1 \ldots u_p}_{v_1 \ldots v_q}$.

Hence $(T)_z \equiv \dfrac{\delta}{\delta z}\displaystyle\int_{t_0}^{z} T\,\delta t$

$$= \sum_{n=1}^{m} \frac{(z-t_0)^{n-1}}{(n-1)!}\,({}^{(n-1)}U)_z + \frac{\delta}{\delta z}\int_{t_0}^{z} \frac{(z-t)^m}{m!}\,\frac{\delta^m T}{\delta t^m}\,\delta t.$$

This gives the required series, which converges if

$$\frac{\delta}{\delta z}\int_{t_0}^{z} \frac{(z-t)^m}{m!}\,\frac{\delta^m T}{\delta t^m}\,\delta t \to 0$$

when $m \to \infty$.

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