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ON THE INEQUALITY

$$\sum_{i=1}^{n} p_i \frac{f_i(p_i)}{f_i(q_i)} \leq 1$$

BY

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1. **Introduction.** In this paper, we are concerned with the functional inequality

$$\sum_{i=1}^{n} p_i \frac{f_i(p_i)}{f_i(q_i)} \leq 1 \tag{1}$$

where $0<p_i<1$, $0<q_i<1$, $f_i(p)\neq 0$, for $0<p<1$, $(i=1,2,\ldots,n)$ $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, and n is a fixed positive integer, $n\geq 2$.

Inequality (1) was studied by Rényi and Fischer, (see [1], [3]) in the special case

$$\sum_{i=1}^{n} p_i \frac{f(p_i)}{f(q_i)} \leq 1 \tag{2}$$

and this provided a characterization of Rényi's entropy. Aczél considered a similar generalization of a similar but simpler and more fundamental inequality in [4].

Fischer has shown [1] that the general positive solution of (2) for $n\geq 3$ has the form

$$f(p)=dp^c \quad \text{where} \quad d>0 \quad \text{and} \quad -1\leq c\leq 0$$

and he also investigated (2) for f which may change signs. For $n=2$ in (2), Fischer proved that the general positive solution is monotone decreasing and continuous and in this case he also gave the general monotone decreasing solution with non-constant sign.

In this article, we give the general solution, with constant sign, to inequality (1) for fixed $n\geq 2$ and to inequality (2) when $n=2$. No regularity assumptions will be imposed on the functions.

2. **The case** $n\geq 3$. Our first theorem extends the results in [1] and [2].

THEOREM 1. *Let* $r\in(0,1]$ *be fixed and let* $f_i:(0,1)\to R$, $i=1,2$, *satisfy the*

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inequality

$$p\frac{f_1(p)}{f_1(q)}+(r-p)\frac{f_2(r-p)}{f_2(r-q)}\le r \tag{3}$$

for all $p\in(0,r)$ *and* $q\in(0,r)$. *If* f_i *do not change signs, then each of the following hold:*

(i) f_i *is monotonic decreasing (increasing) on* $(0,r)$ *if* f_i *is positive (negative) on* $(0,1)$.

(ii) $p\to pf_i(p)$ *is increasing (decreasing) on* $(0,r)$ *if* f_i *is positive (negative) on* $(0,1)$.

(iii) f_i *is locally absolutely continuous on* $(0,r)$*.

(iv) *if* f_1 *is differentiable at* p *then* f_2 *is differentiable at* $r-p$ *and the following relation is valid:*

$$p\frac{f_1'(p)}{f_1(p)}=(r-p)\frac{f_2'(r-p}{f_2(e-p)}. \tag{4}$$

Proof. We interchange p and q in (3) and obtain

$$q\frac{f_1(q)}{f_1(p)}+(r-q)\frac{f_2(r-q)}{f_2(r-p)}\le r. \tag{5}$$

We can write (3) and (5) in the forms

$$\frac{f_2(r-p)}{f_2(r-q)}\le\frac{r-[pf_1(p)/f_1(q)]}{r-p}\quad\text{and}\quad\frac{f_2(r-q)}{f_2(r-p)}\le\frac{r-[qf_1(q)/f_1(p)]}{r-q}.$$

When we multiply these two inequalities, we get

$$1\le\frac{[rf_1(q)-pf_1(p)][rf_1(p)-qf_1(q)]}{(r-p)(r-q)f_1(p)f_1(q)}$$

or, as f_1 does not change signs, that

$$r[f_1(q)-f_1(p)][qf_1(q)-pf_1(p)]\le 0.$$

Hence,

$$[f_i(q)-f_i(p)][qf_i(q)-pf_i(p)]\le 0 \tag{6}$$

for $i=1$, and by symmetry, for $i=2$. We shall now show (i), (ii), and (iii) in the case when f_i is positive. If $f_i(p)<f_i(q)$ for some $p<q<r$ then the left-hand side of (6) would be positive. The contradiction implies that f_i is decreasing on $(0,r)$. Moreover, if $p<q$ then $f_i(p)\ge f_i(q)$ and hence, by (6), $pf_i(p)\le qf_i(q)$.

We prove that f_i is locally absolutely continuous on $(0,r)$ in the case when f_i

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is positive, $i=1, 2$. Let a, b, ε be fixed, $0<\varepsilon<a<b<r$, and let $s, t\in[a, b]$ be any two numbers, $s<t$. It follows from the monotonicity of $p\to pf_i(p)$ and f_i that

$$\begin{aligned}0\le tf_i(t)-sf_i(s)&=(t-s)f_i(t)+s[f_i(t)-f_i(s)]\\&\le(t-s)f_i(t)\\&\le(t-s)f_i(\varepsilon).\end{aligned}$$

Hence

$$|tf_i(t)-sf_i(s)|\le|t-s|\,f_i(\varepsilon)\quad\text{for all}\quad a\le s<t\le b \tag{7}$$

and, by symmetry, (7) also holds for all $t<s$. Thus, $p\to pf_i(p)$ is Lipschitz on $[a, b]$ and therefore f_i is locally absolutely continuous on $(0, r)$.

We prove (iv) in the case when $f_2>0$ on $(0, 1)$. We may write (3) and (5) as

$$(r-p)\frac{f_2(r-p)-f_2(r-q)}{f_2(r-q)}\le p\frac{f_1(q)-f_1(p)}{f_1(q)}$$

and

$$(r-q)\frac{f_2(r-q)-f_2(r-p)}{f_2(r-p)}\le q\frac{f_1(p)-f_1(q)}{f_1(p)}$$

respectively. We deduce from these inequalities, if $r>q>p$, that

$$\begin{aligned}\frac{f_2(r-p)}{f_1(p)}\cdot\frac{q}{r-q}\,\frac{f_1(q)-f_1(p)}{q-p}&\le\frac{f_2(r-q)-f_2(r-p)}{(r-q)-(r-p)}\\&\le\frac{f_2(r-q)}{f_1(q)}\cdot\frac{p}{r-p}\,\frac{f_1(q)-f_1(p)}{q-p}.\end{aligned}$$

Now, (iv) can be derived by letting $q\to p^+$. The case $q<p<r$, $q\to p^-$ leads similarly to the desired result.

We give the general solution to (1) when each f_i has constant sign and $n\ge3$ in

THEOREM 2. *If* f_i $(i=1,2,\ldots,n)$ *do not change signs, then the general solution to* (1) *for fixed* $n\ge3$ *has the form*

$$f_i(p)=b_ip^a,\qquad i=1,2,\ldots,n \tag{8}$$

where $-1\le a\le0$ *and* $b_i>0$ (<0) *if* $f_i>0$ (<0).

Proof. Put $p_i=q_i$ $(i=3,4,\ldots,n)$ *into* (1). With

$$p_1+p_2=q_1+q_2=r,\ p_1=p,\ q_1=q,\ p_2=r-p,\ q_2=r-q,$$

(1) goes over into (3). Inequality (3) holds for all $p\in(0, r)$, $q\in(0, r)$, and each $r\in(0, 1)$. But then Theorem 1 (iv) means the following. If f_1 were not differentiable at a point p_0, then f_2 would not be differentaible at any $r-p_0$

($r \in (p_0, 1)$), that is, on the interval $(0, 1-p_0)$. Since f_2 is monotonic, f_2 is differentiable almost everywhere on (0, 1). Hence f_1 and, by symmetry, also f_2 are differentiable on (0, 1) and (4) implies that

$$p\frac{f_1'(p)}{f_1(p)} = p\frac{f_2'(p)}{f_2(p)} = a \quad \text{for all} \quad p \in (0, 1).$$

By solving for f_i we obtain (8), $i = 1, 2$. Similarly, we can pair f_1 in turn with f_i, $i = 3, 4, \ldots, n$ and find that all f_i are given by (8). Also, by Hölder's inequality,

$$\sum_{i=1}^{n} p_i \frac{p_i^a}{q_i^a} = \sum_{i=1}^{n} p_i^{a+1} q_i^{-a} \leq \left(\sum_{i=1}^{n} p_i\right)^{a+1} \left(\sum_{i=1}^{n} q_i\right)^{-a} = 1$$

for $-1 \leq a \leq 0$ and in fact the opposite inequality holds when $a < -1$ or $a > 0$.

3. **The case** $n = 2$. In this section we give the general solution to (1) and (2) for $n = 2$, when the functions do not change signs. For $n = 2$, inequality (1) goes over into (3) with $r = 1$.

THEOREM 3. *All solutions f_i that do not change signs on* (0, 1), $i = 1, 2$, *of the inequality*

$$p\frac{f_1(p)}{f_1(q)} + (1-p)\frac{f_2(1-p)}{f_2(1-q)} \leq 1, \qquad 0 < p < 1, 0 < q < 1, \tag{9}$$

are of the form

$$f_1(p) = a \exp\left(\int_c^p \frac{(1-t)g'(1-t)}{tg(1-t)}\, dt\right), \qquad f_2(p) = bg(p), p \in (0, 1), \tag{10}$$

where a, b, and c are arbitrary, $ab \neq 0$, $c \in (0, 1)$, with g arbitrary continuous, positive, decreasing, and $p \to pg(p)$ increasing on (0, 1).

Proof. Let f_i be solutions to (9) that do not change signs, say $f_i > 0$, $i = 1, 2$. By Theorem 1, f_2 is decreasing and continuous while $pf_2(p)$ is increasing on (0, 1). It follows from Theorem 1 (iii) and (iv) that

$$p\frac{f_1'(p)}{f_1(p)} = (1-p)\frac{f_2'(1-p)}{f_2(1-p)} \tag{11}$$

almost everywhere on (0, 1). Since f_2 is locally absolutely continuous on (0, 1), therefore $((1-t)/t)(f_2'(1-t)/f_2(1-t))$ is locally integrable on (0, 1) and we solve for f_1 in (11) to obtain (10) with $a = b = 1$, and $f_2 = g$. To prove the converse, it is enough to demonstrate that (10) satisfies (9) when $a = b = 1$. Let g be an arbitrary continuous, positive, decreasing function such that $pg(p)$ is increasing. The argument in Theorem 1 (iii) shows that g is locally absolutely continuous on (0, 1). We can prove that $pf_1(p)$ as defined in (10) is increasing. Indeed, for

$c>0$,

$$pf_1(p)=p\exp\left(\int_c^p \frac{(1-t)g'(1-t)}{tg(1-t)}\,dt\right)$$

(12)
$$=c\exp\left(\int_c^p \frac{(1-t)g'(1-t)}{tg(1-t)}\,dt+\int_c^p \frac{1}{t}\,dt\right)$$

$$=c\exp\left(\int_c^p \frac{(1-t)g'(1-t)+g(1-t)}{tg(1-t)}\,dt\right).$$

As $pg(p)$ is increasing, $d[pg(p)]/dp=pg'(p)+g(p)\geq 0$ almost everywhere and, as g is positive,

(13)
$$\int_{p_1}^{p_2} \frac{(1-t)g'(1-t)+g(1-t)}{tg(1-t)}\geq 0$$

for all $0<p_1<p_2<1$. From (12) and (13) we have

$$p_1f_1(p_1)\leq p_2f_1(p_2) \quad \text{for} \quad 0<p_1<p_2<1$$

and thus $pf_1(p)$ is increasing. By differentiating f_1 in (10), we obtain

$$\frac{tf_1'(t)}{f_1(t)}=(1-t)\frac{g'(1-t)}{g(1-t)}$$

a.e. on (0, 1), say for all $t\in A$. Define

$$H(t)=\frac{tf_1'(t)}{f_1(t)}=(1-t)\frac{g'(1-t)}{g(1-t)},\ t\in A;$$

then $H(t)\leq 0$, $t\in A$. Let p, q be fixed, $1>q>p>0$. Because $pf_1(p)$ and $pg(p)$ are increasing, we find in logical sequence, for $t\in A\cap[p,q]$, that

$$\frac{pf_1(p)}{tf_1(t)}\leq 1\leq \frac{(1-p)g(1-p)}{(1-t)g(1-t)},$$

$$H(t)\frac{pf_1(p)}{tf_1(t)}\geq H(t)\frac{(1-p)g(1-p)}{(1-t)g(1-t)},$$

$$p\frac{f_1(p)f_1'(t)}{f_1(t)^2}\geq(1-p)\frac{g(1-p)g'(1-t)}{g(1-t)^2},$$

$$pf_1(p)\int_p^q \frac{f_1'(t)}{f_1(t)^2}\,dt\geq(1-p)g(1-p)\int_p^q \frac{g'(1-t)}{g(1-t)^2}\,dt,$$

$$pf_1(p)\left[-\frac{1}{f_1(t)}\right]_p^q\geq(1-p)g(1-p)\left[-\frac{1}{g(t)}\right]_{1-q}^{1-p},$$

and that

$$1\geq p\frac{f_1(p)}{f_1(q)}+(1-p)\frac{g(1-p)}{g(1-q)}=p\frac{f_1(p)}{f_1(q)}+(1-p)\frac{f_2(1-p)}{f_2(1-q)},$$

which is (9). Similarly, (9) holds if $q<p$.

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We now give the general solution to inequality (2), when $n=2$, if f has constant sign.

THEOREM 4. *All solutions that do not change signs on* (0, 1), *of the inequality*

$$p\frac{f(p)}{f(q)}+(1-p)\frac{f(1-p)}{f(1-q)}\le 1, \qquad 0<p<1, \qquad 0<q<1, \tag{14}$$

are of the form

$$f(p)=a\exp\left(\int_b^p \frac{G(t)}{t}\,dt\right), \qquad p\in(0,1), \tag{15}$$

where $a\neq 0$, $b\in(0,1)$ *with* G *arbitrary measurable on* (0, 1) *and satisfying for almost all* $p\in(0,1)$

$$G(1-p)=G(p) \tag{16}$$

and

$$-1\le G(p)\le 0. \tag{17}$$

Proof. We may suppose that $f>0$ on (0, 1). We shall use Theorem 1 with $f=f_1=f_2$ and $r=1$. It follows from (i) that f is differentiable a.e. on (0, 1). Then, by (iv) we have that

$$p\frac{f'(p)}{f(p)}=(1-p)\frac{f'(1-p)}{f(1-p)}$$

for almost all p on (0, 1), say on A. Define

$$G(p)=p\frac{f'(p)}{f(p)}, \qquad p\in A;$$

then (16) holds. Moreover, by (ii), $pf(p)$ is increasing so that

$$f(p)+pf'(p)\ge 0, \quad \text{for} \quad p\in A.$$

Thus

$$1+G(p)\ge 0, \qquad p\in A,$$

and since $f'(p)\le 0$, (17) is valid. We obtain from (iii) that f is locally absolutely continuous on (0, 1). Therefore G is measurable and

$$\frac{G(p)}{p}=\frac{f'(p)}{f(p)} \tag{18}$$

is locally integrable on (0, 1). We derive (15) by integrating (18). It can be shown, as in Theorem 3, that all f given by (15) satisfy (14).

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