PERSISTENT RANDOM WALKS MAY HAVE ARBITRARILY LARGE TAILS

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Abstract

We give a probabilistic proof of a result of Shepp, that a symmetric random walk may have jump size distribution with arbitrarily large tails and yet still be persistent.

PERSISTENCE; TRANSIENCE; TAIL BEHAVIOUR

1. Introduction and statement of result

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, which we shall take to be integer valued and symmetrically distributed about zero. Define the random walk $\{S_n\}$ by

$$S_n = X_1 + X_2 + \cdots + X_n \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

The random walk is called *persistent* if

$$P(S_n = 0 \text{ for some } n \geq 1) = 1.$$

Otherwise it is called *transient*. The one-dimensional Chung–Fuchs theorem (see Section 3) tells us that for the random walk to be transient it is necessary that $E|X_1| = \infty$. If the tail $G(x) = P(|X_1| > x)$ is assumed to be regularly varying, then using the analytic necessary and sufficient condition for persistence (Spitzer (1964)) and Tauberian theorems of Pitman (1968) it is possible to obtain necessary and sufficient conditions in terms of G. However, Shepp (1964) proved the following surprising result.

Theorem. Given a function ε with $\varepsilon(x) \downarrow 0$ as $x \to \infty$, there exists a persistent random walk with $G(x) \geq \varepsilon(x)$ for all large enough x.

This result suggests that the size of the tail of the distribution of X_1 does not tell the whole story. It may be regarded as a counterexample in the spirit of Stoyanov (1987).

The purpose of this note is to provide a probabilistic proof of Shepp's theorem, which was originally proved by analytic methods involving characteristic functions. The motivation for considering this problem was the resemblance to a similar problem in branching processes (Grey (1989)).

2. A probabilistic proof

Let $(p_1, p_2 \cdots)$ be a probability distribution with $p_i > 0$ for all $i = 1, 2, \cdots$, and let $(x_1, x_2, \cdots)$ be an increasing sequence of positive integers. Let W_k be the *defective* random walk which takes jumps of sizes $\pm x_i$ with probabilities $\frac{1}{2}p_i$ for $i = 1, 2, \cdots, k$, and which 'dies' at any step with probability $q_k = 1 - \sum_{i=1}^{k} p_i$. We show below that it is possible to choose the pairs $(p_1, x_1), (p_2, x_2), \cdots$ successively, in the style of Grey (1978), in such a way that

(1) W_k has probability at least $1 - k^{-1}$ of returning to zero before dying;

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and

(2) $$q_k \geqq \varepsilon(x_k).$$

It then follows, if $W_1, W_2, \cdots$ are constructed on the same probability space in an obvious way, that $W = \lim_{k \to \infty} W_k$ exists in a strong sense, is a *proper* random walk taking jumps of sizes $\pm x_i$ with probabilities $\frac{1}{2}p_i$ for $i = 1, 2, \cdots$, and, because of (1) and (2), satisfies the requirements of the theorem.

The proof is by induction. The choice of (p_1, x_1) poses no problem. Suppose, therefore, that we have chosen $(p_1, x_1), \cdots, (p_k, x_k)$. Consider the proper random walk on the two-dimensional integer lattice $\mathbb{Z}^2$ taking jumps of sizes $\pm(x_i, 0)$ with probabilities $\frac{1}{2}p_i$ and jumps of sizes $\pm(0, 1)$ each with probability $\frac{1}{2}q_k$. This random walk is persistent, by the two-dimensional Chung–Fuchs theorem (see Section 3). Hence it has countably many partial sample paths returning to $(0, 0)$ and with total probability 1. If we reduce the probability of jumps of sizes $\pm(0, 1)$ by a suitably small amount to $\frac{1}{2}p_{k+1}$ say, thereby making the random walk defective, we can ensure that these partial sample paths still have total probability at least $1 - (k + 1)^{-1}$. If we now choose $x_{k+1} > x_k$ such that $\varepsilon(x_{k+1}) \leqq q_{k+1}$ where $q_{k+1} = 1 - \sum_{i=1}^{k+1} p_i$, and map this defective random walk linearly into one dimension using the transformation $(x, y) \to x + yx_{k+1}$, we have constructed W_{k+1}. This completes the induction.

3. The Chung–Fuchs theorem

The above proof depends upon the following result, which we state for completeness. Here $\|\cdot\|$ denotes Euclidean length.

Theorem. (Chung and Fuchs (1951); see also Chung and Ornstein (1962)).
(a) If $EX_1 = 0$ then the random walk $\{S_n\}$ in one dimension is persistent.
(b) If $EX_1 = 0$ and $E\|X_1\|^2 < \infty$ then the random walk $\{S_n\}$ in two dimensions is persistent.

Acknowledgement

I should like to thank Professor G. E. H. Reuter for drawing to my attention the result of Shepp (1964).

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