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THE APPROXIMATE SYMMETRIC INTEGRAL

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By a symmetric integral is understood an integral obtained from some kind of symmetric derivation process. Such integrals arise most naturally in the study of trigonometric series and in particular to handle the following problem. Suppose that a trigonometric series

$$(1) \qquad a_0/2 + \sum_{k=1}^{+\infty} a_k \cos kx + b_k \sin kx$$

converges everywhere to a function f. It is known that this may occur without f being integrable in any of the more familiar senses so that the series may not be considered as a Fourier series of f; indeed Denjoy [**4**] has shown that if b_n is a sequence of real numbers decreasing to zero but with $\sum b_n/n = +\infty$ then the function $f(x) = \sum b_n \sin nx$ is not Denjoy-integrable. It is natural to ask then for an integration procedure that can be applied to f in order that the series be the Fourier series of f with respect to this integral.

A solution, based on symmetric integrals, follows from the well known observation of Riemann: the series

$$(2) \qquad a_0x^2/4 - \sum_{k=1}^{+\infty}(a_k \cos kx + b_k \sin kx)/k^2$$

obtained by two formal integrations of series (1) converges uniformly and absolutely to a continuous function G from which the function f may be obtained by a second order symmetric derivation,

$$f(x) = \lim_{h\searrow 0} \frac{G(x+h) + G(x-h) - 2G(x)}{h^2}.$$

This suggests the development of an integral that can recover a function G from its second symmetric derivative. This program has been followed by Denjoy [**4**] and James [**7**] both of whom produce second order integrals from this derivation process, James by a Perron-type approach and Denjoy by a transfinite totalization process. Taylor [**20**] introduced his AP-integral to solve this same question when the series is given to be everywhere Abel-summable and he too uses a second order derivation process, but to produce a certain kind of first order integral. The articles of Cross [**3**] and Skvorcov [**19**] and [**18**] should be consulted for the relation between Taylor's integral and the James P^2-integral.

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A first order integral that should integrate all everywhere convergent trigonometric series can be developed from the same principle by using the fact that the series of obtained by a single formal integration of the series (1),

$$F(x) = xa_0/2 + \sum_{k=1}^{+\infty}(b_k \cos kx - a_k \sin kx)/k, \tag{3}$$

converges in general only on a set of full measure, but then f may be recovered from F by using either

$$\lim_{h \searrow 0} \frac{1}{h^2}\left\{\int_x^{x+h} F(t)dt - \int_{x-h}^{x} F(t)dt\right\} = f(x)$$

or

$$\lim_{h \searrow 0} \frac{1}{h}\int_x^{x+h} \frac{F(x+t) - F(x-t)}{2t}dt = f(x).$$

This suggests an integration process that is based on these derivation notions and hence having the feature that indefinite integrals are defined only almost everywhere. First order integrals based on such considerations were advanced by Verblunksy [**23**], who used an approximate version of the Denjoy integral, and Burkill [**1**], who used a version of the Perron integral (Cesàro-Perron integral), but both require additional restrictions beyond the convergence of the series. Marcinkiewicz and Zygmund [**13**] introduced a Perron-type integral specifically for this problem and later Burkill [**2**] provided a "symmetric" version of his Cesàro-Perron integral, also following the Perron scheme, that solves the problem.

For a first order integral that should integrate all everywhere convergent trigonometric series one might instead use this fact from Zygmund [**24**, Theorem 2.22, p. 324]: if the trigonometric series (1) converges everywhere, then the series (3) converges on a set of full measure to a function F and f may be obtained from F by an approximate symmetric derivation. None of the previous approaches to this problem have utilized the fact that an everywhere convergent trigonometric series is the approximate symmetric derivative of its formally integrated series, which is a deeper observation than the above-mentioned Riemann theory. Perhaps the main reason this derivation process did not sponsor an integration procedure to solve the trigonometric series problem lies in the fact that a monotonicity theorem for the approximate symmetric derivative has been long awaited and also somewhat controversial given that several false proofs have appeared. With the appearance of the work of Freiling and Rinne [**5**] we now have such a monotonicity theorem with an apparently valid proof. It is this work which has led to the present paper.

In this article we develop an integral based on the approximate symmetric derivative and which, in particular, integrates all approximate symmetric derivatives and each everywhere convergent trigonometric series. The integral is defined as a limit of Riemann sums as in the Henstock-Kurzweil theory of integration; the key technical fact that allows the development of such an integral is the main covering theorem of Section 2 proving that partitions of almost every interval are available from any approximate symmetric covering relation. A number of descriptive characterizations of the integral are provided as well as a Perron type approach based on the monotonicity theorem of Freiling and Rinne.

For readers interested mainly in the application of this integral to the trigonometric series problem, Section 11 gives a very simple Riemann sums definition of an integral of a 2π-periodic function over a period and an entirely elementary proof that this integral inverts the approximate symmetric derivative. Then Section 12 gives the application to trigonometric series. If one grants just two deep theorems (the covering theorem that justifies the integral and Zygmund's theorem on the approximate symmetric derivation of the formally integrated series) then a completely elementary and intuitive account for the solution of the trigonometric series problem is available.

1. The approximate symmetric basis. Our study of the integral depends directly on the approximate symmetric derivative; for a measurable function F defined almost everywhere we define the $\mathcal{A}$-derivative at a point x as the approximate limit

$$\mathcal{A} - \mathrm{D}F(x) = \text{ap-lim}_{h\searrow 0}\frac{F(x+h)-F(x-h)}{h}.$$

The extreme A-derivates are defined in the obvious way. We shall express this notion in the language of abstract differentiation bases (see [**21**] for example).

An *interval-point relation* is a collection β of pairs (I,x) where I is a closed non-degenerate interval and x a point in I. For any interval-point relation β and any set E we write:

$$\beta(E) = \{(I,x) \in \beta; I \subset E\},$$
$$\beta[E] = \{(I,x) \in \beta; x \in E\}$$

and

$$\sigma(\beta) = \cup\{I; (I,x) \in \beta\}.$$

A finite interval-point relation π is said to be a *packing* if for any distinct pairs (I_1,x_1) and (I_2,x_2) from π the intervals I_1 and I_2 do not overlap. A *partition* of an interval $[a,b]$ is a packing π with

$$[a,b] = \bigcup_{(I,x)\in\pi} I.$$

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For the study of the approximate symmetric derivative one naturally considers interval-point relations containing pairs $([x-t, x+t], x)$ for sufficiently many $t > 0$. The quantity measuring the asymmetry of an interval point relation β at a point x is given by the formula

$$\rho[\beta](x) = \limsup_{h \searrow 0} |\{t \in (0,h); ([x-t, x+t], x) \notin \beta\}|/h \tag{4}$$

where here and elsewhere $|E|$ denotes the Lebesgue outer measure of the set E. As will become clear from the example 3.1, the most natural definition of an "approximately symmetric interval-point relation", namely the requirement that $\rho[\beta](x) = 0$ for every x, is not the right one. What is missing is a measurability condition. (In general by measurability we mean Lebesgue measurability in one or two dimensions.) Therefore, we are led to the following definition.

Definition 1.1. An interval-point relation β is said to be a *measurable approximate symmetric interval-point relation* if there is a measurable set $T \subset \mathbf{R} \times (0, \infty)$ such that

(i) $([x-t, x+t], x) \in \beta$ whenever $(x, t) \in T$, and

(ii) for every $x \in \mathbf{R}$

$$\limsup_{h \searrow 0} |\{t \in (0,h); (x,t) \notin T\}|/h = 0.$$

The family of all measurable approximate symmetric interval-point relations is denoted by $\mathcal{A}$ and we write as well $\mathcal{A}[E] = \{\beta[E]; \beta \in \mathcal{A}\}$ and $\mathcal{A}(E) = \{\beta(E); \beta \in \mathcal{A}\}$.

The following five lemmas express the fundamental properties of the derivation basis $\mathcal{A}$. The proofs of the first three are immediate and may be obtained directly from the definition; the main covering Lemma 1.6 follows directly from Theorem 2.1 whose proof appears in Section 2.

LEMMA 1.2. *If $\beta_1, \beta_2 \in \mathcal{A}$ then $\beta_1 \cap \beta_2 \in \mathcal{A}$.*

LEMMA 1.3. *If $E_1, E_2, E_3, \ldots$ is a sequence of disjoint measurable sets and $\beta_1, \beta_2, \beta_3, \ldots$ is a sequence of elements of $\mathcal{A}$ then*

$$\beta = \bigcup_{i=1}^{\infty} \beta_i[E_i]$$

belongs to $\mathcal{A}[E]$ where $E = \bigcup_{i=1}^{\infty} E_i$.

LEMMA 1.4. *If $\beta \in \mathcal{A}$ and G is open then $\beta(G) \in \mathcal{A}[G]$.*

LEMMA 1.5. *If $\beta \in \mathcal{A}$ then for almost every point x*

$$\limsup_{h \searrow 0} |\{t \in (0,h); ([x, x+t], x+t/2) \in \beta\}|/h = 1 \tag{5}$$

and

$$(6) \qquad \limsup_{h \searrow 0} |\{t \in (0,h); ([x-t,x], x-t/2) \in \beta\}|/h = 1.$$

Proof. Let $\beta \in \mathcal{A}$ and choose a measurable set $T \subset \mathbf{R} \times (0,\infty)$ such that $([x-t,x+t],x) \in \beta$ whenever $(x,t) \in T$; let χ be the characteristic function of $\mathbf{R} \times (0,\infty) \setminus T$, and define the sequences of functions

$$f_n(x) = n|\{t \in (0,1/n); (x,t) \notin T\}|$$

and

$$g_n(x) = n|\{t \in (0,1/n); (x+t,t) \notin T\}|.$$

We compute on any interval $[a,b]$, using the Fubini theorem and a change of variables,

$$\begin{aligned}\int_a^b g_n(x)dx &= n\int_a^b \int_0^{1/n} \chi(x+t,t)dtdx \\ &\leqq n\int_a^{b+1}\int_0^{1/n} \chi(u,v)dvdu = \int_a^{b+1} f_n(u)du.\end{aligned}$$

As $0 \leqq f_n \leqq 1$ and $f_n \longrightarrow 0$ (because of the density requirements on T) we conclude that $\int_a^b g_n(x)dx \longrightarrow 0$ so that g_n converges to 0 in measure. Consequently we may pass to a subsequence converging almost everywhere on $[a,b]$. This gives equation (5) at almost every point of $[a,b]$ and hence almost everywhere on $\mathbf{R}$; assertion (6) is similarly proved.

LEMMA 1.6. *For every sequence $\{\beta_n\} \subset \mathcal{A}$ there is a set B of full measure so that each β_n contains a partition of every interval $[a,b]$ with endpoints in B.*

The dual basis to $\mathcal{A}$ is denoted as $\mathcal{A}^*$ and defined as follows: an interval-point relation β belongs to $\mathcal{A}^*$ if and only if for every point x and every $\beta_1 \in \mathcal{A}$

$$\beta \cap \beta_1[\{x\}] \neq \emptyset.$$

As before the families $\mathcal{A}^*[E]$ and $\mathcal{A}^*(E)$ are defined. Note that $\mathcal{A}^*$ may also be described as the collection of all interval-point relations β with the following property: for some set $T \subset \mathbf{R} \times (0,+\infty)$ (not necessarily measurable this time) β contains every pair (I,x) with $I = [x-h,x+h]$ and $(x,h) \in T$ and each $T_x = \{t; (x,t) \in T\}$ has positive upper density on the right at 0.

The basic properties of $\mathcal{A}^*$ are similar to those for $\mathcal{A}$ except that, of course, there are no analogues for Lemmas 1.5 and 1.6. Note that, because of Lemma 1.2, $\mathcal{A}$ is a filterbase while evidently $\mathcal{A}^*$ is not. Lemma 1.7 is a weak replacement.

LEMMA 1.7. *If $\beta_1 \in \mathcal{A}$ and $\beta_2 \in \mathcal{A}^*$ then $\beta_1 \cap \beta_2 \in \mathcal{A}^*$.*

LEMMA 1.8. *Let $E_1, E_2, E_3, \dots$ be a sequence of disjoint sets and let $\beta_1, \beta_2, \beta_3, \dots$ be a sequence of elements of $\mathcal{A}^*$. Then*

$$\beta = \bigcup_{i=1}^{\infty} \beta_i[E_i]$$

belongs to $\mathcal{A}^*[E]$ where $E = \bigcup_{i=1}^{\infty} E_i$..

LEMMA 1.9. *If $\beta \in \mathcal{A}^*$ and G is open then $\beta(G) \in \mathcal{A}^*[G]$.*

2. Covering theorem. In this section we prove the key result enabling us to develop the notion of approximate symmetric integral. Lemma 1.6 follows directly from this.

THEOREM 2.1. *For every measurable approximate symmetric interval-point relation β there is a set N of measure zero such that every interval with endpoints in $\mathbf{R} \setminus N$ has a partition contained in β.*

The proof is obtained by a series of lemmas containing the main computations. Proposition 2.7 gives a covering property for approximate symmetric covering relations that are not necessarily measurable and then the proof of Theorem 2.1 for the measurable case follows almost immediately by using Lemma 1.5. It is possible to prove this theorem under very slightly weaker density assumptions but we see no application at present.

We begin with some notations. The interval concentric with a given bounded interval I and having length $\kappa|I|$ is denoted $\kappa * I$. The following special notations are needed just for the proofs in this section.

Notation 2.2. Suppose that E is a measurable subset of the real line, $x \in \mathbf{R}$ and $\kappa \in (0,1)$.

(i) We denote by $\partial_+(E)$ the set of all $x \in \mathbf{R}$ such that every neighbourhood of x contains points $u < v < w$ with $|(u,v) \cap E| > 0$ and $|(v,w) \setminus E| > 0$.

(ii) We denote by $\partial_-(E)$ the set of all $x \in \mathbf{R}$ such that every neighbourhood of x contains points $u < v < w$ with $|(u,v) \setminus E| > 0$ and $|E \cap (v,w)| > 0$.

(iii) We denote by $\delta(E,x,\kappa)$ the supremum of the lengths of all those open intervals J containing x for which $|J \setminus E| < \kappa|J|$. If there is no such interval, we let $\delta(E,x,\kappa) = 0$.

(iv) We denote by $\Delta_+(E,x,\kappa)$ the supremum of all $t > 0$ such that

$$|[(x, x+h) \cap (2x - E)] \setminus E| < \kappa h$$

for every $h \in (0,t)$. It there is no such t, we let $\Delta_+(E,x,\kappa) = 0$.

(v) We denote by $\Delta_-(E,x,\kappa)$ the supremum of all $t > 0$ such that

$$|[(x, x-h) \cap (2x - E)] \setminus E| < \kappa h$$

for every $h \in (0, t)$. If there is no such t, we let $\Delta_-(E, x, \kappa) = 0$.

LEMMA 2.3. *Suppose that E is a measurable subset of the real line. Then*
(i) $\partial_-(E) = -\partial_+(-E)$,
(ii) *The sets $\partial_+(E)$ and $\partial_-(E)$ are closed subsets of* $\mathbf{R}$,
(iii) *for every open interval I on the real line $I \cap \partial_+(E) = I \cap \partial_+(I \cap E)$ and $I \cap \partial_-(E) = I \cap \partial_-(I \cap E)$,*
(iv) *for every interval $I = (a, b)$ on the real line $I \cap \partial_+(E) = \emptyset$ if and only if there are $c \in [a, b]$ and a set $N \subset I$ of Lebesgue measure zero such that*

$$(c, b) \setminus N \subset E \subset (c, b) \cup N,$$

(v) *for every interval $I = (a, b)$ on the real line $I \cap \partial_-(E) = \emptyset$ if and only if there are $c \in [a, b]$ and a set $N \subset I$ of Lebesgue measure zero such that*

$$(a, c) \setminus N \subset E \subset (a, c) \cup N,$$

(vi) *for every $\kappa \in (0, 1)$ and every $\delta \in \mathbf{R}$ the set $\{x \in \mathbf{R}; \delta(E, x, \kappa) > \delta\}$ is open,*
(vii) $\Delta_-(E, x, \kappa) = \Delta_+(-E, -x, \kappa)$ *for every $\kappa \in (0, 1)$ and every $x \in \mathbf{R}$ and*
(viii) $\Delta_+(\mathbf{R} \setminus E, x, \kappa) = \Delta_-(E, x, \kappa)$ *and* $\Delta_-(\mathbf{R} \setminus E, x, \kappa) = \Delta_+(E, x, \kappa)$ *for every $\kappa \in (0, 1)$ and every $x \in \mathbf{R}$.*

Proof. All statements of the lemma are obvious.

LEMMA 2.4. *Suppose that A and B are measurable subsets of bounded open intervals I and J, respectively, and that $\kappa \in (0, 1)$ and $\delta \in (0, |I|)$. Then there is $t \in J - I$ such that*

$$|[A + s] \cap B| \geqq \left[\frac{|B|}{|I| + |J|} - \frac{s - t|}{\delta} - 2\kappa\right] |\{x \in I; \delta(A, x, \kappa) > \delta\}|$$

for every $s \in \mathbf{R}$.

Proof. Since the statement is obvious if $|A| = 0$ or if $|B| = 0$ we shall assume that the sets A and B have positive measure. Let $\mathcal{G}$ be the family of all open subintervals K of I such that $|K| > \delta$ and $|K \setminus A| \leqq \kappa|K|$, and let G be the union of this family. Then G is an open subset of I each of whose components has length at least δ. Moreover, there is a (necessarily finite) subfamily of $\mathcal{G}$ covering G such that every point of I belongs to at most two of its members. Hence $|G \setminus A| \leqq 2\kappa|G|$. Since $[G + t] \cap B = \emptyset$ if $t \notin J - I$, and since

$$\int_{-\infty}^{\infty} |[G + t] \cap B| dt = |G||B|,$$

there is $t \in J - I$ such that

$$|[G + t] \cap B| \geqq |G||B|/(|I| + |J|).$$

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Since each component of G has length at least δ,

$$|[G+t] \setminus [G+s]| \leqq |s-t||G|/\delta$$

for every $s \in \mathbf{R}$. We also observe that

$$G \supset \{x \in A; \delta(a, x, \kappa) > \delta\},$$

since, if K is an interval, $K \setminus I \neq \emptyset$, and if $\tilde{K}$ is a subinterval of I such that $K \cap I \subset \tilde{K}$ and $|\tilde{K}| = \min[|K|, |I|]$, then $|K \setminus A|/|K| \leqq |\tilde{K} \setminus A|/|\tilde{K}|$.

Combining the previous estimates, we see that

$$\begin{aligned} |[A+s] \cap B| &\geqq |[G+t] \cap B| - |[G+t] \setminus [G+s]| - |G \setminus A| \\ &\geqq \left[\frac{|B|}{|I|+|J|} - \frac{|s-t|}{\delta} - 2\kappa\right]|G| \\ &\geqq \left[\frac{|B|}{|I|+|J|} - \frac{|s-t|}{\delta} - 2\kappa\right]|\{x \in A; \delta(A, x, \kappa) > \delta\}| \end{aligned}$$

for every $s \in \mathbf{R}$.

LEMMA 2.5. *Suppose that I is a bounded open interval, E is a measurable subset of $\mathbf{R}$ and $\epsilon \in (0, 1/2)$. Suppose further that the center w of I belongs to the closure of the set*

$$Q = \{x \in I; \Delta_+(E, x, \epsilon) > 3|I|\},$$

that $|J \cap E| \leqq |J|/2$ whenever J is a subinterval of $5 * I$ with left end point w, and that $|J \cap E| \geqq |J|/2$ whenever J is a subinterval $5 * I$ with right end point w.

Then the following statements hold.

(i) *Whenever w belongs to the closure of a subinterval J of $5 * I$ then*

$$|J|/2 - \epsilon|J| \leqq |J \cap E| \leqq |J|/2 + \epsilon|J|.$$

(ii) *Whenever J is a subinterval of $5 * I$ then*

$$|J|/2 - 2\epsilon \operatorname{diam}(\{w\} \cup J) \leqq |J \cap E| \leqq |J|/2 + 2\epsilon \operatorname{diam}(\{w\} \cup J).$$

(iii) *Each component of $I \setminus \partial_+(E)$ has length at most $8\epsilon|I|$.*

(iv) *Whenever $x \in Q$ then*

$$|(x - 2|I|, x + 2|I|) \cap [(2x - E)E]| \leqq 16\epsilon|I|.$$

(v) *Whenever $x, y \in Q$ then*

$$|[2(y-x) + (I \cap E)] \setminus E| \leqq 28\epsilon|I|.$$

Proof. (i) Assume first that $J = (a, w)$. For every $\tau \in (0, \min(|I|, |J|/2)$ we find a point $x \in (w - \tau, w + \tau) \cap Q$, we let $\tilde{J} = (a, x)$, and we use the inequality

$$|[(2x - \tilde{J}) \cap (2x - E)] \setminus E < |\epsilon|\tilde{J}|$$

to conclude that

$$\begin{aligned} |J|/2 &\leqq |J \cap E| \leqq |\tilde{J} \cap E| + \tau = |(2x - \tilde{J}) \cap (2x - E)| + \tau \\ &\leqq |[(2x - \tilde{J}) \cap (2x - E)] \setminus E| + |(2x - \tilde{J}) \cap E| + \tau \\ &\leqq |(2x - J) \cap E| + \epsilon|\tilde{J}| + 2\tau \leqq |J|/2 + \epsilon|J| + 3\tau. \end{aligned}$$

Hence

$$|J|/2 \leqq |J \cap E| \leqq |J|/2 + \epsilon|J| + 3\tau,$$

and

$$|J|/2 \geqq |(2x - J) \cap E| \geqq |J|/2 - \epsilon|J| - 3\tau,$$

which shows that the inequality required in (i) holds provided that w is an endpoint of J. The general case follows easily by a decomposition.

(ii) If w belongs to the closure of J, this follows from (i). Otherwise we write $J = J_1 \setminus J_2$, where J_1 and J_2 are intervals with endpoint w, and we estimate

$$\begin{aligned} |J \cap E| &= |J_1 \cap E| - |J_2 \cap E| \leqq |J|/2 + \epsilon(|J_1| + |J_2|) \\ &\leqq |J|/2 + 2\epsilon \operatorname{diam}(\{w\} \cup J) \end{aligned}$$

and

$$\begin{aligned} |J \cap E| &= |J_1 \cap E| - |J_2 \cap E| \geqq |J|/2 - \epsilon(|J_1| + |J_2|) \\ &\geqq |J|/2 - 2\epsilon \operatorname{diam}(\{w\} \cup J) \end{aligned}$$

(iii) According to 2.3(iv) every component L of $I \setminus \partial_+(E)$ contains an interval J such that $|J| \geqq |L|/2$ and $|J \cap E| = 0$ or $|J \setminus E| = 0$. Hence (ii) implies that $|L| \leqq 2|J| \leqq 8\epsilon|I|$.

(iv) Since the interval $(x - |I|, x + |I|)$ contains w, (ii) implies that

$$|(x, x + 2|I|) \cap E| \leqq (1 + 6\epsilon)|I|$$

and

$$|(x, x + 2|I|) \cap (2x - E)| \geqq (1 - 6\epsilon)|I|.$$

Recalling that $\Delta_+(E, x, \epsilon) > 3|I|$ implies

$$|[(x, x + 2|I|) \cap (2x - E)] \setminus E| < 2\epsilon|I|,$$

we combine all these inequalities to get

$$\begin{aligned}&|(x, x+2|I|) \cap E \cap (2x-E)| = |(x, x+2|I|) \cap (2x-E)| \\ &- |[(x, x+2|I|) \cap (2x-E)] \setminus E| \geqq (1-6\epsilon)|I| - 2\epsilon|I| = (1-8\epsilon)|I|.\end{aligned}$$

Hence

$$\begin{aligned}&|(x-2|I|, x) \cap [(2x-E) \setminus E]| = |[(x, x+2|I|) \cap E] \setminus (2x-E)| \\ &= |(x, x+2|I|) \cap E| - |(x, x+2|I|) \cap E \cap (2x-E)| \\ &\leqq (1+6\epsilon)|I| - (1-8\epsilon)|I| = 14\epsilon|I|,\end{aligned}$$

which immediately implies (iv).

(v) From (iv) we see that there is a set N_1 with measure at most $14\epsilon|I|$ such that

$$2x - (I \cap E) \subset E \cup N_1.$$

Using (iv) once more, we find a set N_2 with measure at most $14\epsilon|I|$ such that

$$(y-2|I|, y+2|I|) \cap (2y-E) \subset E \cup N_2.$$

Finally, we use that $2x - (I \cap E) \subset (y-2|I|, y+2|I|)$ to infer that

$$\begin{aligned}2(y-x)+E &= 2y-(2x-E) \\ &\subset [(y-2|I|, y+2|I|) \cap (2y-E)] \cup (2y-N_1) \\ &\subset E \cup N_2 \cup (2y-N_1).\end{aligned}$$

LEMMA 2.6. *Suppose that E is a measurable subset of the real line, I is a bounded open interval with center w belonging to $\partial_+(E)$, $\epsilon \in (0, 1/2)$, and that the set*

$$Q = \{x \in I \cap \partial_+(E); \Delta_+(E, x, \epsilon) > 3|I|\}$$

is dense in $I \cap \partial_+(E)$. Let γ be defined as the smallest nonnegative number such that

$$|\{z \in I; \operatorname{dist}(z, I \cap \partial_+(E)) \leqq \gamma\}| \geqq 3|I|/4.$$

Then the following statements hold.

(i) *For every s in $(-|I|/2, |I|/2)$ and every $\tau > 0$ there are $x, y \in I \cap Q$ such that $|(y-x)-s| < 2\gamma + \tau$.*

(ii) *For every $\kappa \in (0, 1/40$ either*

$$|\{z \in I; \delta(I \cap E, z, \kappa) > \gamma/2\}| \geqq |I|/8$$

or

$$|\{z \in I; \delta(I \setminus E, z, \delta) > \gamma/2\}| \geqq |I|/8.$$

(iii) *Whenever* $\delta \in (0, |I|/8), x \in (0, 1/2)$, *and* $x \in (1 - 8\delta/|I|) * I$, *then*

$$\delta(I \cap E, x, \kappa + \epsilon/2) \geqq 2(1 - \kappa)\delta$$

provided that $\delta(I \cap E, x, \kappa) > \delta$, *and*

$$\delta(I \setminus E, x, \kappa + \epsilon/2) \geqq 2(1 - \kappa)\delta$$

provided that $\delta(I \setminus E, x, \kappa) > \delta$.

(iv) *If* $j = 1, 2, \ldots$ *is such that* $\epsilon \geqq j^{-2}$, *if* $\delta \in (0, 2^{-j-3}|I|)$, *and if* $x \in (1 - 2^{j+3}\delta/|I|) * I$, *then*

$$\delta(I \cap E, x, (j+1)\epsilon/2) > 2^{j-1}\delta$$

provided that $\delta(I \cap E, x, \epsilon/2) > \delta$ *and*

$$\delta(I \setminus E, x, (j+1)\epsilon/2) > 2^{j-1}\delta$$

provided that $\delta(I \setminus E, x, \epsilon/2) > \delta$.

(v) *If* $j = 1, 2, \ldots$ *is such that* $\epsilon \leqq j^{-2}$ *and such that* $\gamma \leqq 2^{-j-7}|I|$ *then*

$$|\{z \in I; \delta(I \cap E, z, (j+1)\epsilon/2) > 2^{j-2}\gamma\}| \geqq |I|/16$$

or

$$|\{z \in I; \delta(I \setminus E, z, (j+1)\epsilon/2) > 2^{j-2}\gamma\}| \geqq |I|/16.$$

Proof. (i) Since the set $M = \{z \in I;\ \mathrm{dist}(z, I \cap Q) \leqq \gamma\}$ has measure at least $3|I|/4$, the assumption that $M \cap (s + M) = \emptyset$ would imply

$$3|I|/2 = |M \cup (s + M)| \leqq |I \cup (s + I)| < 3|I|/2.$$

Hence there is $u \in M \cap (s + M)$ and it suffices to consider $x, y \in I \cap Q$ such that $|u - y| < \gamma + \tau/2$ and $|(u - s) - x| < \gamma + \tau/2$.

(ii) Since this statement is obvious if $\gamma = 0$ (one just notes, that the sets whose measure we are estimating contain all density points of $I \cap E$ and $I \setminus E$, respectively), we shall assume that $\gamma > 0$. Let G be the union of all bounded components J of $I \setminus \partial_+(E)$ having length at least γ and satisfying $|J \cap E| \geqq |J|/2$ and let H be the union of all bounded components J of $I \setminus \partial_+(E)$ having length at least γ and satisfying $|J \setminus E| \geqq |J|/2$. Noting that γ is the smallest nonnegative number such that

$$|\{z \in I; \operatorname{dist}(z, (I \cap Q)) \leqq \gamma\}| \geqq 3|I|/4$$

and that $\gamma > 0$, we see that $|G \cup H| \geqq |I|/4$. Moreover, for every component J of G we may use that $J \cap \partial_+(E) = \emptyset$ and 2.3(iv) to conclude that

$$\delta(I \cap E, x, \kappa) > |J \cap E| \geqq |J|/2$$

for every $x \in J \cap E$ and every $\kappa > 0$. Hence the first inequality from (ii) holds in case $|G| \geqq |G \cup H|/2$. A similar argument shows that the second inequality from (ii) holds in case $|H| \geqq |G \cup H|/2$.

(iii) We prove the first inequality only. The second inequality may be proved in a similar way, or one can note that 2.3(vii) and (viii) imply that each of the inequalities in question follows from the other one.

Since the statement is obvious in case $\delta((I \cap E), x, \kappa) > 2(1-\kappa)\delta$ we may assume that $\delta((I \cap E), x, \kappa) \in (\delta, 2(1-\kappa)\delta)$.

Let $\tau \in (0, \kappa)$ be such that $\delta((I \cap E), x, \kappa - \tau) > \delta$. We let

$$\sigma - \min[\tau\delta/4, \delta((I \cap E), x, \kappa - \tau) - \delta],$$

and we find an open interval $J = (a, b)$ containing x such that $|J \setminus E| < (\kappa - \tau)|J|$ and $|J| > \delta((I \cap E), x, \kappa - \tau) - \sigma$. Noting that

$$|J| \leqq \delta(I \cap E, x, \kappa) \leqq 2(1-\kappa)\delta$$

implies $2(b - a + \sigma) \leqq 4(1-\kappa)\delta + 2\kappa\delta < 4\delta$, and that $x \in [(1 - 8\delta/|I|) * I] \cap J$, we infer that $(a, 2b - a + \sigma) \subset I$.

Let c denote the smallest number from $[a, b]$ such that

$$|\{y \in E; c < y < b\}| = 0.$$

Since $|J \setminus E| < (\kappa - \tau)|J|$, it follows that $c > a + (1 - (\kappa - \tau))|J|$. Hence $c - \sigma > a$ and

$$(c - \sigma, c + \sigma) \subset (a, 2b - a + \sigma) \subset I.$$

We claim that $(c - \sigma, c + \sigma) \cap \partial_+(E) \neq \emptyset$. Indeed, otherwise we could use 2.3(iv) and the fact that $|(c - \sigma, c) \cap E| > 0$ to infer that E contains almost all of $(c, c + \sigma)$. But this would imply that $c = b$ and

$$|(a, b + \sigma) \setminus E| \leqq |J \setminus E| < (\kappa - \tau)|J| < (\kappa - \tau)((b + \sigma) - a).$$

Consequently

$$\delta((I \cap E), x, \kappa - \tau) \geqq (b + \sigma) - a = |J| + \sigma > \delta((I \cap E), x, \kappa - \tau),$$

which would be a contradiction.

Thus our claim is proved and we may use it together with the above observation that $(c-\sigma, c+\sigma) \subset I$ to find a point $z \in (c-\sigma, c+\sigma) \cap Q$. Then

$$|(a, 2z-a)| = 2(z-a) > 2(|J| - (\kappa - \tau)|J - \sigma) > 2(1-\kappa)\delta,$$

$$\begin{aligned}|(a, 2z-a) \setminus E| &= |(a,z) \setminus E| + |(z, 2z-a) \setminus E| \leqq |(a,z) \setminus E| \\ &\quad + |(z, 2z-a) \setminus (2z-E)| + |[(z, 2z-a) \cap (2z-E)] \setminus E| \\ &= 2|(a,z) \setminus E| + |[(z, 2z-a) \cap (2z-E)] \setminus E| \\ &< 2((\kappa-\tau)|J| - (b-c) + \sigma) + \epsilon(|J| - (b-c) + \sigma) \\ &\leqq (2\kappa + \epsilon)(z-a) + 2((1+\kappa+\epsilon)\sigma - \tau|J|) \\ &\leqq (2\kappa+\epsilon)(z-a),\end{aligned}$$

and

$$x \in (a,b) \subset (a, 2z-a) \subset I,$$

which implies that

$$\delta(I \cap E, x, \kappa + \epsilon/2) \geqq |(a, 2z-a)| > 2(1-\kappa)\delta.$$

(iv) Using (iii) inductively, we see that

$$\delta(I \cap E, x, (i+1)\epsilon/2) > 2^i(1 - i^2\epsilon/2)\delta$$

for each $i = 1, 2, \ldots, j$, provided that $\delta(I \cap E, x, \epsilon/2) > \delta$. The second inequality is handled similarly.

(v) This statement follows immediately from (iv) with $\kappa = \epsilon/2$ and from (ii) with $\delta = \gamma/2$.

PROPOSITION 2.7. *There is a constant $\epsilon > 0$ with the following property: Whenever β is an interval-point relation on an interval $(a,b) \subset \mathbf{R}$ such that $\rho[\beta](x) < \epsilon$ for every $x \in (a,b)$, and whenever E is a measurable subset of (a,b), then for almost every $v \in (a,b)$ for which $|(a,v) \cap E| > 0$ there is $u \in (a,v) \cap E$ such that the interval $[u,v]$ admits a partition contained in β.*

Proof. We prove that the statement holds with $\epsilon = 2^{-18}$. We may clearly assume that the set E is open in the density topology. (A set is density open if it is measurable and has density 1 at each of its points.) Also, we may find sets $T_x \subset (0, \infty)$ open in the density topology such that, for each $x \in \mathbf{R}$ the right lower density of T_x at 0 is greater than $1-\epsilon$ and $([x-t, x+t], x) \in \beta$ for every $t \in T_x$.

Instead of β it will be more convenient to consider the interval-point relation $\tilde{\beta}$ defined by the requirement $(I, x) \in \tilde{\beta}$ if and only if $I = [x-t, x+t]$ for some $t \in T_x$. Then we easily see that the set $\tilde{E}$ of all $v \in (a,b)$ for which there is

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$u \in E \cap (a,v)$ such that the interval $[u,v]$ admits a partition contained in $\tilde{\beta}$ is density open. Consequently, $\tilde{E}$ is measurable. We also observe that $x+t \in \tilde{E}$ whenever $x \in (a,b), t \in T_x \cap (0, \min[b-x, x-a])$, and $x-t \in \tilde{E}$. Thus $\Delta_+(\tilde{E},x,\epsilon) > 0$ for every $x \in (a,b)$.

From 2.3(iv) we see that the statement of the proposition is equivalent to $\partial_+(\tilde{E}) = \emptyset$. Thus, in order to find a contradiction, we shall assume that $\partial_+(\tilde{E}) \neq \emptyset$. Then we may use 2.3(ii) and the Baire Category Theorem to find $\Delta > 0$ and an open interval $I_0 \subset (a,b)$ such that $I_0 \cap \partial_+(\tilde{E}) \neq \emptyset$ and the set

$$\{x \in I_0 \cap \partial_+(\tilde{E}); \Delta_+(\tilde{E},x,\epsilon) > \Delta\}$$

is dense in $I_0 \cap \partial_+(\tilde{E})$.

Since $I_0 \cap \partial_+(\tilde{E}) \neq \emptyset$, there are points $u_0 \in I_0 \cap \tilde{E}$ and $v_0 \in I_0 \setminus \tilde{E}$ such that v_0 is a density point of $I_0 \setminus \tilde{E}$ and $u_0 < v_0 < u_0 + \Delta$. Let $w \in [u_0, v_0]$ be a point at which the function $x \longmapsto |(u_0,x) \cap \tilde{E}| - x/2$ attains its maximum on $[u_0,v_0]$. Noting that

$$|(w,x) \cap \tilde{E}| \leqq (x-w)/2 \text{ whenever } w \leqq x \leqq v_0$$

and

$$|(x,w) \cap \tilde{E}| \geqq (w-x)/2 \text{ whenever } u_0 \leqq x \leqq w,$$

we use the facts that u_0 is a density point of $\tilde{E}$ and v_0 is a density point of $I_0 \setminus \tilde{E}$ to deduce that $w \in (u_0,v_0) \cap \partial_+(\tilde{E})$.

Let I be an open interval with center w such that $|I| < \Delta/3$ and $5 * I \subset (u,v)$. The above inequalities imply that the assumptions of 2.5 hold. As in 2.6 and 2.5 we denote

$$Q = \{x \in I; \Delta_+(\tilde{E},x,\epsilon) > 3|I|\},$$

and we let γ be the smallest nonnegative number such that

$$|\{z \in I; \operatorname{dist}(z, I \cap \partial_+(\tilde{E})) \leqq \gamma\}| \geqq 3|I|/4.$$

From 2.5(iii) we immediately see that $\gamma \leqq 8\epsilon|I|$. Letting $j = 8$ and noting that $2^{j+7}\gamma \leqq |I|$, we deduce from 2.6(v) that either

$$|\{z \in I; \delta(I \cap \tilde{E}, z, (j+1)\epsilon/2) > 2^{j-2}\gamma\}| \geqq |I|/16$$

or

$$|\{z \in I; \delta(I \setminus \tilde{E}, z, (j+1)\epsilon/2) > 2^{j-2}\gamma\}| \geqq |I|/16.$$

Assume, for example, that the first of these alternatives holds. (To handle the second case, one just exchanges the notation A and B in the following argument.) Then we let $A = I \cap \tilde{E}, B = I \setminus \tilde{E}$, and we observe that

$$|\{z \in I; \delta(I \cap A, z, 1/64) > 64\gamma\}| \geqq |I|/16.$$

Thus we may use 2.4 to infer that there is $t \in (-|I|, |I|)$ such that

$$|[A+s] \cap B| \geqq \left[\frac{|B|}{2|I|} - \frac{|s-t|}{64\gamma} - 1/32\right] |I|/16$$

for every $s \in \mathbf{R}$. Consequently, if $|s-t| < 5\gamma$ then

$$|[A+s] \cap B| \geqq [1/4 - \epsilon/2 - 5/64 - 1/32]|I|/16 > 2^{-7}|I|.$$

On the other hand, 2.6(i) implies that there are $x, y \in Q$ such that $|(y-x-t/2| < 5\gamma/2$. Hence, letting $s = 2(y-x)$, we use 2.5(v) to conclude that

$$|[A+s] \cap B| \leqq 28\epsilon|I| < 2^{-7}|I|.$$

Since $|s-t| < 5\gamma$, this gives the required contradiction.

Proof of 2.1. Let $\beta \in \mathcal{A}$. We note first that for every $r > 0$ and for almost every x the sets

$$A(x) = \{y \in (x-r, x); ([y, x], (x+y)/2) \in \beta\}$$

and

$$B(x) = \{y \in (x, x+r); ([x, y], (x+y)/2) \in \beta\}$$

are measurable and have positive measure. Their measurability follows immediately from the fact that $\beta \in \mathcal{A}$ while the fact that they have, for almost every point x, positive measure follows from 1.5.

Then, to finish the proof, we just observe that, whenever $x < y, B(x)$ is measurable and has positive measure and $A(y)$ is measurable and has positive measure, we may use 2.7 with $E = (x, (x+y)/2) \cap B(x)$ to find $v \in ((x+y)/2, y) \cap A(x)$ and $u \in (x, (x+y)/2) \cap B(x)$ such that the interval $[u, v]$ admits a partition contained in β. This partition can be, in an obvious way, extended to a partition of the whole interval $[x, y]$.

3. Measurability questions. There are a number of measurability problems that arise naturally in our study. We have already indicated (in Section 1) that the approximate symmetric covering relations that are to be used to define the integral must be taken to be measurable. The main covering Theorem 2.1 is not

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valid without the measurability assumption; this is a consequence of 3.1 proved below.

We wish too to know what measurability properties may be determined from statements about approximate symmetric limits. For the ordinary symmetric limit the situtation has been long studied and the results are satisfying. For example an everywhere symmetrically continuous function is a.e. continuous and hence measurable ([**16**]). An almost everywhere symmetrically differentiable function is measurable ([**22**]) as is its derivative.

The situation for approximate symmetric limits may appear at first quite startling. Indeed, from 3.1 we shall see that an approximately symmetrically differentiable function need not be measurable. However from 3.4 we shall see that its derivative is necessarily measurable. Similarly striking facts one gets also in the general case. Again, 3.1 implies that the set of points at which an approximate symmetric derivative exists need not be measurable. However, from 3.3 we see that the approximate symmetric derivative is always a measurable function with respect to this (possibly nonmeasurable) set. These measurability results will also play a role in Section 8 showing that all the functions that arise in the integration theory are measurable.

Example 3.1. Under the Continuum Hypothesis the following two statements hold for an arbitrary linear subspace E of $\mathbf{R}$ over the field of rational numbers.

(i) There is set $Y \subset \mathbf{R}$ such that both the sets Y and $\mathbf{R} \setminus Y$ are of full outer measure and such that for every $x \in E$ the set $Y \setminus (2x - Y)$ is countable, and for every $x \in \mathbf{R} \setminus E$ the set $Y \setminus (2x - Y)$ is of full outer measure.

(ii) There is a nonmeasurable function $f : \mathbf{R} \longmapsto \mathbf{R}$ such that for every $x \in E$ there is a countable set S_x such that $f(x+h) - f(x-h) = 0$ for every $h \in \mathbf{R} \setminus S_x$, and such that $\mathcal{A} - \overline{D}f(x) = +\infty$ and $\mathcal{A} - \underline{D}f(x) = -\infty$ for every $x \in \mathbf{R} \setminus E$.

Proof. To simplify the notation, let $Q[Z]$ denote the linear span of the set $Z \subset \mathbf{R}$ over the field of rational numbers.

Under the continuum hypothesis we may arrange the set of all pairs (x, C), where $x \in \mathbf{R}$ and C is an uncountable compact subset of $\mathbf{R}$ into a sequence (x_τ, C_τ) indexed by countable ordinals τ. By transfinite induction we choose points y_τ and z_τ, and sets Y_τ and Z_τ as follows. (As usual, to include the first step of the construction in the general description, we set the union over an empty family of indices to an empty set.)

As y_τ we always choose an arbitrary point of

$$C_\tau \setminus Q\left[\{x_\tau\} \cup \bigcup_{\sigma > \tau} Z_\sigma\right].$$

Then we define

$$Y_\tau = Q[\{y_\tau\} \cup \{x_\sigma; \sigma \leqq \tau, x_\sigma \in E\}] \setminus \bigcup_{\sigma < \tau} Z_\sigma,$$

we choose

$$z_\tau \in C_\tau \setminus Q\left[\{x_\tau, y_\tau\} \cup \bigcup_{\sigma<\tau} Z_\sigma\right],$$

and we put

$$Z_\tau = Q\left[\{x_\tau, y_\tau, z_\tau\} \cup \bigcup_{\sigma<\tau} Z_\sigma\right].$$

Let $Y = \cup_{\sigma<w_1} Y_\sigma$. Observing that $Y_\tau \subset Z_\tau = Q[Z_\tau] \subset Z_\sigma$ and $Y_\sigma \cap Z_\tau = \emptyset$ whenever $\tau < \sigma$, we easily see that $y_\tau \in Y \cap C_\tau$ and $z_\tau \in (\mathbf{R} \setminus Y) \cap C_\tau$ for each countable ordinal τ. This implies that the sets $\mathbf{R} \setminus Y$ and Y have full outer measure.

If $x \in E$, we find τ such that $x = x_\tau$ and we obtain $Y \setminus (2x - Y)$ is countable by showing that

$$Y \setminus (2x - Y) \subset \bigcup_{\sigma \leqq \tau} Y_\sigma.$$

To prove this, assume that $y \in Y_\nu, 2x_\tau - y \notin Y$, and $\nu > \tau$. Then y and x_τ belong to

$$Q[\{y_\nu\} \cup \{x_\sigma; \sigma \leqq \nu, x_\sigma \in E\}],$$

hence

$$2x_\tau - y \in Q[\{y_\nu\} \cup \{x_\sigma; \sigma \leqq \nu, x_\sigma \in E\}].$$

However, since $2x_\tau - y$ does not belong to Y_ν, this can happen only if

$$2x_\tau - y \in \bigcup_{\eta<\nu} Z_\eta.$$

Let $\tau \leqq \eta < \nu$ be such that $2x_\tau - y \in Z_\eta$. Then, since $x_\tau \in Z_\eta$ and since $Z_\eta = Q[Z_\eta]$, we infer that $y \in Z_\eta$. But this contradicts $y \in Y_\nu$.

We finish the proof of the first statement by showing that for every $x \in \mathbf{R} \setminus E$ the set $Y \setminus (2x - Y)$ intersects every uncountable compact subset C of $\mathbf{R}$. To prove this, we find a countable ordinal τ such that $(x, C) = (x_\tau, C_\tau)$. Since $y_\tau \in C \cap Y$, we just need to prove that $2x_\tau - y_\tau \notin Y$. But this is almost obvious, since $2x_\tau - y_\tau$ belongs to Z_τ but not to $\cup_{\sigma<\tau} Z_\sigma$ otherwise

$$y_\tau \in Q\left[\{x_\tau\} \cup \bigcup_{\sigma<\tau} Z_\sigma\right].$$

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This means that $2x_\tau - y_\tau$ can be in Y only if it is in Y_τ. However, since

$$y_\tau \notin Q\left[\{x_\tau\} \cup \bigcup_{\sigma<\tau} Z_\sigma\right],$$

$2x_\tau - y_\tau \in Y_\tau$ implies that

$$x = x_\tau \in Q[\{x_\sigma; \sigma \leqq \tau, x_\sigma \in E\}] \subset E.$$

To prove the second statement of our remark, we let f be the characteristic function of the set Y. For each $x \in E$ we put

$$S_x = [(Y \setminus (2x - Y)) - x] \cup [x - (Y \setminus (2x - Y))].$$

Then clearly each of the sets S_x is countable and the first statement follows from the observation that $f(x+h) - f(x-h) \neq 0$ implies $h \in S_x$. If $x \notin E$, we observe that the sets

$$U = \{t \in (0,\infty); x + t \in Y, x - t \notin Y\}$$

and

$$V = \{t \in (0,\infty); x + t \notin Y, x - t \in Y\}$$

have full outer measure in $(0,\infty)$, and that

$$\lim_{t\searrow 0, t\in U} (f(x+t) - f(x-t))/2t = +\infty$$

and

$$\lim_{t\searrow 0, t\in V} (f(x+t) - f(x-t))/2t = -\infty$$

which completes the proof.

We obtain our measurability results as separation properties. Recall that sets A and B are said to be separated by a set M if $A \subset M$ and $B \cap M = \emptyset$.

LEMMA 3.2. *Suppose that the subsets U and V of the reals cannot be separated by a measurable set and that h is a positive function defined on $U \cup V$. Then there are a positive number ϵ and a nonempty compact subset P of* **R** *with the following properties.*

(i) *The intersection $I \cap P$ has positive measure whenever I is an open interval meeting P.*

(ii) *The sets $\{x \in U \cap P; h(x) > \epsilon\}$ and $\{x \in V \cap P; h(x) > \epsilon\}$ are both of full outer measure in P.*

Proof. If for each $k = 1, 2, \ldots$ the sets $\{x \in U; h(x) > 1/k\}$ and $\{x \in V; h(x) > 1/k\}$ could be separated by measurable sets, say, M_k, then U and V would be separated by $\bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} M_k$. Hence there is a positive number ϵ such that the sets $\{x \in U; h(x) > \epsilon\}$ and $\{x \in V; h(x) > \epsilon\}$ cannot be separated by a measurable set.

Next we observe that whenever $E_1, E_2, \ldots$ are measurable sets such that the sets $A \cap E_i$ and $B \cap E_i$ are separated by measurable sets, say, M_i, then the sets $A \cap \cup E_i$ and $B \cap \cup E_i$ are separated by $\cup(E_i \cap M_i)$. From this we easily deduce that there is a measurable set $A \subset \mathbf{R}$ such that the sets $\{x \in U \cap A; h(x) > \epsilon\}$ and $\{x \in V \cap A; h(x) > \epsilon\}$ can be separated by a measurable set and that the set $\mathbf{R} \setminus A$ contains no measurable set B of positive measure for which the sets $\{x \in U \cap B; h(x) > \epsilon\}$ and $\{x \in V \cap B; h(x) > \epsilon\}$ would be separated by a measurable set.

Since the set $\mathbf{R} \setminus A$ has positive measure, it contains a nonempty compact set P for which (i) holds. Clearly, P is the required set, since the assumption that, for example, the set $P \setminus \{x \in U \cap P; h(x) > \epsilon\}$ contains a measurable set B of positive measure immediately implies that the sets $\{x \in U \cap B; h(x) > \epsilon\}$ and $\{x \in V \cap B; h(x) > \epsilon\}$ are separated by B.

THEOREM 3.3. *Let f be an arbitrary function defined on a measurable set S and let $a < b$ be real numbers. Then the sets $\{x \in S; \mathcal{A} - \overline{D}f(x) < a\}$ and $\{x \in S; \mathcal{A} - \underline{D}f(x) > b\}$ can be separated by a measurable set.*

Proof. Let us assume, to the contrary, that the sets

$$U = \{x \in S; \mathcal{A} - \overline{D}f(x) < a\}$$

and

$$U = \{x \in S; \mathcal{A} - \underline{D}f(x) < a\}$$

cannot be separated by a measurable set. For each $x \in U$ we choose $h(x) > 0$ such that for every $0 < h < h(x)$ the set

$$\left\{t \in (0, h); \frac{f(x+t) - f(x-t)}{2t} \geqq a\right\}$$

has outer measure at most $h/72$. Similarly, for each $x \in V$ we choose $h(x) > 0$ such that for every $0 < h < h(x)$ the set

$$\left\{t \in (0, h); \frac{f(x+t) - f(x-t)}{2t} \leqq b\right\}$$

has outer measure at most $h/72$.

Since U and V cannot be separated by a measurable set, we may use 3.2 to find a measurable set P of positive measure such that the sets

$$U_0 = \{x \in U \cap P; h(x) > \epsilon\}$$

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and

$$V_0 = \{x \in V \cap P; h(x) > \epsilon\}$$

are both of full outer measure in P.

Let $x \in U_0$ be a density point of P and let $y \in U_0 \cap (x, x + \epsilon/2)$ be such that the set $(x, y) \setminus P$ has measure at most $(y - x)/5$. Then we can find a point z belonging to the set

$$(3x/4 + y/4, (x + y)/2) \cap V_0.$$

From the definition of the function $h(x)$ we infer that there is a measurable subset T of the interval $(0, 2(y - x))$ with measure at least

$$2(y - x) - 3[2(y - x)]/72 = 2(y - x) - (y - x)/12$$

such that for each $t \in T$ the inequalities $f(x+t)-f(x-t) < 2at$, $f(y+t)-f(y-t) < 2at$, and $f(z+t)-f(z-t) > 2bt$ hold. Let N be the set of all points $u \in (y, 2y-x)$ such that $u-y \in T, 2y-u-x \in T, z-2x+2y-u \in T, u-z \in T, x-2z+u \in T$, and $y - 2x + 2z - u \in T$. Observing that for every $u \in (y, 2y - x)$ all the points $u-y, 2y-u-x, z-2x+2y+u, u-z, x-2z+u$, and $y-2x+2y-u$ belong to $(0, 2(y-x))$, we easily infer that N has measure at least $(y-x)-6[(y-x)/12] > 0$. We conclude that $N \neq \emptyset$. On the other hand, using the above inequalities, we easily see that every $u \in N$ fulfils

$$\begin{aligned} f(u) &< f(2y - u) + 2a(u - y) < f(2x - 2y + u) + 2a(y - x) \\ &< f(2z - 2x + 2y - u) + 2a(y - x) - 2b(z - 2x + 2y - u) \end{aligned}$$

as well as

$$\begin{aligned} f(u) &> f(2z - u) + 2b(u - z) > f(2x - 2z + u) + 2b(z - u) \\ &- 2a(x - 2z + u) > f(2y - 2x + 2z - u) + 2b(u - z) - 2a(y - x). \end{aligned}$$

Thus $2b(u - z) - 2a(y - x) < 2a(y - x) - 2b(z - 2x + 2y - u)$, which is $4b(y - x) < 4a(y - x)$. This finishes the proof, since we know that $a < b$ and $y - x > 0$. (These computations are closely related to those in [**9**, p. 590]).

COROLLARY 3.4. *If an almost everywhere defined function is almost everywhere approximately symmetrically differentiable, then its approximate symmetric derivative is a measurable function.*

Proof. The statement follows immediately from 3.3.

4. The variation. By an *interval* function we shall understand a real-valued function h defined for all pairs (a, b) with $a < b$ in a set of full measure on the line. We write the values as $h(a,b)$ or occasionally as $h([a,b])$ or $h(I), I = [a,b]$. We call such a function *measurable* provided the function

$$(x,y) \longrightarrow h(x,y)$$

is measurable as a function of two variables. It is said to be $\mathcal{A}$-continuous (or approximately symmetrically continuous) at a point x if

$$\text{ap-lim}_{y \searrow 0}|h(x-y, x+y)| = 0$$

and to be $\mathcal{A}*$-continuous (or weakly approximately symmetrically continuous) at a point x if

$$\text{ap-liminf}_{y \searrow 0}|h(x-y, x+y)| = 0.$$

By an *additive interval function* we shall mean an interval function H such that $H(a,b) + H(b,c) = H(a,c)$ for all points $a < b < c$ in a set of full measure. If F is a real function defined almost everywhere then the interval function $(x,y) \longrightarrow F(y) - F(x)$ is an additive interval function that we denote as ΔF; of course every additive interval functionis of this form.

These interval functions permit the ordinary manipulations; thus we can write $h + k, c_1 h + c_2 K, hk, h/k$ etc. for interval functions h and k with obvious interpretations. We have a special convention for the product of a real function f (defined everywhere) with an interval function h; this is defined so that

$$fh(a,b) = f((a+b)/2)h(a,b)$$

for every pair for which h is defined. The inequality $h \leqq k$ for interval functions h and k shall mean that $h(a,b) \leqq k(a,b)$ for all pairs a, b in a set of full measure and the equality $h = k$ that $h(a,b) - k(a,b)$ again for all pairs a, b in a set of full measure.

Definition 4.1. Let β be a collection of interval-point pairs and let h be an interval function defined at least for all I with $(I,x) \in \beta$. Then we write

$$\text{Var}(h, \beta) = \sup_{\pi \subset \beta} \sum_{(I,x) \in \pi} |h(I)|$$

where the supremum is with regard to all packings π contained in β.

Definition 4.2. Let h be an interval function and E a set of real numbers. Then we write

$$V(h, \mathcal{A}[E]) = \inf_{\beta \in \mathcal{A}} \text{Var}(h, \beta[E]).$$

Definition 4.3. Let h be an interval function and E and A sets of real numbers. Then we write

$$V_A(h, \mathcal{A}[E]) = \inf_{\beta \in \mathcal{A}} \mathrm{Var}(h, \beta[E](A)).$$

The basic properties of the variation that we require in our development of the theory are produced in the ensuing statements.

LEMMA 4.4 *For any interval functions h_1 and h_2 and any set E of real numbers*

$$V(h_1 + h_2, \mathcal{A}[E]) \leqq V(h_1, \mathcal{A}[E]) + V(h_2, \mathcal{A}[E]).$$

Proof. The proof follows directly from the property of Lemma 1.2.

LEMMA 4.5. *Let h be an interval function that is $\mathcal{A}$-continuous at each point, let K be a finite union of closed intervals and let H be the complement of the interior of K. Then for any set E*

$$V(h, \mathcal{A}[E]) = V_K(h, \mathcal{A}[E]) + V_H(h, \mathcal{A}[E]).$$

Proof. Let β_1 and β_2 be arbitrary elements of $\mathcal{A}[E]$ and for any $\epsilon > 0$ choose a $\beta_3 \in \mathcal{A}$ so that $\mathrm{Var}(h, \beta_3[K \cap H]) < \epsilon$. This is possible since $K \cap H$ is finite and h is everywhere $\mathcal{A}$-continuous. The collection

$$\beta_4 = \beta_1(K) \cup \beta_3[K \cap H]$$

is also an element of $\mathcal{A}[E]$ and hence

$$V(h, \mathcal{A}[E]) \leqq \mathrm{Var}(h, \beta_4) \leqq \mathrm{Var}(h, \beta_1(K)) + \mathrm{Var}(h, \beta_2(H)) + \epsilon.$$

Then, since ϵ is arbitrary and β_1 and β_2 are arbitrary elements of $\mathcal{A}[E]$,

$$V(h, \mathcal{A}[E] \leqq V_K(h, \mathcal{A}[E]) + V_H(h, \mathcal{A}[E]).$$

The opposite inequality is clear and so the lemma is proved.

LEMMA 4.6. *Let h be an interval function that is $\mathcal{A}$-continuous at each point, let $\epsilon > 0$ and suppose that $V(h, \mathcal{A}[E]) < +\infty$. For any element $\beta \in \mathcal{A}[E]$ if the inequality*

$$\mathrm{Var}(h, \beta) \leqq V(h, \mathcal{A}[E]) + \epsilon$$

holds, then for any set K that is a finite union of closed intervals,

$$\mathrm{Var}(h, \beta(K)) \leqq V_K(h, \mathcal{A}[E]) + \epsilon.$$

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Proof. If we have the inequality

$$\mathrm{Var}(h, \beta) \leqq V(h, \mathcal{A}[E]) + \epsilon,$$

if K is a finite union of intervals, and if H is the complement of the interior of K then, by lemma 4.5,

$$\begin{aligned}\mathrm{Var}(h, \beta(K)) &\leqq \mathrm{Var}(h, \beta) - \mathrm{Var}(h, \beta(H)) \\ &\leqq V(h, \mathcal{A}[E]) + \epsilon - V_H(h, \mathcal{A}[E]) \\ &= V_K(h, \mathcal{A}[E]) + \epsilon\end{aligned}$$

as required.

THEOREM 4.7. *Let h be an $\mathcal{A}$-continuous interval function and suppose that $f, g_1, g_2, \ldots$ is a sequence of nonnegative measurable real functions with $g_1 \leqq g_2 \leqq \ldots$ and $f \leqq \sup_n g_n$. Then for any measurable set E,*

$$V(fh, \mathcal{A}[E]) \leqq \lim_{n \to +\infty} V(g_n h, \mathcal{A}[E]).$$

Proof. We may suppose that $V(g_n h, \mathcal{A}[E])$ is finite for each n. Let $\epsilon > 0$ and $0 < c < 1$. Then for each point x there is a least integer $n(x)$ for which $cf(x) \leqq g_m(x)$ if $m \geqq n(x)$. Choose a sequence $\{\beta_n\}$ from $\mathcal{A}[E]$ with the property that

$$\mathrm{Var}(g_n h, \beta) \leqq V(g_n h, \mathcal{A}[E]) + \epsilon/2^n.$$

By Lemma 4.6 this gives

$$\mathrm{Var}(g_n h, \beta(K)) \leqq V_K(g_n h, \mathcal{A}[E]) + \epsilon/2^n$$

for every finite union of intervals K. Define the sets

$$X_n = \{x \in E; n(x) = n\}$$

and define the collection

$$\beta = \bigcup_{n=1}^{\infty} \beta_n[X_n].$$

Since the sequence of sets $X_1, X_2, \ldots$ is disjointed, measurable and covers E, β must be in $\mathcal{A}[E]$ by Lemma 1.3. We now estimate $V(fh, \mathcal{A}[E])$ by computing $\mathrm{Var}(fh, \beta)$. Let π denote an arbitrary packing contained in β and write $K_n =$

$\sigma(\pi_n)$ where $\pi_n = \pi[X_n]$. There is a first integer N so that π_m is empty for all $m > N$. We compute

$$\begin{aligned}
\sum_{(I,x)\in\pi} |f(x)h(I)| &= \sum_{i=1}^{N} \sum_{(I,x)\in\pi_n} |f(x)h(I)| \\
&\leqq c^{-1} \sum_{i=1}^{N} \sum_{(I,x)\in\pi_n} |g_n(x)h(I)| \\
&\leqq c^{-1} \sum_{i=1}^{N} \mathrm{Var}(g_n h, \beta_n(K_n)) \\
&\leqq \epsilon c^{-1} + c^{-1} \left\{ \sum_{i=1}^{N} V_{K_n}(g_n h, \mathcal{A}[E]) \right\} \\
&\leqq \epsilon c^{-1} + c^{-1} \left\{ \sum_{i=1}^{N} V_{K_n}(g_N h, \mathcal{A}[E]) \right\} \\
&\leqq \epsilon c^{-1} + c^{-1} \{ V(g_N h, \mathcal{A}[E]) \}.
\end{aligned}$$

As this holds for all packings $\pi \subset \beta$, we have

$$V(fh, \mathcal{A}[E]) \leqq \mathrm{Var}(fh, \beta) \leqq \epsilon c^{-1} + c^{-1} \{\sup_N V(g_n h, \mathcal{A}[E])\}.$$

Letting $\epsilon \searrow 0$ and $c \nearrow 1$ in this inequality we obtain the result.

COROLLARY 4.8. *Let h be an $\mathcal{A}$-continuous interval function. Then*

$$V(h, \mathcal{A}[E]) = \sup_J V_J(h, \mathcal{A}[E])$$

where the supremum is taken over all intervals J.

Proof. Take g_n as the characteristic function of the interval $[-n, n]$ and apply Theorem 4.7.

COROLLARY 4.9. *Let h be an $\mathcal{A}$-continuous interval function and suppose that $f, g_1, g_2, \ldots$ is a sequence of measurable functions such that*

$$0 \leqq |f(x)| \leqq \sum_{n=1}^{\infty} |g_n(x)|$$

everywhere in a measurable set E. Then

$$V(fh, \mathcal{A}[E]) \leqq \sum_{n=1}^{\infty} V(g_n h, \mathcal{A}[E])$$

Proof. This is evidently true for finite sums and the extension to infinite sums requires merely an application of 4.7.

Dual versions of the above variational concepts are also defined using the dual basis $\mathcal{A}^*$ and we use the notation $V(h, \mathcal{A}^*[E])$. Most of the results stated above hold true for this variation with minor changes. Note that Lemma 4.4 does not have an analogue here since the basis $\mathcal{A}^*$ is not filtering; this means too that Corollary 4.9 also cannot hold for this basis. Otherwise all the other properties hold: the results in 4.5, 4.6 and 4.7 hold for interval functions h that are at least weakly approximately symmetrically continuous and the measurability assumptions on the functions may be dropped.

5. Variational measures. For integration theories developed along the lines we follow here there is an associated measure theory that is frequently required. We present this here.

Definition 5.1. Let h be an interval function. Then we define the outer measure h^* on the class of Lebesgue measurable sets by writing, for any Lebesgue measurable set E,

$$h^*(E) = V(h, \mathcal{A}[E]).$$

Definition 5.2. Let h be an interval function. Then we define the outer measure h_* on the class of all subsets of the real line by writing, for any set E,

$$h_*(E) = V(h, \mathcal{A}^*[E]).$$

Note here that the set function h_* is defined for all subsets of the real line whereas we wish h^* to be defined only on measurable sets; both are outer measures subject to this interpretation, i.e., each is a countably subadditive, monotone set function on its domain. For additive interval functions F the measure F^* represents a generalization of the Lebesgue-Stieltjes measures; indeed if f is continuous and monotonic and F is its associated interval function it can be shown that this measure (and F_* too) is precisely the usual Lebesgue-Stieltjes measures associated with f. The interval function ℓ defined by setting $\ell(I) = |I|$ generates a measure ℓ^* that gives exactly the Lebesgue measure (see 5.6 below).

LEMMA 5.3. *Let h be an interval function and x_0 a real number. Then*

$$h^*(x_0) = \text{ap} - \limsup_{t \searrow 0+} |h(x_0 - t, x_0 + t)|$$

and

$$h_*(x_0) = \text{ap} - \liminf_{t \searrow 0+} |h(x_0 - t, x_0 + t)|.$$

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Note that a measurable function F defined almost everywhere is approximately symmetrically continuous ($\mathcal{A}$-continuous) at a point x_0 if and only if $\Delta F^*(x_0) = 0$, and is weakly approximately symmetrically continuous ($\mathcal{A}^*$-continuous) at a point x_0 if and only if $\Delta F_*(x_0) = 0$.

LEMMA 5.4. *For every Lebesgue measurable set* $E, h_*(E) \leqq h^*(E)$.

The set functions h^* and h_* associated with an interval function h are genuine outer measures on the real line that have nice topological properties. We develop these ideas next.

THEOREM 5.5. *For any interval function* h *the set functions* h^* *and* h_* *are metric outer measures.*

Proof. It is clear that the set function h^* is nonnegative and monotone and thus, to show that it is an outer measure, we need to show that it is countably subadditive on the class of Lebesgue measurable sets. Let $X, Y_1, Y_2, Y_3, \dots$ be a sequence of measurable sets for which $X \subset \bigcup_{i=1}^{\infty} Y_i$. We may suppose that the sequence of sets $Y_1, Y_2, Y_3, \dots$ is disjointed. Suppose that a positive number ϵ has been given and choose $\beta_i \in \mathcal{A}[Y_i]$ in such a way that

$$\operatorname{Var}(h, \beta_i) \leqq h^*(Y_i) + \epsilon/2^i.$$

Define β as the union of the families β_i then, by Lemma 1.3, $\beta \in \mathcal{A}[X]$. Therefore

$$h^*(X) \leqq \operatorname{Var}(h, \beta) \leqq \sum_{i=1}^{\infty} \operatorname{Var}(h, \beta_i) \leqq \sum_{i=1}^{\infty} h^*(Y_i) + \epsilon.$$

As ϵ is an arbitrary positive number we have then the inequality $h^*(X) \leqq \sum_{i=1}^{\infty} h^*(Y_i)$ as required to establish that h^* is an outer measure on the measurable sets. In precisely the same manner it may be shown that h_* is also an outer measure.

To see that h^* is a metric outer measure suppose that two measurable sets X_1 and X_2 are separated in such a way that there are open sets G_1 and G_2 with $X_1 \subset G_1, X_2 \subset G_2$, and $G_1 \cap G_2 = \emptyset$. Then if $\beta \in \mathcal{A}[X_1 \cup_2]$ chosen in such a way that

$$\operatorname{Var}(h, \beta) \leqq h^*(X_1 \cup X_2) + \epsilon,$$

then we may define the collections $\beta_1 = \beta(G_1)$ and $\beta_2 = \beta(G_2)$ which, by Lemma 1.4 belong to $\mathcal{A}[X_1]$ and $\mathcal{A}[X_2]$. Now we compute

$$\begin{aligned} h^*(X_1) + h^*(X_2) &\leqq \operatorname{Var}(h, \beta_1) + \operatorname{Var}(h, \beta_2) \\ &\leqq \operatorname{Var}(h, \beta) \\ &\leqq H^*(X_1 \cup X_2) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we must have the equality

$$h^*(X_1 \cup X_2) = h^*(X_1) + h^*(X_2)$$

as required.

The same arguments apply to h_* and so the proof is complete.

THEOREM 5.6. *The measure ℓ^* is precisely Lebesgue measure and ℓ_* is the Lebesgue outer measure.*

Proof. This is a direct consequence of the Vitali covering theorem for Lebesgue measure.

6. Differential equivalence. Most of the results that we discuss concern derivatives and integrals of interval functions, and are identical for those interval functions that belong to the same "variational" equivalence class. The terminology for this equivalence relation and the general idea of exploiting it are due to Kolmogorov [**8**]. Henstock [**6**] uses the same idea in his concept of "variational equivalence" and Leader [**11**] and [**10**] exploits this notion to define a class of "differentials".

Definition 6.1. Let h_1 and h_2 be interval functions. We say that h_1 and h_2 are *differentially equivalent* and we write $h_1 \equiv h_2$ provided that

$$V(h_1 - h_2, \mathcal{A}[\mathbf{R}]) = 0.$$

This is evidently an equivalence relation because of 4.4. We develop now some of the more immediate properties of this equivalence relation. Throughout h_1 and h_2 are interval functions and f is a real function defined everywhere. Recall that the product fh is defined as an interval function $(a, b) \longrightarrow f((a+b)/2)h(a, b)$.

LEMMA 6.2. *If $h_1 \equiv h_2$ then, for any measurable real function f, the equivalence $fh_1 \equiv fh_2$ holds.*

LEMMA 6.3. *Let $h_1 \equiv h_2$ and k be an interval function for which*

$$g(x) = \text{ap}-\limsup_{h \searrow 0} |k(x - h, x + h)| < +\infty$$

everywhere. Then, if g is measurable, $kh_1 \equiv kh_2$.

Proof. Evidently 6.2 is included in 6.3; to prove the equivalence relation in the latter let X_n denote the set of points x at which $g(x) < n$ and let β_n be the collection of all interval-point pairs (I, x) for which $|k(I)| < n$, if $x \in X_n$ and x the midpoint of I. We easily check that $\beta_n \in \mathcal{A}[X_n]$ and hence, for any other $\beta \in \mathcal{A}$, we have

$$\text{Var}(kh_1 - kh_2, \beta \cap \beta_n[X_n]) \leqq n\text{Var}(h_1 - h_2, \beta).$$

From this we obtain $V(kh_1 - kh_2, \mathcal{A}[X_n]) = 0$ for all integers n and, since the sets X_n evidently cover the real line and are measurable, the conclusion of the lemma will follow.

THEOREM 6.4. *For any interval function h and any measurable function f the relation $fh \equiv 0$ holds if and only if $f(x) = 0$ for h^*-almost every point x.*

Proof. Let X_n denote the set of points x at which $|f(x)| > n^{-1}$. Then it is easy to show that $V(fh, \mathcal{A}[X_n]) \geqq n^{-1}h^*(X_n)$ for each n. Since we assume here that $V(fh, \mathcal{A}[X_n]) = 0$ and that each X_n is measurable it follows from the fact that h^* is an outer measure that h^* must vanish on the set of points at which f differs from 0.

THEOREM 6.5. *For any interval function h and any measurable function f the relation $fh \equiv gh$ holds if and only if $f(x) = g(x)$ for h^*-almost every points x.*

Proof. This follows from 6.4.

THEOREM 6.6. *For an additive interval function H the relation $H \equiv 0$ holds if and only if $H = 0$.*

Proof. Recall that $H = 0$ means that there is a set of full measure B so that $H(a, b) = 0$ for each pair in B. If $H \equiv 0$ then for every integer n there is an element $\beta_n \in \mathcal{A}$ with $\mathrm{Var}(H, \beta_n) < 1/n$. By Lemma 1.6 there is a set B of full measure so that every β_n contains a partition of any interval with endpoints in B. Evidently then, since H is additive,

$$|H(a, b)| \leqq \mathrm{Var}(H, \beta) < n^{-1}$$

for all $a, b \in B$ so that $H(a, b) = 0$ for each pair in B as required.

THEOREM 6.7. *For any interval functions h and k if $h \equiv k$ then the variational measures for h and k are identical: $h^* = k^*$ and $h_* = k_*$.*

Proof. If $h \equiv k$ then for any measurable set E,

$$V(h - k, \mathcal{A}[E]) = 0.$$

Let $\epsilon > 0$ and choose $\beta \in \mathcal{A}[E]$ so that

$$\mathrm{Var}(h - k, \beta) < \epsilon.$$

Then if $\beta_1 \in \mathcal{A}[E]$ the collection $\beta_1 \cap \beta$ is in $\mathcal{A}[E]$ and

$$\begin{aligned} \mathrm{Var}(h, \beta \cap \beta_1) &\leqq \mathrm{Var}(k, \beta \cap \beta_1) + \mathrm{Var}(h - k, \beta) \\ &< \mathrm{Var}(k, \beta_1) + \epsilon. \end{aligned}$$

This shows that $h^*(E) \leqq k^*(E)$ and so, by symmetry, equality must hold. Similarly the identity $h_* = k_*$ may be proved.

THEOREM 6.8. *Let h and k be interval functions and f a measurable function. If* $h \equiv fk$ *and* k^* *is σ-finite on a measurable set E then so too is* h^* *and* h^* *vanishes on every subset of E of* k^**-measure zero.*

Proof. This is immediate if f is bounded, otherwise just apply this to each of the measurable sets $\{x; |f(x)| < n\}$ for $n = 1, 2, \ldots$.

THEOREM 6.9. *Let h and k be interval functions, f a real function and suppose that* $h \equiv fk$. *Then at* k_**-almost every point x,*

$$\text{ap-}\lim_{t \searrow 0} \frac{h(x-t,x+t)}{k(x-t,x+t)} = f(x).$$

Proof. For each integer n let β_n denote the collection of pairs $([x-t, x+t], x)$ satisfying the inequality

$$|h(x-t,x+t) - f(x)k(x-t,x+t)| \geqq |k(x-t,x+t)|/n$$

and let Y_n denote the set of all points x such that β_n is in $\mathcal{A}^*[\{x\}]$. Let Y denote the union of the sequence of sets Y_n; we shall show that $k_*(Y) = 0$ and that for every point x not in Y the limits stated in the theorem must hold.

Let $\epsilon > 0$. Since $h \equiv fk$ we may select an element $\alpha \in \mathcal{A}$ so that

$$\mathrm{Var}(h - fk, \alpha) < \epsilon.$$

Each collection $\alpha \cap \beta_n$ is in $\mathcal{A}^*[Y_n]$ and hence

$$\begin{aligned} k_*(Y_n) &= V(k, \mathcal{A}^*[Y_n]) \\ &\leqq \mathrm{Var}(k, \alpha \cap \beta_n) \\ &\leqq \mathrm{Var}(n(h - fk), \alpha \cap \beta_n) \\ &\leqq n\mathrm{Var}(h - fk, \alpha) \\ &\leqq n\epsilon. \end{aligned}$$

Consequently each Y_n has k_*-measure zero and so $k_*(Y) = 0$ as stated.

Now for each $x \notin Y$ and for each integer n the set V_t of t for which

$$|h(x-t,x+t) - f(x)k(x-t,x+t)| < |k(x-t,x+t)|/n.$$

has density 1 at 0 and from this at each point x in $\mathbf{R} \setminus Y$ we easily verify the required limits.

THEOREM 6.10. *Let h and k be measurable interval functions, let f be a measurable function, suppose that the measure* k^* *is σ-finite, and suppose that the limit*

$$\text{ap-}\lim_{t \searrow 0} \frac{h(x-t,x+t)}{k(x-t,x+t)} = f(x)$$

holds at h^*-almost every and k^*-almost every point x. Then $h \equiv fk$.

Proof. Let X be the set of points x at which the above stated limits hold. Then for every integer n the collection β_n of pairs $([x-t, x+t], x)$ satisfying the inequality

$$|h(x-t, x+t) - f(x)k(x-t, x+t)| \leqq n^{-1}|k(x-t, x+t)|$$

must be in $\mathcal{A}[X]$. From this we may deduce

$$V(h - fk, \mathcal{A}[Y]) \leqq n^{-1}k^*(Y)$$

for every subset Y of X. As n is arbitrary and k^* is σ-finite on X we must have $V(h - fk, \mathcal{A}[X]) = 0$, and this gives

$$V(h - fk, \mathcal{A}[\mathbf{R}]) \leqq V(h - fk, \mathcal{A}[X]) + h^*(\mathbf{R} \setminus X) + (fh)^*(\mathbf{R} \setminus X) = 0$$

which is equivalent to the relation $h \equiv fk$ that we wished to prove.

7. Integrable interval functions. We define a general notion of integral for interval functions. For any interval function h we define a notion of integrability and an integral $\int_a^b dh$. This is then specialized in Section 8 for interval functions of the more traditional form $(x-t, x+t) \longrightarrow 2f(x)t$. We prefer the notation $\int_a^b f\, dh$ for an integral of the interval function fh.

Definition 7.1. A real-valued interval function h is said to be *$\mathcal{A}$-integrable* if for every $\epsilon > 0$ there is an element $\beta \in \mathcal{A}$ such that for every pair of partitions $\pi_1, \pi_2 \subset \beta$ of the same interval

$$\left| \sum_{(I,x)\in\pi_1} h(I) - \sum_{(I,x)\in\pi_2} h(I) \right| < \epsilon.$$

Our fundamental lemma now gives a useful necessary and sufficient condition for integrability and provides the tools needed for defining the integral itself. We use this lemma below to define the notion of "indefinite integral".

LEMMA 7.2. *Let h be an interval function. Then h is $\mathcal{A}$-integrable if and only if there is a set B of full measure and an additive interval function H defined on all pairs (a, b) with $a, b \in B$ so that for every $\epsilon > 0$ there is an element $\beta \in \mathcal{A}$ that contains a partition of every interval with endpoints in B and such that for every packing $\pi \subset \beta$,*

$$\sum_{([y,z],x)\in\pi} |H(y, z) - h(y, z)| < \epsilon.$$

Proof. If this condition holds for such a function H then for every $\epsilon > 0$ there is an element $\beta \in \mathcal{A}$ so that $\mathrm{Var}(H - h, \beta) < \epsilon/2$. Then for every pair of partitions $\pi_1, \pi_2 \subset \beta$ of the same interval

$$\left| \sum_{(I,x)\in\pi_1} h(I) - \sum_{(I,x)\in\pi_2} h(I) \right|$$

$$\leqq \left| \sum_{(I,x)\in\pi_1} h(I) - \sum_{(I,x)\in\pi_1} H(I) \right| + \left| \sum_{(I,x)\in\pi_2} h(I) - \sum_{(I,x)\in\pi_2} H(I) \right|$$

$$\leqq 2\mathrm{Var}(H - h, \beta) < \epsilon.$$

And this shows that h is integrable as required.

In the other direction let us choose, for each n, an element $\beta_n \in \mathcal{A}$ in such a way that for every pair of partitions $\pi_1, \pi_2 \subset \beta_n$ of the same interval

$$\left| \sum_{(I,x)\in\pi_1} h(I) - \sum_{(I,x)\in\pi_2} h(I) \right| < 1/n.$$

In view of Lemma 1.2 we may suppose that always $\beta_n \subset \beta_{n-1}$. By Lemma 1.6 there is a set B of full measure so that each β_n contains a partition of every interval with endpoints in B. For such an interval $[a, b]$ let $S_n(h, [a, b])$ denote the closure of the set of all sums $\sum_{(I,x)\in\pi} h(I)$ for any partition $\pi \subset \beta_n$ of $[a, b]$. These sets form a decreasing sequence of nonempty closed sets with diameter shrinking to zero. Let $H(a, b)$ denote the limit point.

We claim that H is an additive interval function (defined on intervals with endpoints in B), and that $\mathrm{Var}(H - h, \beta_n) \leqq 2/n$. Certainly H is defined for all such intervals and for these intervals

$$\left| H(a, b) - \sum_{\pi} h(I) \right| \leqq 2/n$$

for all partitions π of $[a, b]$ from β_n. Take then three intervals $[a, b]$, $[b, c]$ and $[a, c]$ with endpoints in B; we may choose partitions π_1 and π_2 of $[a, b]$ and $[a, c]$ respectively from β_n. Then $\pi_3 = \pi_1 \cup \pi_1$ is a partition of $[a, c]$ and the three sums $\sum_{\pi_i} h(I)$ for $i = 1, 2, 3$ are within $2/n$ of $H(a, b)$, $H(b, c)$ and $H(a, c)$ respectively. Consequently $|H(a, b) + H(b, c) - H(a, c)| \leqq 6/n$ for all n. It follows that H is additive and that $\mathrm{Var}(H - h, \beta_n) \leqq 2/n$. It is clear now that H and B satisfy the conditions stated in the lemma and so the proof is complete.

We shall say for any such pair that H is an *indefinite integral* of h. In that case we write

$$(\mathcal{A}) \int_a^b dh = H(a, b)$$

with the understanding that this is defined for all a, b in a set of full measure. Note that there is no canonical choice of the set B for any choice of h and so the indefinite integral is simply an interval function defined almost everywhere. The lemma gives a procedure whereby the value $H(a, b)$ may be determined but the set B may not be prescribed. The prefix $(\mathcal{A})$ may be dropped where it is clear so that we may write $\int_a^b dh$.

This lemma may also be expressed more briefly.

LEMMA 7.3. *Let h be an interval function. Then h is $\mathcal{A}$-integrable if and only if there is an additive interval function H so that $H \equiv h$.*

We now develop the most basic properties of the integral. Lemmas 7.4–7.6 are immediate.

LEMMA 7.4. *If h is $\mathcal{A}$-integrable with an indefinite integral H then $H - h$ is $\mathcal{A}$-continuous at every point. H is [weakly] approximate symmetrically continous at any point at which h is.*

LEMMA 7.5. *If h_1 and h_2 are $\mathcal{A}$-integrable then so too is any linear combination $c_1h_1 + c_2h_2$ and*

$$\int_a^b d(c_1h_1 + c_2h_2) = c_1 \int_a^b dh_1 + c_2 \int_a^b dh_2$$

for all a and b in a set of full measure.

LEMMA 7.6. *Let h be $\mathcal{A}$-integrable. Then for every interval $[a, b]$ with endpoints in a set of full measure there is a sequence of Riemann sums*

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} h(x_{i-1}^{(n)}, x_i^{(n)}) = \int_a^b dh$$

where $\{x_0^{(n)}, x_1^{(n)}, \dots, x_{m_n}^{(n)}\}$ is a partition of $[a, b]$ and $\max\{x_i^{(n)} - x_{i-1}^{(n)}\} \to 0$.

8. The integral. We investigate now an integral that directly generalizes the Riemann integral by specializing the integral of the preceding section and we develop the basic properties.

Definition 8.1. A real-valued function f defined everywhere is said to be $\mathcal{A}$-*integrable* if for every $\epsilon > 0$ there is an element $\beta \in \mathcal{A}$ such that for every pair of partitions $\pi_1, \pi_2 \subset \beta$ of the same interval

$$\left| \sum_{(I,x)\in\pi_1} f(x)|I| - \sum_{(I,x)\in\pi_1} f(x)|I| \right| < \epsilon.$$

LEMMA 8.2. *A real-valued function f is $\mathcal{A}$-integrable if and only if there is a set B of full measure and an additive interval function F defined for pairs in*

B so that for every $\epsilon > 0$ there is an element $\beta \in \mathcal{A}$ that contains a partition of every interval with endpoints in B and such that for every packing $\pi \subset \beta$,

$$\sum_{([y,z],x)\in\pi} |F(y,z) - f((y+z)/2)(y-z)| < \epsilon.$$

We shall say for any such pair that F is an *indefinite integral* of F. In that case we write

$$(\mathcal{A}) \int_a^b f(x)dx = F(a,b)$$

for any pair of numbers a and b with $a, b \in B$. The prefix $(\mathcal{A})$ may be dropped where it is clear so that the integral is simply written $\int_a^b f(x)dx$; this will not interfere with classical notations since when the integral exists in any of many well-known senses (Riemann, Lebesgue, Denjoy-Perron) it will exist in this present sense and with the same value. In the case of periodic functions (with period p say) this evidently allows integration over a period $\int_a^{a+p} f(x)dx$ for almost every a. For periodic functions the integral may be presented in an essentially simpler way that more closely mimics the definition of the Riemann integral. See the material in Section 11 for a complete discussion.

As we shall see the integrability of a function in this sense will require that it be measurable (see Theorem 8.3). Two measurable functions that are Lebesgue equivalent (i.e., equal almost everywhere) are interchangable as far as integrability and the integral are concerned (see Theorem 8.6) thus it will be natural to require for an integral $\int_a^b f(x)dx$ only that f is measurable and defined almost everywhere.

THEOREM 8.3. *If a real-valued function f is $\mathcal{A}$-integrable with an indefinite integral F then f and F are measurable and F is $\mathcal{A}$-continuous at every point.*

Proof. By 6.9 and 5.6 the function f is almost everywhere equal to the approximate symmetric derivative of F. Hence the measurability of f follows immediately from 3.4. That F is approximately symmetrically continuous everywhere follows directly from Lemma 7.4.

Assume for the moment that F is not measurable. Then there are real numbers $a < b$ such that the sets

$$U = \{x \in \mathbf{R}; F(x) > b\}$$

and

$$V = \{x \in \mathbf{R}; F(x) < a\}$$

cannot be separated by a measurable set. Thus 3.2 with $h = 1/(1 + |f|)$ provides us with a measurable set P of positive measure and with a positive constant c

such that the sets $\{x \in U \cap P; |f(x)| < c\}$ and $\{x \in V \cap P; |f(x)| < c\}$ are of full outer measure in P. Since f is measurable, we infer that $|f(x)| < c$ for almost all $x \in P$.

Using 8.2 with $\epsilon = (b-a)/4$, we find an element $\beta \in \mathcal{A}$ such that for every $([y,z],x) \in \beta$,

$$|F(y) - F(z) - f((y+z)/2)(y-z)| < (b-a)/4.$$

From 1.5 we may take a density point x of P which has the property that the set

$$\{t; ([x, x+t], x+t/2) \in \beta\}$$

has upper density 1 on the right at 0. We find $y \in (x, x+(b-a)/4c)$ such that $|P \cap (x,y)| > (y-x)/4$ and

$$|\{u \in (x,y); ([x,y],(x+u)/2) \in \beta\}| > (y-x)/4.$$

Let N be the set of all those points $u \in P \cap (x,y)$ such that $([x,y],(x+u)/2) \in \beta$ and $|f((x+u)/2)| < c$. Then N is a measurable subset of P of positive measure. However, for every $u \in N$ we have

$$\begin{aligned}|(F(u) - F(x))| &< |(F(u) - F(x)) - f((x+u)/2)(u-x)| \\ &\quad + |f((x+u)/2)|(u-x) \\ &< (b-a)/4 + c[(b-a)/4c] = (b-a)/2.\end{aligned}$$

But this implies that at least one of the sets $U \cap N$ and $V \cap N$ is empty, which contradicts that they both have full outer measure in P. This contradiction shows that F is measurable and completes the proof.

THEOREM 8.4. *If f_1 and f_2 are $\mathcal{A}$-integrable then so too is any linear combination $c_1 f_1 + c_2 f_2$ and*

$$\int_a^b (c_1 f_1(x) + c_2 f_2(x))dx = c_1 \int_a^b f_1(x)dx + c_2 \int_a^b f_2(x)dx$$

for all a, b in a set of full measure.

THEOREM 8.5. *Let f be integrable. Then for every interval $[a,b]$ with endpoints in a set of full measure there is a sequence of Riemann sums*

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} f\left((x_{i-1}^{(n)} + x_i^{(n)})/2\right)\left(x_i^{(n)} - x_{i-1}^{(n)}\right) = \int_a^b f(x)dx$$

where $\{x_0^{(n)}, x_1^{(n)}, \ldots, x_{m_n}^{(n)}\}$ is a partition of $[a,b]$ and $\max\{x_i^{(n)} - x_{i-1}^{(n)}\} \to 0$.

THEOREM 8.6. *If $f = g$ almost everywhere and f is $\mathcal{A}$-integrable then g is $\mathcal{A}$-integrable and*

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

for all a, b in a set of full measure.

Proof. We know that f is measurable and that $f\ell \equiv F$ for some additive interval function F. By 6.5 and 5.6 we have $g\ell \equiv f\ell$ and consequently $g\ell \equiv F$. By Lemma 8.2 then g is integrable and also has F as an indefinite integral.

THEOREM 8.7. *If f and g are $\mathcal{A}$-integrable functions for which*

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

for all a, b in a set of full measure then $f = g$ almost everywhere.

Proof. We know that f and g are measurable and, because of the indentity of the two indefinite integrals, $f\ell \equiv g\ell$. By Lemmas 6.5 and 5.6 then $f = g$ almost everywhere.

THEOREM 8.8. *In order for a function f to be $\mathcal{A}$-integrable with an indefinite integral F it is necessary and sufficient that the following hold:*

1. *F is a measurable, additive interval function that is everywhere $\mathcal{A}$-continuous.*
2. *f is almost everywhere the approximate symmetric derivative of F.*
3. *F^* is σ-finite and vanishes on every set of Lebesgue measure zero.*

Proof. These conditions are necessary and sufficient in order that $F \equiv f\ell$. See Theorems 6.9, 6.8 and 8.3.

COROLLARY 8.9. *In order for a function f to be $\mathcal{A}$-integrable with an indefinite integral F it is sufficient that F is a measurable, additive interval function for which any one of the following hold:*

1. *f is everywhere the approximate symmetric derivative of F.*
2. *f is the approximate symmetric derivative of F at all but denumerably many points and F is everywhere $\mathcal{A}$-continuous.*
3. *f is almost everywhere and F^*-almost everywhere the approximate symmetric derivative of F.*

THEOREM 8.10. *If f is integrable on every finite interval in the sense of Riemann, Lebesgue or Denjoy-Perron then f is $\mathcal{A}$-integrable and the integrals agree.*

Proof. If f is Denjoy-Perron integrable with an indefinite integral F then for every $\epsilon > 0$ there is a positive measurable function $\delta(x)$ so that every sum of

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the form

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

over a partition of an interval $[a, b]$ with $\xi_i \in [x_{i-1}, x_i]$ and $x_i - x_{i-1} < \delta(\xi_i)$ must be within ϵ of $F(b) - F(a)$ (see [**15**], for example, for a proof). We write $H(a, b) = F(b) - F(a)$ so that H is a measurable, additive interval function.

Let β be the collection of interval-point pairs

$$\{([x - t, x + t], x); 0 < t < \delta(x)\}.$$

This is clearly an element of $\mathcal{A}$ and for any partition $\pi \subset \beta$ of an interval $[a, b]$ the sum

$$\sum_{(I,x)\in\pi} f(x)|I|$$

may be expressed in the form

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

over a partition of an interval $[a, b]$ with $\xi_i \in [x_{i-1}, x_i]$ and $x_i - x_{i-1} < \delta(\xi_i)$ merely by using the identity

$$f(x_i)|[x_i - t_i, x_i + t_i]| = f(x_i)|[x_i - t_i, x_i]| + f(x_i)|[x_i, x_i + t_i]|.$$

From this we can deduce $\operatorname{Var}(f\ell - H, \beta) < \epsilon$; and hence that $f\ell \equiv H$. The theorem evidently follows now from Lemma 8.2.

THEOREM 8.11. *If a function f is $\mathcal{A}$-integrable and nonnegative then f is Lebesgue integrable on every finite interval.*

Proof. If F is an indefinite integral for f then it is easy to see, from Lemma 8.2 for example, that $F(a, b) \geqq 0$ for all pairs in a set B of full measure. Thus we may extend F to a continuous monotonic function defined everywhere and, since f is almost everywhere the approximate symmetric derivative of F, evidently $F' = f$ almost everywhere. Consequently f is Lebesgue integrable on each interval.

9. Integration by parts. While an adequate integration by parts formula is available for the $\mathcal{A}$-integral there are a number of differences between this theory and others. The integral here is very fragile; for example a function f may be integrable without the product fg having an integral even for very smooth functions g. Thus if $F(x) = x^{-2}$ then $F'(x)$ is integrable but $F'(x) \sin x$ and $F(x)$

are not. We focus then on the validity of the usual integration by parts formula rather than on conditions that supply integrability.

The integration by parts formula is obtained from a trivial identity. Let F and G denote functions defined everywhere and let ΔF and ΔG denote the corresponding interval functions. We use the following elementary computation

$$\begin{aligned} G(x)[F(x+h) - F(x-h)] + F(x)[G(x+h) - G(x-h)] \\ - [F(x+h)G(x+h) - F(x-h)G(x-h)] \\ = [F(x-h) - F(x)][G(x-h) - G(x)] \\ + [F(x+h) - F(x)][G(x+h) - G(x)] \end{aligned}$$

to obtain the identifity

$$(7) \qquad G\Delta F + f\Delta G - \Delta FG = L_{FG} - R_{FG}$$

where the interval functions L_{FG} and R_{FG} are defined by the statements

$$L_{FG}(x-h, x+h) = [F(x-h) - F(X)][G(x-h) - G(x)]$$

and

$$R_{FG}(x-h, x+h) = [F(x+h) - F(x)][G(x+h)_G(x)].$$

From this identity we obtain immediately that the condition under which the integration by parts formula holds is merely that $V(L_{FG} - R_{FG}, \mathcal{A}[\mathbf{R}]) = 0$. It is easy to prove that $V(L_{FG}, \mathcal{A}[E]) = V(R_{FG}, \mathcal{A}[E])$ if F is approximately continuous at every point of E and if G has bounded variation (or even if G is VBG_*) and this may be used to obtain special conditions under which the forumla is valid.

LEMMA 9.1. *Let the functions F and G be defined everywhere. Then, on the assumption that the integrals exist, the integration by parts formula*

$$\int_a^b F(x)dG(x) = F(b)G(b) - F(a)G(a) - \int_a^b G(x)dF(x)$$

holds for a and b in a set of full measure if and only if

$$V(L_{FG} - R_{FG}, \mathcal{A}[\mathbf{R}]) = 0.$$

For a further application we obtain another variant under certain natural conditions; of course other versions are possible.

LEMMA 9.2. *Let the real functions f and g be integrable where the functions F and G are their indefinite integrals. Suppose that the following integrals*

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exist and that G has bounded variation on every compact interval. Then the integration by parts formula

$$(8) \qquad \int_a^b F(x)g(x) = F(b)G(b) - F(a)G(a) - \int_a^b G(x)dF(x)$$

holds for a and b in a set of full measure if and only if the measure $(\Delta FG)^*$ vanishes on every set of Lebesgue measure zero.

Proof. The necessity is clear, If (8) holds then $\Delta FG \equiv Fg\ell + Gf\ell \equiv F\Delta G + G\Delta F$. Thus for any set N of Lebesgue measure zero (i.e., with $\ell^*(N) = 0$)$(F\Delta G)^*(N) = (Fg\ell)^*(N) = (Fg\ell)^*(N) = 0$ and $(G\Delta F)^*(N) = (Gf\ell)^*(N) = 0$ so that $(\Delta FG)^*(N) = 0$.

For the sufficiency F is approximately continuous on a set of full measure B and so $V(L_{FG}, \mathcal{A}[B]) = V(R_{FG}, \mathcal{A}[B]) = 0$ since G has bounded variation. If $N = \mathbf{R} \setminus B$ then, since N has Lebesgue measure zero (i.e., $\ell^*(N) = 0), (F\Delta G)^*(N) = (Fg\ell)^*(N) = 0), (F\Delta G)^*(N) = (Fg\ell)^*(N) = 0$ and $(G\Delta F)^*(N) = (Gf\ell)^*(N) = 0$. Thus if, in addition, $(\Delta FG)^*(N) = 0$ it follows from the identity (7) that $V(L_{FG} - R_{FG}, \mathcal{A}[N]) = 0$. Consequently $V(L_{FG} - R_{FG}, \mathcal{A}[\mathbf{R}]) = 0$ and the present integration by parts formula follows from Lemma 9.1.

10. The $\mathcal{A}$-Perron integral. The fact that a monotonicity theorem is available for the approximate symmetric derivative allows a Perron type approach to the integral. In this section we sketch out how such an integral could be defined (a number of variants are possible) and show that the resulting integral is included in the $\mathcal{A}$-integral. We will define the $\mathcal{A}$-Perron integral for an arbitrary measurable interval function h in such a way that applied to such a function of the form $(I, x) \longrightarrow f(x)|I|$ it will produce a direct generalization of the classical Perron integral as defined in Saks [**17**].

There are two natural approaches to a Perron-type integral in this setting. For a measurable function f we can say that an additive, measurable interval function H is a majorant of f provided that the lower approximate symmetric derivative of H exceeds f everywhere; alternatively we might require formally less of a majorant by asking that the lower approximate symmetric derivative of H exceeds f almost everywhere but that it is greater than $-\infty$ everywhere. Minorants G are then defined so that $-G$ is a majorant of $-f$ and the Perron integral is defined by taking extremes over all majorant of $-f$ and the Perron integral is defined by taking extremes over all majorants and minorants. Of course here the extremes are interval functions defined on intervals $[a, b]$ for all a, b in a set of full measure.

Such an approximate-symmetric Perron type integral would be justified by either of the following two monotonicity theorems.

LEMMA 10.1. [Freiling-Rinne] *An additive, measurable interval function H that has everywhere a nonnegative lower approximate symmetric derivative is nonnegative.*

Proof. Under these assumptions the collection

$$\beta_n = \{([x-t, x+t], x); H(x-t, x+t) > -t/n\}$$

is an element of $\mathcal{A}$ for every integer n. If β_n contains a partition of $[a, b]$ then evidently $H(a, b) > -(b-a)/(2n)$. Thus, using 1.6, choose a set of full measure B so that every β_n contains a partition of every interval with endpoints in B. Then $F(a, b) \geqq 0$ for all such intervals which is what it means for $F \geqq 0$.

LEMMA 10.2. *An additive, measurable interval function H that has almost everywhere a nonnegative lower approximate symmetric derivative and everywhere a lower approximate symmetric derivative greater than $-\infty$ must be nonnegative.*

Proof. The proof here is similar to the proof of 10.1 but requires handling this set of measure zero. Let f be the lower approximate symmetric derivative of F and define the sets $E_0 = \{x; f(x) \geqq 0\}, E_1 = \{x; 0 > f(x) \geqq -1\}$, $E_2 = \{x; -1 > f(x) \geqq -2\}$ and so on. These are measurable sets since, by hypothesis, for $n \geqq 1$ each E_n has measure zero.

Let $\epsilon > 0$ and choose open sets G_n containing E_n for $n = 1, 2, \ldots$ so that $|G_n| < \epsilon/n2^n$. Let β_0 be the collection $\{(I, x)); F(I) > -\epsilon|I|\}$ and, for each $n = 1, 2, \ldots$ let β_n be the collection $\{(I, x)); F(I) > -n|I|\}$. The collections $\beta_n(G_n)$ are in $\mathcal{A}[E_n]$ (using for convenience $G_0 = \mathbf{R}$). Thus

$$\beta = \bigcup_{n=0}^{\infty} \beta_n(G_n)$$

belongs to $\mathcal{A}$. If β contains a partition of an interval $[a, b]$ then evidently

$$F(a, b) \geqq -\epsilon(b-a) - \sum_{1}^{\infty} n\epsilon/n2^n = -2\epsilon(b-a).$$

The proof may now be completed as in the proof of 10.1.

We develop the integral in some greater generality since there are few extra complications that arise. Let f be a measurable real function defined almost everywhere and let h be a positive measurable interval function. We shall assume always that h^* is σ-finite, i.e., that there is a sequence of measurable sets covering the real line on each of which h^* is finite.

An additive, measurable interval function H is said to be an $\mathcal{A}$-major function for f, h if

$$\operatorname{ap-}\liminf_{t \searrow 0} \frac{H(x-t, x+t)}{h(x-t, x+t)} \geqq f(x)$$

for almost every point x and

$$\operatorname{ap-}\liminf_{t \searrow 0} \frac{H(x-t, x+t)}{h(x-t, x+t)} > -\infty$$

everywhere. Similarly an additive, measurable interval function G is said to be an $\mathcal{A}$-minor function for f, h if

$$\text{ap-}\limsup_{t \searrow 0} \frac{G(x-t, x+t)}{h(x-t, x+t)} \leqq f(x)$$

for almost every point x and

$$\text{ap-}\limsup_{t \searrow 0} \frac{G(x-t, x+t)}{h(x-t, x+t)} < +\infty$$

everywhere.

For such functions f, h we define the functions

$$\overline{fdh} = \inf\{H;\ \text{H a major function for } f, h\}$$

and

$$\underline{fdh} = \sup\{H;\ \text{H a minor function for } f, h\}$$

We claim that these functions are well defined interval functions in our sense. The extrema are taken in the complete lattice of measurable interval functions and may hence be realized as the limit of a sequence. (See, for example, [**12**, Example 23.3(iv), p. 126].) This allows the limit functions to be defined on all intervals $[a, b]$ with endpoints in a set of full measure. Lemma 10.3 below will show that invariably $\underline{fdh} \leqq \overline{fdh}$.

We say that f, h is $\mathcal{A}$-Perron integrable if

$$-\infty < \underline{fdh} = \overline{fdh} < +\infty$$

i.e., if the functions agree and are finite valued almost everywhere. In this case then either is taken as the indefinite integral in the $\mathcal{A}$-Perron sense. If f, h is $\mathcal{A}$-Perron integrable then the integral $\int_a^b fdh = \overline{fdh}(a, b)$ is defined for all a, b in a set of full measure.

To justify the definition of the $\mathcal{A}$-Perron integral in general we require the following "monotonicity" lemma. If F is an $\mathcal{A}$-major function for f, h and G is an $\mathcal{A}$-minor function for f, h then, because of Lemma 10.3 below and the approximate continuity of F and G on B we must have $0 \leqq F(a, b) - G(a, b)$ for every interval with endpoints in a set of full measure.

LEMMA 10.3. *Let f be a measurable function and let h be a measurable positive interval function for which H^* is σ-finite. If F is an $\mathcal{A}$-major function for f, h and G is an $\mathcal{A}$-minor function for h, h then $F \leqq G$.*

Proof. Let $\epsilon > 0$ and suppose that $\{E_n\}$ is a disjointed sequence of measurable sets covering the line with each $h^*(E_n)$ finite. Choose a sequence of positive

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numbers $\{c_n\}$ so that $\sum_{n=1}^{\infty} c_n h^*(E_n) < \epsilon$. For each n choose an element β_n of $\mathcal{A}[E_n]$ so that

$$\mathrm{Var}(h, \beta_n) < h^*(E_n) + \epsilon/(2^n c_n)$$

and so that both inequalities

$$F(I) \geqq (f(x) - c_n)h(I)$$

and

$$G(I) \leqq (f(x) + c_n)h(I)$$

hold for all pairs $(I, x) \in \beta_n$. Note that this means that

$$F(I) - G(I) \geqq -2c_n h(I)$$

for such pairs. Then, by Lemma 1.3, the collection

$$\beta = \bigcup_{n=1}^{\infty} \beta_n[E_n]$$

is an element of $\mathcal{A}$. For any interval $[a, b]$ for which β contains a partition we compute

$$F(a, b) - G(a, b) \geqq -\sum_{n=1}^{\infty} 2c_n h^*(E_n) + \epsilon/2^n \geqq -3\epsilon.$$

As ϵ is arbitrary we may pass as before to a set of full measure with the property that $0 \leqq F(a, b) - G(a, b)$ for every interval with endpoints in that set.

THEOREM 10.4. *Let f be a measurable function and let h be a measurable positive interval function for which h^* is σ-finite. If f, h is $\mathcal{A}$-Perron integrable then fh is $\mathcal{A}$-integrable and the integrals agree.*

Proof. Let $\epsilon > 0$ and let $H(a, b) = \int_a^b f dh$ for this integral. Let F and G be major and minor functions for f, h with $0 \leqq F(a, b) - G(a, b) < \epsilon$ for a fixed pair a, b.

We give the proof for h^* finite but if h^* is only σ-finite a proof using the ideas of the proof of Lemma 10.2 may be fashioned. The collections

$$\beta_1 = \{(I, x); F(I) \geqq (f(x) - \epsilon)h(I)\}$$

and

$$\beta_2 = \{(I, x); G(I) \leqq (f(x) + \epsilon)h(I)\}$$

are elements of $\mathcal{A}$. Choose $\beta \subset \beta_1 \cap \beta_2$ in $\mathcal{A}$ so that $\mathrm{Var}(h, \beta) < h^*(\mathbf{R}) + 1$. For pairs $(I, x) \in \beta$

$$|H(I) - f(x)h(I)| \leqq F(I) - G(I) + \epsilon h(I)$$

so that

$$\mathrm{Var}(H - fh, \beta([a, b])) \leqq \epsilon(2 + h^*(\mathbf{R})).$$

From this it follows that $V_{[a,b]}(H - fh, \mathcal{A}[\mathbf{R}]) = 0$ and then finally that $V(H - fh, \mathcal{A}[\mathbf{R}]) = 0$. Since $H \equiv fh$ the result now follows from 7.3.

11. The periodic integral. Let f be a 2π-periodic function. We may directly define a Riemann-type integral as a limit of Riemann sums over a partition of any period; here the limit is taken in an "approximate symmetric" sense but otherwise the integral is very much a familiar looking Riemann integral that generalizes the classical integral.

Recall that a measurable approximate symmetric interval-point relation β (definition 1.1) is a collection of pairs $([x - t, x + t], x)$ satisfying the requirement that there is a measurable set $T \subset \mathbf{R} \times (0, \infty)$ such that $([x - t, x + t], x) \in \beta$ whenever $(x, t) \in T$, and for every $x \in \mathbf{R}$

$$\limsup_{h \searrow 0} |\{t \in (0, h); (x, t) \notin T\}|/h = 0.$$

A finite sequence

$$x_0 < x_1 < \ldots < x_{n-1} < x_n$$

such that $x_0 = a$ and $x_n = b$ is a partition of $[a, b]$ subordinated to β if

$$([x_{i-1}, x_i], (x_i + x_{i-1})/2) \in \beta$$

for every $i = 1, 2, \ldots, n$. The main result of Section 2 asserts that for every such β there is a partition of $[a, b]$ subordinated to β for all a, b in a set of full measure. This observation allows us to define a simple Riemann-type integral for 2π-periodic functions.

Definition 11.1. Let f be a 2π-periodic function. We say that f has a *periodic integral* if there is a number c such that for every $\epsilon > 0$ there is a measurable approximate symmetric covering relation β such that, for any partition

$$x_0 < x_1 < \ldots < x_{n-1} < x_n = x_0 + 2\pi$$

that is subordinated to β,

$$\left| \sum_{i=1}^{n} f((x_{i-1} + x_i)/2)(x_i - x_{i-1}) - c \right| < \epsilon.$$

Because such partitions exist this number c, if it exists, is unique and we may write it as $\int_a^b f(x)dx$. It is immediately clear that a 2π-periodic function that is Riemann integrable is integrable in this sense and to the same value. This justifies using this simple notation for the concept; as we shall see this integral generalizes most familiar integration procedures and is in fact precisely the integral defined in Section 8. Directly from the definition it is possible to see that the integral has some considerable generality in that it integrates the exact approximate symmetric derivative of measurable functions; in particular then it integrates exact ordinary derivatives. Indeed if f is a 2π-periodic measurable functions with f everywhere the approximate symmetric derivative of a measurable function F then the collection

$$\beta = \left\{ ([x-h, x+h], x); \left| \frac{F(x+h) - F(x-h)}{2h} - f(x) \right| < \epsilon \right\}$$

is, for every $\epsilon > 0$, a measurable approximate symmetric covering relation. For this any sum

$$\sum_{i=1}^{n} f((x_{i-1} + x_i)/2)(x_i - x_{i-1})$$

taken over a partition of length 2π subordinated to β cannot differ from $F(2\pi + t) - F(0)$ by more than $2\pi\epsilon$.

THEOREM 11.2. *Let f be a 2π-periodic function that has a periodic integral. Then f is $\mathcal{A}$-integrable and*

$$\int_0^{2\pi} f(x)dx = (\mathcal{A}) \int_c^{c+2\pi} f(x)dx$$

for all c in a set of full measure.

Proof. By definition, since f has a periodic integral, for every $\epsilon > 0$ there is an element $\beta \in \mathcal{A}$ so that

$$\left| \int_0^{2\pi} f(x)dx - \sum_{\gamma} f(x)|I| \right| < \epsilon \tag{9}$$

for every partition γ of an interval of length 2π from β. Let B be a set of full measure such that β contains a partition of every interval with endpoints in B; we may assume that β contains only pairs $([a, b], (a + b)/2)$ with a and b in B.

If γ_1 and γ_2 are partitions of the same interval and both are contained in $\beta([a, a + 2\pi])$ where both a and $a + 2\pi$ are in B then there is a packing γ_3 so that $\alpha_1 = \gamma_1 \cup \gamma_3$ and $\alpha_2 = \gamma_2 \cup \gamma_3$ are partitions of $[a, a + 2\pi]$. Consequently, by (9),

$$\left| \sum_{\alpha_1} f(x)|I| - \sum_{\alpha_2} f(x)|I| \right| \leqq 2\epsilon$$

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which gives

$$\left|\sum_{\gamma_1} f(x)|I| - \sum_{\gamma_2} f(x)|I|\right| \leqq 2\epsilon.$$

If we now argue as in the proof of Lemma 7.2 but just on the interval $J = [a, a+2\pi]$ we may obtain an additive interval function F defined for almost all pairs in that interval and with $V_J(f\ell - F, \mathcal{A}[\mathbf{R}]) = 0$. We can extend F by periodicity and additivity to an additive interval function defined for all pairs in a set of full measure on the real line. For this $V_J(f\ell - F, \mathcal{A}[\mathbf{R}]) = 0$ for every interval J of length 2π so that, by Corollary 4.8, $V(f\ell - F, \mathcal{A}[\mathbf{R}]) = 0$ and consequently f is $\mathcal{A}$-integrable. Certainly $\int_0^{2\pi} f(x)dx = F(a+2\pi) - F(a)$ for almost all a.

For an arbitrary function f one can define a "periodic integral" over an arbitrary interval $[a, b]$ by extending f from $[a, b]$ to a function g so as that $f = g$ there and g is $(b-a)$-periodic. We could write $\int_a^b f(x)dx$ for $\int_a^b g(x)dx$ if g has a periodic integral in this sense. Such a scheme is not entirely satisfying but serves to define an integral defined on all intervals which at first sight might seem to recommend it. Note the following disadvantage of such an "integral". The function F' where $F(x) = |x|^{-1}$ would be integrable by such a method on the interval $[-1, 1]$ and not so on the intervals $[-1, 0]$ and $[0, 1]$. This kind of non-additivity is considered a defect in any proposed integration theory.

This procedure may be shown to be equivalent to the following limit process which can be considered a kind of approximate-symmetric principal value computation: if f is $\mathcal{A}$-integrable then the above periodic integral of f may be written as

$$\int_a^b f(x)dx = \text{ap}-\lim_{h\searrow 0}(\mathcal{A}) - \int_{a+h}^{b-h} f(x)dx.$$

12. Coefficient problem for trigonometric series. Our main application of the integration theory developed in this article is to show that, with this definition of the integral, an everywhere convergent trigonometric series is the Fourier series of its sum. Note, however, that not every $\mathcal{A}$-integrable function has a Fourier series in this sense. For example the function $F(x) = |x|^{-1}$ is an indefinite $\mathcal{A}$-integral but $F'(x)$ sin nx is not integrable for any n.

THEOREM 12.1. *Let the trigonometric series*

$$a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

converge everywhere to a finite value $f(x)$. Then f is $\mathcal{A}$-integrable, and the series is the $\mathcal{A}$-Fourier series for f, i.e., for each n

$$\pi a_n = (\mathcal{A}) \int_p^{p+2\pi} f(t) \cos t\, dt$$

and

$$\pi b_n = (\mathcal{A}) \int_p^{p+2\pi} f(t) \sin t \, dt$$

for all p in a set of full measure.

Proof. The formally integrated series

$$F(x) = xa_0/2 + \sum_{k=0}^{\infty} (b_k \cos kx - a_k \sin kx)/k$$

converges on a set B of full measure and has everywhere $f(x)$ for its $\mathcal{A}$-derivative (see [**24**, Theorem 2.22, p. 324]). Consequently, by Lemma 8.9, f is $\mathcal{A}$-integrable and ΔF is an indefinite integral. Moreover, if we integrate through a period we evidently obtain the formula for a_0.

In much the same way we obtain a formula for the remaining coefficients; we multiply the series

$$a_0/2 + \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$$

by cos kx, use elementary trigonometric identities for cos nx cos kx and for sin nx cos kx, and rearrange the series to obtain a trigonometric series that converges everywhere to $f(x)$ cos kx. For example the term corresponding to a_n cos kx cos nx is replaced by

$$a_n(\cos(n+k)x + \cos(n-k)x)/2.$$

As before, its formalloy integrated series converges on a set B of full measure and has everywhere $f(x)$ cos kx for its $\mathcal{A}$-derivative. Consequently $f(x)$ cos kx is $\mathcal{A}$-integrable and we may integrate it over a period $[p, p+2\pi]$ for an appropriate choice of p. Since the constant term in the series for $f(x)$ cos kx is $a_k/2$ this gives the required formula for a_k. A similar argument supplies the formula for b_k.

13. Relations to other integrals. In this section we wish to point out the relations that hold among the various symmetric integrals that have been defined. Recall that on the one hand we have the (SCP)-integral of Burkill, the $T(P)$-integral of Marcinkiewicz-Zygmund, and the P^2-integral of James all related to the second order symmetric derivative, while our $\mathcal{A}$-integral arises from the approximate symmetric derivative.

Let us say at the outset that while these other symmetric integrals are based on quite similar underlying principles the $\mathcal{A}$-integral, because it is based on the approximate symmetric derivative, belongs to a different family of integrals.

Accordingly one need not expect any close relationship even though all of these integrals are sufficient to integrate everywhere convergent trigonometric series.

Our first observation is that the $\mathcal{A}$-integral is neither contained in nor does it contain the two first order integrals of Burkill and Marcinkiewicz-Zygmund. This arises merely because of the continuity requirements of these integrals and elementary examples suffice to illustrate. Let $F(x) = x^{-2}$ and $f(x) = -2x^{-3}$ its derivative. Because of the symmetry of the function F at the origin and that fact that $F' = f$ everywhere else it is clear that f is $\mathcal{A}$-integrable and that $F(y) - F(x) = \int_x^y f(t)dt$ for all x, y in a set of full measure. But this same relation cannot hold in either of the senses of Burkill or Marcinkiewicz-Zygmund simply because in such a case F would have to be integrable, Denjoy integrable for the Burkill (SCP)-integral and Lebesgue-integrable for the Marcinkiewicz-Zygmund integral. Also the function $G(x) = -x^{-1}$ would represent a (not normalized) second order indefinite P^2-integral of f on any interval excluding the origin; but no adjustment is possible to obtain a continuous such function for intervals that contain the origin and indeed there are no major or minor functions for f in the sense of the P^2-integral. In summary $\mathcal{A}$-integrability does not imply integrability in the sense of any other symmetric integral.

This behaviour at a single point is also enough to cause this problem in the other direction. We may define a function F that is everywhere differentiable except at the origin and such that

$$\lim_{h \searrow 0} 1/h \left\{ \int_0^h F(t)dt - \int_{-h}^0 F(t)dt \right\} = 0$$

and yet F is not approximately symmetrically continuous at the origin. In this case F' is (SCP)-integrable and hence also P^2-integrable but certainly not $\mathcal{A}$-integrable.

For example let each interval $[1/(n+1), 1/n]$ be split into the three adjacent intervals I_{1n}, I_{2n} and I_{3n} (in that order) by choosing a small centered interval I_{2n} of length 2^{-n} times the length of $[1/(n+1), 1/n]$. Define $F(x) = (-1)^n$ on each I_{1n}, $F(x) = (-1)^{n+1}$ on each I_{3n}, and choose F on I_{2n} so that $|F(x)| \geqq 1$, F is continuously differentiable on $[1/(n+1), 1/n]$ and

$$\int_{1/(n+1)}^{1/n} F(t)dt = 0.$$

This defines F on $(0, 1)$; extend F by writing $F(-x) = -F(x)$ and $F(0) = 0$. We evidently have

$$\lim_{h \searrow 0} 1/h \int_0^h F(t)dt = \lim_{h \searrow 0} 1/h \int_{0h}^0 F(t)dt = 0$$

and F' exists except at 0. But the set $\{h > 0; |F(h) - F(-h)| \geqq 1\}$ has density 1 at 0 and so F is far from symmetrically approximately continuous here. Thus,

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while F serves as an indefinite (SCP)-integral of its derivative, F' cannot be $\mathcal{A}$-integrable.

While these trivial differences separate the integrals there is still a compatibility problem: if a function f is integrable in two different symmetric senses then do the integrals computed by the two different procedures yield the same values? The answer is, perhaps surprisingly, that they do not. We may produce a function f integrable in both the $\mathcal{A}$-sense and in the (SCP)-sense but with different indefinite integrals in the two senses. Again this is not deep but is just a reflection of the fact that the continuity requirements of the two integrals differ.

To construct such an example we again let I_{1n}, I_{2n} and I_{3n} denote the intervals defined above; let F vanish on each interval I_{1n} and I_{3n} and be defined on I_{2n} so as to be nonnegative and continuously differentiable on $[1/)n+1), 1/n]$ with

$$\int_{1/(n+1)}^{1/n} F(t)dt = 1/n - 1/(n+1).$$

We set F_1 to vanish on $[-1, 0]$ and to equal F on $(0, 1]$ and we define F_2 to have the constant value 1 on $[-1, 0]$ and to equal F on $(0, 1]$. Note that F_1 has been choosen so as to be approximately continuous at 0 while F_2 has been chosen so as to be (SC)-continuous in Burkill's sense. Let $f = F_1' = F_2'$. Then F_1 is an indefinite $\mathcal{A}$-integral for f and F_2 is an indefinite (SCP)-integral for f which exhibits the essential incompatibility of the integrals. This same example illustrates compatibility problems with the P^2-integral and the Marcinkiewicz-Zygmund integral.

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