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A MULTIPLE CHARACTER SUM EVALUATION

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We evaluate in a simple and direct manner a multiple character sum, a special case of which can also be derived from the Möbius inversion and a result of Hanlon.

1. INTRODUCTION

Let Γ be a finite Abelian group with operation written multiplicatively, and let $\chi : \Gamma \longrightarrow \mathbb{C}^\times$ be a character of order m. Then we are interested in evaluating the following multiple character sum

$$S_{n,m} = \sum_{\substack{(\gamma_1,\dots,\gamma_n)\in\Gamma^n \\ \gamma_i\neq\gamma_j}} \chi(\gamma_1\cdots\gamma_n)\ , \tag{1}$$

where the sum is over all $(\gamma_1,\dots,\gamma_n)\in\Gamma^n$ satisfying $\gamma_i\neq\gamma_j$ for all i,j $(1\leqslant i,j\leqslant n)$ with $i\neq j$.

A special case of the sum (1) with $n=m$ was introduced by Professor Fernando Rodriguez Villegas in the number theory seminar on February 5, 2004 of University of Texas at Austin. I would like to thank him for drawing my attention to this problem. He evaluated the sum (1) for $n=m$ by using Möbius inversion and a result of Hanlon in the early 1980's (see [**2**, Theorem 4, p. 338]). We shall briefly go over his method for the special case of $n=m$.

A partition β of $[n]=\{1,2,\dots,n\}$ is a collection $\beta=B_1\,|B_2|\cdots|\,B_k$ of nonempty, disjoint subsets of $[n]$ whose union is $[n]$. The set of all partitions of $[n]$ is denoted by Π_n. Π_n is partially ordered by the relation:

$$\beta\leqslant\beta' \Longleftrightarrow \beta \text{ is a refinement of } \beta'.$$

Obviously, $(\Pi_n,\leqslant)$ has the unique maximal element $\beta_1=12\cdots n$, and the unique minimal element $\beta_0=1\,|2|\cdots|\,n$. For $\beta=B_1\,|B_2|\cdots|\,B_k\in\Pi_n$, let

$$\sum\nolimits_\beta = \bigl\{\sigma \mid \sigma:[n]\longrightarrow\Gamma, \sigma|_{B_i}=\text{constant}\bigr\},$$

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$$\sum\nolimits_{\beta}' = \sum\nolimits_{\beta} \setminus \bigcup_{\beta<\beta'} \sum\nolimits_{\beta'},$$

$$f(\beta) = \sum_{\sigma\in\sum'_\beta} \chi\big(\sigma(1)\cdots\sigma(n)\big).$$

Then

(2) $$S_{n,n} = f(\beta_0).$$

Put

(3) $$g(\beta) = \sum_{\beta\leqslant\beta'} f(\beta') = \sum_{\sigma\in\sum_\beta} \chi\big(\sigma(1)\cdots\sigma(n)\big).$$

Then we claim that

(4) $$g(\beta) = \begin{cases} |\Gamma|, & \text{for } \beta = \beta_1 \\ 0, & \text{otherwise.} \end{cases}$$

For $\beta = B_1\,|B_2|\cdots\mid B_k$ and $\sigma \in \sum_{\beta}$, let $n_i = |B_i|$, $\sigma|_{B_i} = \sigma_i$, for $i = 1,\ldots,k$. As χ has order n, $\sigma \mapsto \chi\big(\sigma(1)\cdots\sigma(n)\big) = \prod_{i=1}^{k} \chi^{n_i}(\sigma_i) : \sum_\beta \longrightarrow \mathbb{C}^\times$ is trivial $\Leftrightarrow n = n_i$, for all $i \Leftrightarrow k = 1 \Leftrightarrow \beta = \beta_1$. This shows the claim in (4). Now, by (2), (3), (4), and Möbius inversion,

(5) $$S_{n,n} = f(\beta_0) = \sum_{\beta\in\Pi_n} \mu(\beta)g(\beta) = \mu(\beta_1)\,|\Gamma|,$$

where μ is the Möbius function of the poset $(\Pi_n, \leqslant)$. The following is a special case of a result of Hanlon (see [**2**, Theorem 4, p. 338]) which had been used repeatedly in subsequent papers (see [**1**, Theorem 4.3, p. 293], [**3**, Theorem 2.4, p. 447], [**4**, Theorem 2.1.12, p. 7]).

THEOREM 1. (Hanlon) $\mu(\beta_1) = (-1)^{n-1}(n-1)!$.

From (5) and Theorem 1, we get the following corollary.

COROLLARY 2. $S_{n,n} = (-1)^{n-1}(n-1)!\,|\Gamma|$.

In the present paper, we show the following more general theorem in a direct and simple manner.

THEOREM 3. *Let Γ be a finite Abelian group with operation written multiplicatively, and let $\chi : \Gamma \longrightarrow \mathbb{C}^\times$ be a character of order m. Then the multiple character sum in* (1)

$$S_{n,m} = \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\Gamma^n \\ \gamma_i\neq\gamma_j}} \chi(\gamma_1\cdots\gamma_n),$$

summing over all $(\gamma_1,\ldots,\gamma_n)\in\Gamma^n$ *satisfying* $\gamma_i\neq\gamma_j$ *for all* $i,j(1\leqslant i,j\leqslant n)$ *with* $i\neq j$, *is given by*

$$(6)\qquad S_{n,m}=\begin{cases}\dfrac{(-1)^{(n/m)(m-1)}(n-1)!\prod_{j=1}^{n/m}(|\Gamma|-(j-1)m)}{m^{(n/m)-1}((n/m)-1)!}, & \text{if } m\mid n\\ 0, & \text{otherwise.}\end{cases}$$

2. Proof of the theorem

The following lemma is elementary but will be useful.

Lemma 4. *For an integer* $n>1$, *and* $\gamma\in\Gamma$, *let*

$$S'_{n,m}(\gamma)=\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\Gamma^n\\ \gamma_i\neq\gamma_j,\gamma_i\neq\gamma}}\chi(\gamma_1\cdots\gamma_n).$$

Here the sum is over all $(\gamma_1,\ldots,\gamma_n)\in\Gamma^n$ *satisfying* $\gamma_i\neq\gamma_j$ *for all* $i,j(1\leqslant i,j\leqslant n)$ *with* $i\neq j$ *and all* $\gamma_i\neq\gamma$. *Then*

(a) $S'_{n,m}(\gamma)\ =\ S_{n,m}-n\chi(\gamma)S'_{n-1,m}(\gamma)$,

(b) $S_{n,m}=\sum_{\gamma\in\Gamma}\chi(\gamma)S'_{n-1,m}(\gamma)$.

As the result (6) in Theorem 3 for $m=1$ is trivial, we may assume that $m>1$. Using (a), (b) of Lemma 4, the sum in (1) can be written as:

$$(7)\quad S_{n,m}=\sum_{\gamma\in\Gamma}\chi(\gamma)\{S_{n-1,m}-(n-1)\chi(\gamma)S'_{n-2,m}(\gamma)\}=-(n-1)\sum_{\gamma\in\Gamma}\chi^2(\gamma)S'_{n-2,m}(\gamma),$$

since χ has order $m>1$ and hence is nontrivial.

Case 1. $m>n$. Applying (7) $n-2$ times with (a) of Lemma 4 in mind, we have:

$$\begin{aligned}S_{n,m}&=(-1)^{n-2}(n-1)!\sum_{\gamma\in\Gamma}\chi^{n-1}(\gamma)S'_{1,m}(\gamma)\\&=(-1)^{n-1}(n-1)!\sum_{\gamma\in\Gamma}\chi^n(\gamma)\\&=0,\end{aligned}$$

as $S'_{1,m}(\gamma)=\sum_{\gamma'\neq\gamma}\chi(\gamma')=-\chi(\gamma)$ and χ^n is nontrivial.

Case 2. $m=n$.

$$S_{n,m}=(-1)^{n-1}(n-1)!\sum_{\gamma\in\Gamma}\chi^n(\gamma)=(-1)^{n-1}(n-1)!\,|\Gamma|,$$

as χ has order $n=m$. This agrees with Corollary 2.

CASE 3. $m < n$. Write $n = lm + r$, $0 \leqslant r < m$. Applying (7) $m-1$ times, we get:

$$\begin{aligned} S_{n,m} &= (-1)^{m-1}(n-1)\cdots(n-(m-1))\sum_{\gamma\in\Gamma}\chi^m(\gamma)S'_{n-m,m}(\gamma) \\ &= (-1)^{m-1}(n-1)\cdots(n-(m-1)) \times \sum_{\gamma\in\Gamma}\{S_{n-m,m} - (n-m)\chi(\gamma)S'_{n-m-1,m}(\gamma)\} \\ &= (-1)^{m-1}(n-1)\cdots(n-(m-1))(|\Gamma| - (n-m))S_{n-m,m}, \qquad (8) \end{aligned}$$

by using (a), (b) in Lemma 4.

CASE 3(a). $r = 0$ Applying (8) $l-1$ times, we obtain:

$$\begin{aligned} S_{n,m} &= \prod_{j=1}^{l-1}\Big\{(-1)^{m-1}(n-(j-1)m-1)\cdots(n-(j-1)m-(m-1)) \\ &\qquad\qquad \times(|\Gamma| - (l-j)m)\Big\}S_{m,m} \\ &= \prod_{j=1}^{l}\Big\{(-1)^{m-1}(n-(j-1)m-1)\cdots(n-(j-1)m-(m-1)) \\ &\qquad\qquad \times(|\Gamma| - (l-j)m)\Big\} \\ &= \frac{(-1)^{l(m-1)}(n-1)!\prod_{j=1}^{l}(|\Gamma| - (j-1)m)}{m^{l-1}(l-1)!}, \end{aligned}$$

in view of Corollary 2.

CASE 3(b). $r > 0$ Applying (8) l times, we have:

$$\begin{aligned} S_{n,m} &= \prod_{j=1}^{l}\Big\{(-1)^{m-1}(n-(j-1)m-1)\cdots(n-(j-1)m-(m-1)) \\ &\qquad\qquad \times(|\Gamma| - (l-j)mj)\Big\}S_{r,m} \\ &= 0 \end{aligned}$$

by Case 1 above. This completes the proof of Theorem 3.

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