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THE BISECTION WIDTH OF CUBIC GRAPHS

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For a graph G, define the bisection width $bw(G)$ of G as $\min\{ e_G(A,B) : \{A,B\}$ partitions $V(G)$ with $||A|-|B|| \leqslant 1 \}$ where $e_G(A,B)$ denotes the number of edges in G with one end in A and one end in B. We show almost every cubic graph G of order n has $bw(G) \geqslant n/11$ while every such graph has $bw(G) \leqslant (n+138)/3$. We also show that almost every r-regular graph G of order n has $bw(G) \geqslant c_r n$ where $c_r \to r/4$ as $r \to \infty$. Our last result is asymtotically correct.

1. INTRODUCTION

For a graph G, define the bisection width $bw(G)$ of G by

$$bw(G) = \min\{e_G(A,B) : \{A,B\} \text{ partitions } V(G) \text{ with } |\,|A|-|B|\,| \leqslant 1\}$$

where $e_G(A,B)$ denotes the number of edges in G with one end in A and one end in B.

The problem of finding the bisection width of a graph is of fundamental importance in many divide-and-conquer stratagems and, as such, is the subject of an extensive literature. (See [**4**, **9**, **10**, **13**, **15**, **18**] for general results and [**6**, **11**] for results regarding VLSI design.)

Unfortunately, the bisection problem for graphs, in general, is NP-complete [**12**] and remains so for r-regular graphs [**9**]. Polynomial-time algorithms which give exact solutions are known only for trees and bounded-width planar graphs [**9**] while polynomial-time algorithms which give approximate solutions may give solutions which are far from exact [**18**]. Consequently, heuristic algorithms which hopefully give nearly exact solutions most of the time have been developed in [**9**, **13**, **14**, **16**, **18**].

In [**9**] a method was given for transforming a regular graph G of order n into a cubic graph G^* of order $\Theta(n^6)$ so that any minimum bisection of G^* uses only edges of G. As a result, we content ourselves mainly with an examination of cubic graphs. As usual, we say that *almost every* graph has a property Q provided the probability that a graph of order n has property Q tends to 1 as $n \to \infty$.

We show that almost every cubic graph G of order n has $bw(G) \geqslant n/11$ while every such graph has $bw(G) \leqslant (n+138)/3$. We also show that almost every r-regular

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graph G of order n has $bw(G) \geqslant c_r n$ where $c_r \to r/4$ as $r \to \infty$. (Note that absolute lower bounds for the bisection width of a graph are not particularly informative, since they must be nearly zero.)

Our notation and terminology follows Bollobás [7].

2. An Upper Bound for the Bisection Width of a Cubic Graph

We give now an upper bound for the bisection width of a cubic graph.

Theorem 1. *Every cubic graph G of order n has $bw(G) \leqslant (n+138)/3$.*

Proof: Let $\{A, B\}$ be an equisized partition of $V(G)$ with $bw(G) = e_G(A, B)$. Set $A_i = \{v \in A : e_G(v, B) = i\}$ for $0 \leqslant i \leqslant 3$ and $A_{1i} = \{v \in A_1 : e_G(v, A - A_1) = i\}$ for $0 \leqslant i \leqslant 2$. (Define B_i and B_{1i} similarly.)

Suppose $x \in A_3$ and $y \in B_1 \cup B_2 \cup B_3$ with $xy \notin E(G)$; exchanging x with y shows $\{A, B\}$ is not an optimal partition, which is a contradiction. Consequently, $|B_1 \cup B_2 \cup B_3| \leqslant 3$ and $bw(G) \leqslant 9 \leqslant (n+138)/3$. We assume $|A_3| = |B_3| = 0$.

Suppose $|B_2| \geqslant 4$. When $|A_2| \neq 0$, there exists $x \in A_2$ and $y \in B_2$ with $xy \notin E(G)$; exchanging x with y shows $\{A, B\}$ is not an optimal partition. Consequently, $|A_2| = 0$. When $G[A_{10} \cup A_{11}]$ is empty, we have $|A_{10} \cup A_{11}| \leqslant 1$. Then

$$3|A_0| \geqslant e_G(A_0, A_1) \geqslant 2|A_1| - 2$$

so that

$$n/2 = |A_0| + |A_1| \geqslant (5|A_1| - 2)/3$$

and

$$bw(G) \leqslant (3n+4)/10 \leqslant (n+138)/3 .$$

When $G[A_{10} \cup A_{11}]$ is nonempty, there exist an edge x_1x_2 in $G[A_{10} \cup A_{11}]$ and $y_1, y_2 \in B_2$ with $e_G(\{x_1, x_2\}, \{y_1, y_2\}) = 0$; exchanging $\{x_1, x_2\}$ with $\{y_1, y_2\}$ shows $\{A, B\}$ is not an optimal partition. We assume $|A_2|, |B_2| \leqslant 3$.

Denote a path (cycle) of order n by $P_n(C_n)$. Let

$a = \max |\{E_1, \dots, E_t\}|$ where $\{E_1, \dots, E_t\}$ is a set of vertex-disjoint subgraphs of $G[A]$ and each

$E_i \cong P_3 \subseteq G[A_{10} \cup A_{11}]$ or

$\cong C_3 \subseteq G[A_0 \cup A_{11}]$ with precisely one vertex in A_0 or

$\cong C_4 \subseteq G[A_0 \cup A_{10} \cup A_{11}]$ with precisely one vertex in A_0 and precisely one vertex in A_{10} or

$\cong P_5 \subseteq G[A_0 \cup A_{10} \cup A_{11}]$ with only the centre vertex in A_0 and let $A_j^* = \bigcup\{V(E_i) \cap A_j : 1 \leqslant i \leqslant a\}$ for $0 \leqslant j \leqslant 1$. (Define b, $\{F_1, \dots, F_t\}$, B_j^* for $0 \leqslant j \leqslant 1$ similarly.)

Claim. $\min\{a, b\} \leqslant 5$.

Suppose $a, b \geqslant 6$. Choose $e_G(E_i, F_j) = 0$ with $||E_i| - |F_j||$ as large as possible, say $|E_i| \geqslant |F_j|$. When $|E_i| = |F_j|$; exchanging E_i with F_j shows $\{A, B\}$ is not an optimal partition. When $|E_i| = |F_j| + 1$; exchanging E_i' with F_j, where E_i' is the subgraph of E_i contained in $G[A_1]$, shows $\{A, B\}$ is not an optimal partition. When $|E_i| = |F_j| + 2$ then $|E_i| = 5$ and $|F_j| = 3$. Since $b \geqslant 6$, there exist $F_k \neq F_j$ with $e_G(E_i, F_k) = 0$. By the above, $|F_k| = 3$; exchanging E_i with $F_j \cup F_k'$, where F_k' is a subpath of order 2 contained in $G[B_1]$, shows $\{A, B\}$ is not an optimal partition. ∎

We assume $a \leqslant 5$ so that $|A_0^*| \leqslant 5$ and $|A_1^*| \leqslant 20$.

Claim. $|A_{10}| \leqslant 25$.

Note that $G[A_{10} \cup A_{11}]$ is a vertex-disjoint set of paths and cycles when $|A_{10} \cup A_{11}| \neq 0$, since $\delta(G[A_{10} \cup A_{11}]) = 1$ and $\Delta(G[A_{10} \cup A_{11}]) = 2$. Consequently, $|A_{10}| \leqslant 25$ since $a \leqslant 5$ (after breaking paths and cycles apart if necessary). ∎

Let $A_1' = \{w \in A_1 - A_1^* : vw \in E(G) \text{ for some } v \in A_1^*\}$. Clearly, $|A_1'| \leqslant 2 \cdot 5 = 10$. Set $|A_{12}| = c|A_1|$ where $c \in [0, 1]$.

Then

$$|A_{11}| + |A_{12}| \geqslant |A_1| - 25$$

so that

$$|A_{11}| \geqslant (1 - c)|A_1| - 25 .$$

Now

$$3|A_0| \geqslant e_G(A_0, A_1) \geqslant |A_{11}| + 2|A_{12}| - 3$$

so that

$$|A_0| \geqslant [(1 + c)|A_1| - 28]/3 .$$

Then

$$n/2 \geqslant |A_0| + |A_1| \geqslant [(4 + c)|A_1| - 28]/3$$

so that

$$|A_1| \leqslant (3n + 56)/2(4 + c)$$

and

$$bw(G) \leqslant 6 + |A_1| \leqslant 6 + (3n + 56)/2(4 + c) .$$

Also

$$\begin{aligned} |A_{11}| - |A_1^*| - |A_1'| - 5 &\leqslant |A_{11} - (A_1^* \cup A_1')| - 5 \\ &\leqslant |A_0 - A_0^*| = |A_0| - |A_0^*| , \end{aligned}$$

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by the maximality of a, so that

$$|A_0| \geqslant |A_{11}| - 35 \geqslant (1-c)|A_1| - 60 .$$

Then

$$n/2 \geqslant |A_0| + |A_1| \geqslant (2-c)|A_1| - 60$$

so that

$$|A_1| \leqslant (n+120)/2(2-c)$$

and

$$bw(G) \leqslant 6 + |A_1| \leqslant 6 + (n+120)/2(2-c) .$$

Consequently,

$$\begin{aligned} bw(G) &\leqslant \min\{6 + (3n+56)/2(4+c), 6 + (n+120)/2(2-c)\} \\ &\leqslant (n+138)/3 , \end{aligned}$$

since the above minimum is at most $(n+138)/3$ for $n \geqslant 184$ and at most $6 + (3n+56)/8 \leqslant (n+138)/3$ for $n \leqslant 182$. ∎

Remark. In general, if $\{A, B\}$ is a partition of the vertices of an r-regular graph G of order n with $bw(G) = e_G(A,B)$, one would hope that either $G[A]$ or $G[B]$ contains a small number of forbidden subgraphs (see definition of a, b in Theorem 1) which, in turn, impose structure on $G[A]$ or $G[B]$ and give $bw(G) \leqslant c_r n + O(1)$ for some $c_r < r/4$. At present we have only the result of Goldberg and Gardner [**13**] that, for any such graph G, $bw(G) \leqslant r(n+\varepsilon_n)/4$ where $\varepsilon_n = 1$ for odd n and $\varepsilon_n = n/(n-1)$ for even n. There are, however, limitations on how small the ratio $bw(G)/n$ can be made for r-regular graphs G of order n.

An r-regular graph G of order n is an (n,r,c)-*expander* if $|N(X) - X| \geqslant c|X|$ for all $X \subseteq V(G)$ with $|X| \leqslant n/2$. (These and similar graphs have an extensive literature; see the references in [**1**].) Clearly, any (n,r,c)-expander G has $bw(G) \geqslant c\lfloor n/2 \rfloor$.

Let $\lambda_1(G)$ denote the second largest eigenvalue of the adjacency matrix of G in absolute value. Note that $0 < \lambda_1(G) < r$ when G is connected. Alon and Milman [**3**] have shown that any r-regular graph G of order n is an $(n, r, (r - \lambda_1(G))/2r)$-expander while Alon and Boppana [**2**] (see also [**17**]) have shown that $\varliminf_{n\to\infty} \lambda_1(G_n) \geqslant 2\sqrt{r-1}$ for any sequence $\{G_n\}$ of such graphs. Lubotzky, Phillips and Sarnak [**17**] have shown this last result asymptotically correct by constructing infinite families of r-regular graphs G with $\lambda_1(G) \leqslant 2\sqrt{r-1}$ for all primes $r - 1 \equiv 1 (\text{mod } 4)$.

The above results imply that any r-regular graph G of large order n has $bw(G) \geqslant cn$ where c, unfortunately, is rather small. We improve this by showing that almost every r-regular graph G of order n has $bw(G) \geqslant c_r n$ where $c_r \to r/4$ as $r \to \infty$.

3. A Lower Bound for the Bisection Width of Almost Every Cubic Graph

Bender and Canfield [5] gave the first formula for the asymptotic number of labelled r-regular graphs of order n. Bollobás [8] gave a simpler proof of the same formula that, more importantly, contained a model for the set of regular graphs which can be used to study labelled random regular graphs. We describe now this model.

Let rn be even and $q = rn/2$. Let $V = V_1 \cup \cdots \cup V_n$ be a disjoint union of rn labelled vertices where $|V_i| = r$ for $1 \leqslant i \leqslant n$. A *configuration* is a 1-regular graph with vertex set V. Denote the set of configurations by $\Phi = \Phi(n,r)$. Clearly,

$$|\Phi| = (rn)!/2^q q!\ .$$

A configuration is *good* if when we shrink each set V_i to a vertex v_i we obtain a simple graph. Denote the set of good configurations by $\Omega = \Omega(n,r)$ and the set of simple r-regular graphs with vertex set $\{v_1, \ldots, v_n\}$ by $\mathcal{G}_n^{(r)}$. Clearly,

$$|\Omega| = (r!)^n |\mathcal{G}_n^{(r)}|\ .$$

Now regard Φ as a probability space where $P(F) = |\Phi|^{-1}$ for any configuration F. Bollobás [8] showed that

$$P(\text{configuration } F \text{ is good}) \to e^{(1-r^2)/4} \qquad (n \to \infty)$$

and, hence,

$$|\mathcal{G}_n^{(r)}| \sim e^{(1-r^2)/4} |\Phi| \Big/ (r!)^n \qquad (n \to \infty)\ .$$

Finally regard $\mathcal{G}_n^{(r)}$ as a probability space where $P(G) = |\mathcal{G}_n^{(r)}|^{-1}$ for any r-regular graph G with vertex set $\{v_1, \ldots, v_n\}$. An immediate consequence of the preceding is that if the probability that a configuration has a certain property tends to 1 as $n \to \infty$ then the probability that an r-regular graph has the corresponding property also tends to 1 as $n \to \infty$.

For $r \geqslant 3$, let $c = c_r$ be the unique real number in $(0, r/4)$ with $2^{(2-r)} r^r = (2c)^{2c}(r-2c)^{(r-2c)}$. (The constant exists since $x^x(r-x)^{r-x}$ monotonically decreases on $[0, r/2]$.) Note that $c_3 = .0922357\cdots \in (1/11, 1/10)$. We denote $t(t-1)\cdots(t-k+1)$ by $(t)_k$.

We give now

Theorem 2. *Almost every cubic graph G of order n has $bw(G) \geqslant n/11$.*

Proof: Let $n = 2m$. Fix a partition $\{A, B\}$ of $\{1, \ldots, n\}$ with $|A| = |B| = m$. Let $V_A = \bigcup\{V_i : i \in A\}$ (Define V_B similarly). Note that the event $e_F(V_A, V_B) = j$

is a nonempty subset of Φ if and only if $3m$ and j have the same parity. Put $p_j = (3m-j)/2$. Then

$$
\begin{aligned}
P(e_F(V_A,V_B)=j) &= \frac{(3m)_j^2}{j!}\left[\frac{(3m-j)!}{2^{p_j}p_j!}\right]^2|\Phi|^{-1} \\
&= \frac{[(3m)!]^3 2^j}{j![p_j!]^2(rn)!}, \qquad (1)
\end{aligned}
$$

where the left factor of (1) is the number of ways of labelling the ends of the j edges between V_A and V_B and the middle factor of (1) is the number of ways of completing the 1-factor in both V_A and V_B.

For $j \geqslant 2$, we have

$$P(e_F(V_A,V_B)=j-2) = \frac{j(j-1)}{(3m-j+2)^2}P(e_F(V_A,V_B)=j)$$

where $j(j-1)/(3m-j+2)^2$ increases with j. For even $3m$ and $2k \leqslant \lfloor c_3 n \rfloor$, we have

$$
\begin{aligned}
P(e_F(V_A,V_B)\leqslant 2k) &= \sum_{\text{even } j\leqslant 2k} P(e_F(V_A,V_B)=j) \\
&\leqslant P(e_F(V_A,V_B)=2k)\left(1+\alpha+\cdots+\alpha^k\right),
\end{aligned}
$$

where $\alpha = 2k(2k-1)/(3m-2k+2)^2$. Since $\alpha \leqslant (2c_3/3-2c_3)^2 \leqslant 1/2$, we have

$$P(e_F(V_A,V_B)\leqslant 2k) \leqslant 2P(e_F(V_A,V_B)=2k).$$

Then

$$
\begin{aligned}
P(bw(F)\leqslant 2k) &= P(e_F(V_A,V_B)\leqslant 2k \text{ for some } \{A,B\}) \\
&\leqslant \sum_{\{A,B\}} P(e_F(V_A,V_B)\leqslant 2k) \\
&\leqslant \binom{n}{m}\frac{[(3m)!]^3 2^{2k}}{(2k)![p_{2k}!]^2(3n)!}.
\end{aligned}
$$

From $\binom{n}{m} = O\left(2^n m^{-1/2}\right)$ and Stirling's Formula, we obtain

$$P(bw(F)\leqslant 2k) = O\left(\frac{2^{-m}3^{3m}m^{3m+1/2}}{(2k)^{2k+1/2}(3m-2k)^{3m-2k+1}}\right).$$

Now write $2k = 2cm \leqslant \lfloor c_3 n \rfloor$ and we have

$$P(bw(F)\leqslant cn) = O\left(n^{-1}\right).$$

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For odd $3m$, a similar calculation with $2k$ replaced by $2k+1$ gives the same result. Then

$$P(bw(F) \leqslant \lfloor c_3 n \rfloor) \to 0 \qquad (n \to \infty)$$

and, consequently,

$$P\Big(bw\Big(G \in \mathcal{G}_n^{(3)}\Big) \geqslant c_3 n\Big) \to 1 \qquad (n \to \infty)\,.$$

∎

Remark. In general, a similar calculation shows that

$$P\Big(bw\Big(G \in \mathcal{G}_n^{(r)}\Big) \geqslant c_r n\Big) \to 1 \qquad (n \to \infty)\,.$$

Since $(2d)^{2d}(1-2d)^{(1-2d)}$ monotonoically decreases to $1/2$ on $[0,1/4]$, we have $(2d)^{2d}(1-2d)^{(1-2d)} \geqslant 2^{2/r}/2$ for fixed $d \in (0,1/4)$ and all sufficiently large r. Consequently, $c_r \geqslant rd$ so that $c_r \to r/4$ as $r \to \infty$. We summarize this now.

THEOREM 3. *Almost every r-regular graph G of order n has $bw(G) \geqslant c_r n$. Moreover, $c_r \to r/4$ as $r \to \infty$.*

In view of the upper bound for the bisection width given by Goldberg and Gardner [**13**], this last result is asymptotically correct.

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