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A REMARK ON LITTLEWOOD–PALEY g-FUNCTION

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We prove L^p-estimates for the Littlewood–Paley g-function associated with a complex elliptic operator $L = -\operatorname{div} A\nabla$ with bounded measurable coefficients in $\mathbb{R}^n$.

1. INTRODUCTION

Let $A = A(x)$ be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda|\xi|^2 \leqslant \operatorname{Re}\langle A\xi, \xi\rangle \quad \text{and} \quad \big|\langle A\xi, \zeta\rangle\big| \leqslant \Lambda|\xi||\zeta|, \tag{1.1}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leqslant \Lambda < \infty$. Here $\langle A\xi, \zeta\rangle = \sum\limits_{i,j} a_{ij}(x)\xi_i\overline{\zeta_j}$ denotes the usual inner product in $\mathbb{C}^n$. We define a divergence form operator

$$Lf \equiv -\operatorname{div}(A\nabla f), \tag{1.2}$$

which we interpret in the usual weak sense via a sesquilinear form.

By the holomorphic functional calculus theory ([**10**]), $\psi(L)$ is well-defined for any function $\psi \in \Psi(S_\mu)$ (see (2.1) below). We consider the Littlewood–Paley g-function

$$g_L(f)(x) = g_{\psi,L}(f)(x) = \left(\int_0^\infty \big|\psi_s(L)f(x)\big|^2 \frac{ds}{s}\right)^{1/2}, \tag{1.3}$$

where $\psi_s(z) = \psi(sz)$.

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$ and $\psi(z) = z^{1/2}e^{-z^{1/2}}$, then $g_L(f)(x)$ is the classical Littlewood–Paley g-function $g_1(f)(x)$, which is also given by

$$g_1(f)(x) = \left(\int_0^\infty \Big|\frac{\partial}{\partial y}(P_y * f)(x)\Big|^2 y\,dy\right)^{1/2},$$

where $P_y(x) = c_n y\big(y^2 + |x|^2\big)^{-(n+1)/2}$ is the Poisson kernel. It is well-known that $g_1(f)(x)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. See [**11**, Chapter 4].

The main result of this paper is the following theorem.

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THEOREM 1.1. *Let L be as in (1.2). We assume that $n \geqslant 3$ and $(2n/n+2) < p < \left(2n/(n-2)\right)$. If $f \in L^p(\mathbb{R}^n)$, then*

$$c\|f\|_p \leqslant \left\|g_L(f)\right\|_p \leqslant c^{-1}\|f\|_p, \tag{1.4}$$

where $c = c(\psi)$ is a positive constant independent of f.

We remark that when A has real entries, or when $n = 1, 2$ in the case of complex entries, the analytic semigroup e^{-tL} generated by L has a kernel $p_t(x,y)$ which satisfies Gaussian upper bounds, that is,

$$\left|p_t(x,y)\right| \leqslant Ct^{-n/2}\exp\left(-\frac{\beta|x-y|^2}{t}\right) \quad \text{for some } \beta > 0, \tag{1.5}$$

and for all $t > 0$, and all $x,\ y \in \mathbb{R}^n$ (see [**4**, pp. 30-31]). By [**2**, Theorem 4], the estimate (1.4) is true for all $1 < p < \infty$. Unfortunately, in the case of complex entries, (1.5) is no longer true if $n \geqslant 3$. It was proved in [**1**] that there is a complex elliptic operator $L = -\operatorname{div} A\nabla$ which does not have Gaussian upper bounds (1.5) in dimensions $n \geqslant 5$. And then we can not follow the technique in [**2**] to obtain Theorem 1.1. Instead, we need to use some weighted norm estimates for the semigroup e^{-tL} (Lemma 2.2 below). See [**3, 5, 8, 9**].

The paper is organised as follows. In Section 2, we state some known results to be used throughout this paper. In Section 3, we prove a lemma, which plays a key role in the proof of Theorem 1.1. The proof of Theorem 1.1 will be given in Section 4 by using the technique already employed in [**7**] and [**5**].

2. PRELIMINARIES

For $\nu \in (0,\pi]$, we denote by S_ν the open sector $S_\nu = \left\{z \in \mathbb{C} : |\arg z| < \nu\right\}$ and by $H_\infty(S_\nu)$ the set of all bounded holomorphic functions on S_ν. If $\mu \in (\pi/2,\pi)$, we define

$$\Psi(S_\mu) := \left\{g \in H_\infty(S_\mu) : \exists s > 0, \exists c \geqslant 0 : \left|g(z)\right| \leqslant \frac{c|z|^s}{1+|z|^{2s}}\right\}. \tag{2.1}$$

We are given an elliptic operator as in (1.2) with ellipticity constants λ and Λ in (1.1). By the holomorphic functional calculus theory, for any $g \in \Psi(S_\mu)$, $g(L)$ can be computed by the absolutely convergent Cauchy integral

$$g(L) = -\frac{1}{2\pi i}\int_\gamma (\eta I - L)^{-1} g(\eta)\, d\eta, \tag{2.2}$$

where $\mu \in (\pi/2,\pi)$ and the path γ consists of two rays $re^{\pm i\theta}, r \geqslant 0$ and $\pi/2 < \theta < \mu$, described counter-clockwise. We refer to [**10**] for the details.

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Now, we denote by $B(x,r)$ balls in $\mathbb{R}^n$, let $A(x,\sqrt{t},k)$ be the following annulus in $\mathbb{R}^n$:

$$A(x,\sqrt{t},k) = B(x,(k+1)\sqrt{t}) \backslash B(x,k\sqrt{t}).$$

Moreover, we write P_E for the projection obtained by multiplying by the characteristic function of a set E. We consider the Hardy-Littlewood p-maximal operator M_p, defined by $M_p f(x) = \sup_{r>0} N_{p,r} f(x)$, where

$$N_{p,r} f(x) = \left(\left| B(x,r) \right|^{-1} \int_{B(x,r)} \left| f(y) \right|^p dy \right)^{1/p}.$$

If $n \geqslant 3$, we denote

$$p_{\min} = 2n/(n+2) \quad \text{and} \quad p_{\max} = 2n/(n-2).$$

First, Theorem 1.1 is true for $p = 2$ (see [**10**]).

Lemma 2.1. *Let L be as in (1.2) and $n \geqslant 3$. Then, there exists a positive constant $c = c(\psi)$ independent of f such that*

$$c\|f\|_2 \leqslant \left\| g_L(f) \right\|_2 \leqslant c^{-1} \|f\|_2 .$$

Lemma 2.2. *Let L be as in (1.2). Then for all p and q such that $p_{\min} < p < q < p_{\max}$ there exist positive constants b and C such that*

$$\left\| P_{B(x,\sqrt{t})} e^{-tL} P_{A(x,\sqrt{t},k)} \right\|_{L^p \to L^q} \leqslant C \left| B(x,\sqrt{t}) \right|^{1/q-1/p} e^{-bk^2} \tag{2.3}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and $k \in \mathbb{N}$.

Proof: We refer to [**8**, Section 2] and [**5**, Remark 2.2]. ∎

Lemma 2.3. *Suppose that $p_{\min} < p < q < p_{\max}$. Then we have*

(i) *for all $r,s,t > 0$ and $x,z \in \mathbb{R}^n$, there exists $\rho > n+1$ such that*

$$N_{q,\sqrt{t}}\left(P_{B(z,r)} e^{-tL} P_{B(z,s)^c} f\right)(x) \leqslant C \left(\sum_{k > (s-r)\left(\sqrt{t}\right)^{-1}} k^{n-1-\rho} N_{p,k\sqrt{t}} f(x)^p \right)^{1/p};$$

(ii) *for all $r,t > 0$, $f \in L^p(\mathbb{R}^n)$, $z \in \mathbb{R}^n$, $x \in B(z,\sqrt{t}/2)$, there exist $0 < \gamma < \beta$ such that*

$$N_{q,2r}\left(P_{B(z,r)} e^{-tL} P_{B(z,8r)^c} f\right)(x) \leqslant C\left(1 + \frac{r}{\sqrt{t}}\right)^{-\beta} \left(1 + \frac{\sqrt{t}}{r}\right)^{\gamma} M_p f(x).$$

Proof: For any fixed $b > 0$ as in Lemma 2.2, there exists a constant $\rho > n + 1 + (np/q)$ such that $|e^{-b(k-1)^2} - e^{-bk^2}| \leqslant Ck^{-\rho-1}$ for some positive constant C. Let $\beta = (\rho - n)/p$, and $\gamma = n/q$. By [**5**, Lemma 3.3], Lemma 2.3 is proved. ∎

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REMARK.

(i) The paper [8] shows the optimality of the interval $(p_{\min}, p_{\max})$ of the semigroup e^{-tL} in Lemma 2.2 when L is defined as in (1.2);

(ii) when $(p,q) = (1,\infty)$, the weighted norm estimate (2.3) characterises the fact that the operators e^{-tL} have integral kernels $p_t(x,y)$ satisfies certain Poisson upper bounds ([5, Proposition 3.7]).

PROPOSITION 2.4. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function. Then the following are equivalent:*

(a) *For all $x, y \in \mathbb{R}^n, t > 0$ we have*

$$|p_t(x,y)| \leqslant C\left|B(x,\sqrt{t})\right|^{-1} g\big(|x-y|^2/t\big).$$

(b) *For all $x \in \mathbb{R}^n, t > 0, k \in \mathbb{R}^+$ we have*

$$\left\|P_{B(x,\sqrt{t})} e^{-tL} P_{A(x,\sqrt{t},k)}\right\|_{L^1 \to L^\infty} \leqslant C\left|B(x,\sqrt{t})\right|^{-1} g(k^2).$$

3. A KEY LEMMA

Denote $S_0 = I$ and $S_t = e^{-tL}$. For any $m \in \mathbb{N}$, we let $D^m S_t = (I - S_t)^m = \sum_{k=0}^{m} C_m^k (-1)^k S_{kt}$. Let $\delta > 2\beta = 2(\rho - n)/p > 0$. We define

$$C_{\gamma,\delta,s}^{t,\beta} = \int_0^\infty \int_0^\infty \Big(1 + \frac{t}{\mu}\Big)^{-2\beta} \Big(1 + \frac{\mu}{t}\Big)^{2\gamma} e^{-b\nu\mu} \min\big(1, (t\nu)^\delta\big) |\psi_s(\nu)|\, d\nu\, d\mu. \tag{3.1}$$

In order to prove Theorem 1.1, we need the following lemma.

LEMMA 3.1. *Suppose that $p_{\min} < p < 2 < q < p_{\max}$ and $m > 2\beta$. Then, for all $t > 0$, $f \in L^p(\mathbb{R}^n)$, $z \in \mathbb{R}^n$, and $x \in B(z, \sqrt{t}/2)$,*

$$N_{q,\sqrt{t}}\Big(P_{B(z,\sqrt{t}/2)}\big(\psi_s(L) D^m S_t\big) P_{B(z,4\sqrt{t})^c} f\Big)(x) \leqslant C C_{\gamma,\delta,s}^{t,\beta}\, M_p f(x).$$

PROOF: Let $f(L) = \psi_s(L) D^m S_t$ where $f(\lambda) = \psi_s(\lambda)\big(1 - e^{-t\lambda}\big)^m$. We first represent the operator $f(L)$ by using the semigroup $e^{-\mu L}$. As in (2.2), we have

$$f(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda I)^{-1} f(\lambda) d\lambda$$

where the contour $\gamma = \gamma_+ \cup \gamma_-$ is given by $\gamma_+(s) = s e^{iv}$ for $s \geqslant 0$ and $\gamma_-(s) = -s e^{-iv}$ for $s \leqslant 0$, and $v > \pi/2$.

For $\lambda \in \gamma$, substitute:

$$(L - \lambda I)^{-1} = \int_0^\infty e^{\lambda\mu} e^{-\mu L} d\mu.$$

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Changing the order of integration gives

$$f(L) = \int_0^\infty e^{-\mu L} n(\mu)\, d\mu,$$

where

$$n(\mu) = \frac{1}{2\pi i} \int_\gamma e^{\lambda\mu} f(\lambda)\, d\lambda.$$

Consequently, by (ii) of Lemma 2.3 we have

$$\begin{aligned} &N_{q,\sqrt{t}}\Big(P_{B(z,\sqrt{t}/2)}(\psi_s(L)D^m S_t)P_{B(z,4\sqrt{t})^c} f\Big)(x) \\ &\leqslant \int_0^\infty N_{q,\sqrt{t}}\Big(P_{B(z,\sqrt{t}/2)} e^{-\mu L} P_{B(z,4\sqrt{t})^c} f\Big)(x) |n(\mu)|\, d\mu \\ &\leqslant C \int_0^\infty \int_0^\infty |e^{\lambda\mu}\psi_s(\lambda)(1-e^{-t\lambda})^m| N_{q,\sqrt{t}}\Big(P_{B(z,\sqrt{t}/2)} e^{-\mu L} P_{B(z,4\sqrt{t})^c} f\Big)(x) d|\lambda|\, d\mu \\ &\leqslant C\ C^{t,\beta}_{\gamma,\delta,s} M_p f(x), \end{aligned}$$

which completes the proof of Lemma 3.1. ∎

Corollary 3.2. *Let $C^{t,\beta}_{\gamma,\delta,s}$ be as in (3.1). Then, there exists a constant C independent of t, β, γ and δ such that*

$$\int_0^\infty \left(C^{t,\beta}_{\gamma,\delta,s}\right)^2 \frac{ds}{s} \leqslant C < \infty.$$

Proof: We denote

$$C^{t,\beta}_{\gamma,\delta} = \int_0^\infty \int_0^\infty \left(1 + \frac{t}{\mu}\right)^{-2\beta} \left(1 + \frac{\mu}{t}\right)^{2\gamma} e^{-b\nu\mu} \min\big(1, (t\nu)^\delta\big) d\nu d\mu.$$

Since

$$\left(\int_0^\infty |\psi_s(\nu)|^2 \frac{ds}{s}\right)^{1/2} \leqslant C < \infty,$$

the Minkowski inequality implies that

$$\int_0^\infty \left(C^{t,\beta}_{\gamma,\delta,s}\right)^2 \frac{ds}{s} \leqslant C\left(C^{t,\beta}_{\gamma,\delta}\right)^2.$$

Noting that $0 < 2\gamma < 2\beta < \delta$, we have $C^{t,\beta}_{\gamma,\delta} < \infty$ for any $t > 0$ (see [**5**, Lemma 3.6] or [**7**, page 259] where the case $\gamma = 0$). So, the proof of Corollary 3.2 is complete.

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4. Proof of Theorem 1.1

We first state a Calderón–Zygmund decomposition. For its proof, (see [**6**, Theorem 1.1, Chapter 8]).

Lemma 4.1. *Let $\lambda > 0$. Then for any $f(x) \in L^p(\mathbb{R}^n), p \geqslant 1$, there exist a constant C independent of f and λ, and a decomposition*

$$f = h + b = h + \sum_j b_j,$$

so that

(i) $|h(x)| \leqslant C\lambda$ *for all almost $x \in \mathbb{R}^n$;*

(ii) *there exists a sequence of balls Q_j so that the support of each b_j is contained in Q_j and*

$$\int_{\mathbb{R}^n} |b_j(x)|^p \, dx \leqslant C\lambda^p |Q_j|;$$

(iii) $\sum_j |Q_j| \leqslant C\lambda^{-p} \int_{\mathbb{R}^n} |f|^p \, dx$;

(iv) *each point of $\mathbb{R}^n$ is contained in at most a finite number N of the balls Q_i.*

Proof of Theorem 1.1: We first consider the second inequality of (1.4) using an idea of [**7**, Theorem 1] (or [**5**, Theorem 1.1]). For any p such that $p_{\min} = 2n/(n+2) < p \leqslant 2$, we shall prove that $g_L(f)$ satisfies weak type (p,p) estimate. And then the boundedness of $g_L(f)$ from $L^p(\mathbb{R}^n)$ $(p_{\min} < p < 2)$ to itself follows from the Marcinkiewicz interpolation theorem. Using a standard duality argument, $g_L(f)$ is proved to be a bounded operator on $L^p(\mathbb{R}^n)$ for all $2 < p < p_{\max} = 2n/(n-2)$.

For any $\lambda > 0$, there exist a decomposition $f = h + b = h + \sum_j b_j$, and a sequence of balls Q_j as in Lemma 4.1. Denote $Q_j = Q_j(x_j, r_j)$ and $t_j = (2r_j)^2$. Choosing $m > 2\beta$ as in Lemma 3.1, we then decompose $\sum_j b_j = h_1 + h_2$, where

$$h_1 = \sum_j (I - D^m S_{t_j}) b_j, \quad \text{and} \quad h_2 = \sum_j (D^m S_{t_j}) b_j.$$

We have,

$$\Big|\{x : |g_L f(x)| > \lambda\}\Big| \leqslant \Big|\{x : |g_L(h)(x)| > \lambda/3\}\Big| + \sum_{k=1}^{2} \Big|\{x : |g_L(h_k)(x)| > \lambda/3\}\Big|,$$

and we shall estimate the three terms separately, where we write λ instead of $\lambda/3$.

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We start with the first term. Using Lemma 2.1, we obtain

$$\begin{aligned}\left|\{x:\ |g_L(h)(x)|>\lambda\}\right| &\leqslant \lambda^{-2}\int_{\mathbb{R}^n}|g_L(h)(x)|^2\,dx\\ &\leqslant C\lambda^{-2}\int_{\mathbb{R}^n}|h(x)|^2\,dx\\ &\leqslant C\Big(\frac{\|f\|_p}{\lambda}\Big)^p.\end{aligned}$$

We estimate the second term, that is, the term involving $h_1=\sum_j(I-D^mS_{t_j})b_j$. We claim that

$$\|h_1\|_2=\Big\|\sum\nolimits_j(I-D^mS_{t_j})b_j\Big\|_2\leqslant C\lambda^{-1}\Big\|\sum\nolimits_j\chi_{Q_j}\Big\|_2. \tag{4.1}$$

Since $\left\|\sum_j\chi_{Q_j}\right\|_2^2\leqslant C(\|f\|_p/\lambda)^p$ by iv) of Lemma 4.1, we obtain

$$\begin{aligned}\left|\{x:\ |g_L(h_1)(x)|>\lambda\}\right| &\leqslant \lambda^{-2}\int_{\mathbb{R}^n}|g_L(h_1)(x)|^2\,dx\\ &\leqslant C\lambda^{-2}\int_{\mathbb{R}^n}|h_1|^2\,dx\\ &\leqslant C\Big(\frac{\|f\|_p}{\lambda}\Big)^p.\end{aligned}$$

We now prove the claim (4.1). Recall that $t_j=(2r_j)^2$, and let $1/p'+1/p=1$. Note that for any $\phi\in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\left|\langle\phi,(I-D^mS_{t_j})b_j\rangle\right|=\left|\langle(I-D^mS_{t_j})^*\phi,b_j\rangle\right| &\leqslant \left\|\chi_{Q_j}(I-D^mS_{t_j})^*\phi\right\|_{p'}\|b_j\|_p\\ &\leqslant C\lambda^{-1}|Q_j|N_{p',r_j}(I-D^mS_{t_j})^*\phi(x_j)\\ &\leqslant C\lambda^{-1}\int_{Q_j}N_{p',r_j}(I-D^mS_{t_j})^*\phi\,dx.\end{aligned}$$

Observing that $S_0=I$, and $I-D^mS_t=-\sum\limits_{k=1}^{m}C_m^k(-1)^kS_{kt}$ for all $t>0$. Applying (i) of Lemma 2.3 ($q=p'$ and $p=p$), we obtain

$$N_{p',\sqrt{t}}(I-D^mS_t)^*f(x)\leqslant CM_pf(x).$$

So, for any $\phi\in L^2(\mathbb{R}^n)$ we have $\|M_p\phi\|_2\leqslant C\|\phi\|_2$, and then

$$\begin{aligned}\sup_{\|\phi\|_2\leqslant1}\left|\Big\langle\phi,\sum\nolimits_j(I-D^mS_{t_j})b_j\Big\rangle\right| &\leqslant C\lambda^{-1}\sup_{\|\phi\|_2\leqslant1}\int(M_p\phi)\sum\nolimits_j\chi_{Q_j}(x)\,dx\\ &\leqslant C\lambda^{-1}\Big\|\sum\nolimits_j\chi_{Q_j}\Big\|_2\sup_{\|\phi\|_2\leqslant1}\|M_p\phi\|_2\\ &\leqslant C\lambda^{-1}\Big\|\sum\nolimits_j\chi_{Q_j}\Big\|_2,\end{aligned}$$

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which completes the proof of the claim (4.1).

We now turn to estimate the third term, that is, the term which involves h_2. Denoting $Q_j^* = Q_j(x_j, 8r_j)$ we have

$$\begin{aligned}\left|\left\{x:\ |g_L(h_2)(x)| > \lambda\right\}\right| &\leqslant \sum_j |Q_j^*| + \lambda^{-2}\int_{(\cup_j Q_j^*)^c} |g_L(h_2)(x)|^2\,dx\\ &\leqslant C\Big(\frac{\|f\|_p}{\lambda}\Big)^p + \lambda^{-2}\int_{(\cup_j Q_j^*)^c} |g_L(h_2)(x)|^2\,dx.\end{aligned}$$

Denote $G_{j,s} = \chi_{(Q_j^*)^c}\psi_s(L)(D^m S_{t_j})\chi_{Q_j} b_j$. Observe that

$$\begin{aligned}\int_{(\cup_j Q_j^*)^c} |g_L(h_2)(x)|^2\,dx &= \int_{(\cup_j Q_j^*)^c}\int_0^\infty \Big|\sum_j \psi_s(L)(D^m S_{t_j})b_j\Big|^2 \frac{dx\,ds}{s}\\ &\leqslant \int_0^\infty \int_{\mathbb{R}^n} \Big|\sum_j \chi_{(Q_j^*)^c}\psi_s(L)(D^m S_{t_j})\chi_{Q_j} b_j\Big|^2\,dx\frac{ds}{s}\\ &\leqslant \int_0^\infty \Big\|\sum_j G_{j,s}b_j\Big\|_2^2\frac{ds}{s}.\end{aligned}$$

We shall estimate

$$\int_0^\infty \Big\|\sum_j G_{j,s}b_j\Big\|_2^2\frac{ds}{s} \leqslant C\lambda^{-2}\Big\|\sum_j \chi_{Q_j}\Big\|_2^2, \tag{4.2}$$

and can then argue as with the term $h_1(x)$ to obtain

$$\left|\left\{x:\ |g_L(h_2)(x)| > \lambda\right\}\right| \leqslant C\Big(\frac{\|f\|_p}{\lambda}\Big)^p.$$

Now we prove (4.2). Choosing r such that $p_{\min} < r < 2$. Using (ii) of Lemma 2.3 ($q = p'$ and $p = r$), we have

$$\begin{aligned}|\langle \phi, G_{j,s}b_j\rangle| \leqslant \|\chi_{Q_j} G_{j,s}^*\phi\|_{p'}\,\|b_j\|_p &\leqslant C\lambda^{-1}|Q_j| N_{p',r_j}(G_{j,s}^*\phi)(x_j)\\ &\leqslant C\lambda^{-1}\int_{Q_j} N_{p',r_j}(G_{j,s}^*\phi)(x)\,dx\\ &\leqslant C\lambda^{-1}C_{\gamma,\delta,s}^{t,\beta}\int_{Q_j} M_r\phi\,dx.\end{aligned}$$

Noting that $\|M_r\phi\|_2 \leqslant C\|\phi\|_2$, by Corollary 3.2 we obtain

$$\begin{aligned}\int_0^\infty \Big\|\sum_j G_{j,s}b_j\Big\|_2^2\frac{ds}{s} &= \int_0^\infty \Big(\sup_{\|\phi\|_2\leqslant 1}\Big|\Big\langle \phi, \sum_j G_{j,s}b_j\Big\rangle\Big|\Big)^2\frac{ds}{s}\\ &\leqslant C\lambda^{-2}\Big\|\sum_j \chi_{Q_j}\Big\|_2^2 \sup_{\|\phi\|_2\leqslant 1}\|M_r\phi\|_2^2\int_0^\infty \Big(C_{\gamma,\delta,s}^{t,\beta}\Big)^2\frac{ds}{s}\\ &\leqslant C\lambda^{-2}\Big\|\sum_j \chi_{Q_j}\Big\|_2^2,\end{aligned}$$

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which completes the proof of (4.2), and then the second inequality in (1.4) when $p_{\min} < p < p_{\max}$.

The first inequality of (1.4), that is, the reverse square function estimates when $p_{\min} < p \leqslant 2$ and $2 \leqslant p < p_{\max}$ are consequences of the second inequality (that is, the square function estimates) when $2 \leqslant p < p_{\max}$ and $p_{\min} < p \leqslant 2$, respectively.

REFERENCES

[1] P. Auscher, T. Coulhon and Ph. Tchamitchian, 'Absence de principe du maximum pour certaines équations paraboliques complexes', *Colloq. Math.* **171** (1996), 87–95.

[2] P. Auscher, X.T. Duong and A. McIntosh, 'Boundedness of Banach space valued singular integral operators and Hardy spaces' (to appear).

[3] P. Auscher, S. Hofmann, M. Lacey, J. Lewis, A. McIntosh and Ph. Tchamitchian, 'The solution of Kato's conjectures', *C.R. Acad. Sci. Paris Ser. I Math.* **332** (2001), 601–606.

[4] P. Auscher and Ph. Tchamitchian, 'Square root problem for divergence operators and related topics', *Astérisque* **249** (1998), 577–623.

[5] S. Blunck and P.C. Kunstmann, 'Calderón–Zygmund theory for non-integral operators and H^∞ functional calculus' (to appear).

[6] D.G. Deng and Y.S. Han, *Theory of H^p spaces* (Peking Univ. Press, China, 1992).

[7] X.T. Duong and A. McIntosh, 'Singular integral operators with non-smooth kernels on irregular domains', *Rev. Mat. Iberoamericana* **15** (1999), 233–265.

[8] V. Liskevich, Z. Sobol and H. Vogt, 'On L^p-theory of C_0-semigroups associated with second order elliptic operators II' (to appear).

[9] V. Liskevich and H. Vogt, 'On L^p-spectrum and essential spectra of second order elliptic operators', *Proc. London Math. Soc.* **80** (2000), 590–610.

[10] A. McIntosh, 'Operators which have an H_∞-calculus', in *Miniconference on Operator Theory and Partial Differential Equations* (Proceedings of the Centre for Mathematical Analysis, ANU, Canberra, 1986), pp. 210–231.

[11] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series **30** (Princeton University Press, Princeton N.J., 1970).

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