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On the metric dimension of circulant graphs

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Abstract. In this note, we bound the metric dimension of the circulant graphs $C_n(1, 2, \ldots, t)$. We shall prove that if $n = 2tk + t$ and if t is odd, then $\dim(C_n(1, 2, \ldots, t)) = t + 1$, which confirms Conjecture 4.1.1 in Chau and Gosselin (2017, *Opuscula Mathematica* 37, 509–534). In Vetrík (2017, *Canadian Mathematical Bulletin* 60, 206–216; 2020, *Discussiones Mathematicae. Graph Theory* 40, 67–76), the author has shown that $\dim(C_n(1, 2, \ldots, t)) \leq t + \left\lceil \frac{p}{2} \right\rceil$ for $n = 2tk + t + p$, where $t \geq 4$ is even, $1 \leq p \leq t + 1$, and $k \geq 1$. Inspired by his work, we show that $\dim(C_n(1, 2, \ldots, t)) \leq t + \left\lfloor \frac{p}{2} \right\rfloor$ for $n = 2tk + t + p$, where $t \geq 5$ is odd, $2 \leq p \leq t + 1$, and $k \geq 2$.

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected connected graph. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between these two vertices. For an ordered set $W = \{w_1, \ldots, w_k\}$ of k distinct vertices of G, we refer to the k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ as the metric representation of a vertex v with respect to W. The set W is called a *resolving set* of G if $r(u|W) = r(v|W)$ implies that $u = v$ for all $u, v \in V(G)$. A resolving set containing a minimum number of vertices is called a *metric basis* of G, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

Motivated by the problem of uniquely determining the location of an intruder in a network, Slater introduced the notion of metric dimension of a graph in [9], where the resolving sets were referred to as locating sets. Harary and Melter also introduced the idea of the metric dimension of a graph in [5]. It was proved that the metric dimension is an NP-hard graph invariant [8] and has been widely investigated in the last 55 years and it also has applications in many diverse areas [6, 7].

This note is devoted to the study of the metric dimension of circulant graphs. Let n, t, and $a_1, a_2, \ldots, a_t$ be positive integers so that $1 \leq a_1 < a_2 < \cdots < a_t \leq \left\lfloor \frac{n}{2} \right\rfloor$. The circulant graph $C_n(a_1, a_2, \ldots, a_t)$ consists of a vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and an edge set $\{v_i v_{i+a_j} : 0 \leq i \leq n-1, 1 \leq j \leq t\}$, where the indices are taken modulo n. The numbers $a_1, a_2, \ldots, a_t$ are called *generators*. We restrict our attention to special kinds of circulant graphs, i.e., the circulant graphs $C_n(1, 2, \ldots, t)$ with consecutive

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generators. In [1], Borchert and Gosselin studied the metric dimension of $C_n(1,2)$ and $C_n(1,2,3)$, and obtained that for $n \geq 6$,

$$\dim(C_n(1,2)) = \begin{cases} 4, & \text{for } n \equiv 1 \bmod 4, \\ 3, & \text{otherwise}, \end{cases}$$

and that for $n \geq 8$,

$$\dim(C_n(1,2,3)) = \begin{cases} 5, & \text{for } n \equiv 1 \bmod 6, \\ 4, & \text{otherwise}. \end{cases}$$

In [3, 11], the authors studied the metric dimension of $C_n(1,2,3,4)$, and obtained that for $n \geq 20$,

$$\dim(C_n(1,2,3,4)) = \begin{cases} 6, & \text{for } n \equiv 0,1,7 \bmod 8, \\ 5, & \text{for } n \equiv 2,3,5,6 \bmod 8, \\ 4, & \text{for } n \equiv 4 \bmod 8. \end{cases}$$

For the results concerning $\dim(C_n(1,2,\ldots,t))$ with arbitrary integers $t \geq 5$, the reader may refer to [2, 4, 10, 12].

We shall assume throughout this note that $n = 2tk + r$, where $t \geq 4$, $k \geq 2$, and $2 \leq r \leq 2t+1$. When $t \leq r \leq 2t+1$, we may also assume $n = 2tk + t + p$, where $0 \leq p \leq t+1$. It is known that the distance between two vertices v_i and v_j in $C_n(1,2,\ldots,t)$ is

$$d(v_i, v_j) = \min\left\{\left\lceil \frac{|i-j|}{t} \right\rceil, \left\lceil \frac{n-|i-j|}{t} \right\rceil\right\}, \tag{1.1}$$

and that the diameter of $C_n(1,2,\ldots,t)$ is $d := k+1$.

Here, we set forth our notation and terminology. Let W and V be subsets of vertices in $G = C_n(1,2,\ldots,t)$, where V consists of at least two vertices. A vertex w is said to *resolve* a pair of vertices u and v if $d(u,w) \neq d(v,w)$. W is said to *distinguish* V if for any pair of distinct vertices u and v in V, there exists a vertex in W which can resolve u and v. It is easy to see that if W can distinguish $V(G)$, then it is a resolving set of G. Vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+s}$ with consecutive indices are called the *consecutive vertices*. The *outer cycle* of the circulant graph is a spanning subgraph of G in which the vertex v_i is adjacent to exactly the vertices v_{i+1} and v_{i-1}. When $r = 2$, the unique vertex that has distance $k+1$ from w will be called the *opposite vertex* of w, and is denoted by w', and we can then define $W' := \{w' : w \in W\}$ for the vertex set W.

2 Lower bounds

This section deals with the lower bounds for $\dim(C_n(1,2,\ldots,t))$. In [2, 10], the authors have shown that when $3 \leq r \leq t$ and n is sufficiently large, $\dim(C_n(1,2,\ldots,t))$ has a lower bound of t.

Theorem 2.1 *([10, Theorem 2.3]) Let $n = 2tk + r$ where $3 \leq r \leq t$, and $n \geq t^2 + 1$. Then $\dim(C_n(1,2,\ldots,t)) \geq t$.*

Theorem 2.3 improves their result. We begin with the following lemma.

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Lemma 2.2 *Suppose that $r = t$, and that $2 \le x \le t$. If a vertex set W can distinguish x consecutive vertices, then the cardinality of W is at least $x-1$.*

Proof Without loss of generality, assume that W can distinguish $V = \{v_1, v_2, \dots, v_x\}$. Let W_1 be the intersection of W and V, and p the cardinality of W_1. We can assume $p \le x-2$, and then assume $V \backslash W_1 = \{v_{i_1}, \dots, v_{i_{x-p}}\}$, where $i_1 < \cdots < i_{x-p}$. It follows that $W \backslash W_1$ can distinguish $x-p-1$ pairs of vertices $(v_{i_1}, v_{i_2}), \dots, (v_{i_{x-p-1}}, v_{i_{x-p}})$. Suppose $w_j \in W \backslash W_1$ can resolve $(v_{i_j}, v_{i_{j+1}})$ for each such j, then it can resolve two consecutive vertices in the $v_{i_j} - v_{i_{j+1}}$ path of the outer cycle, say $v_{i'_j}$ and $v_{i'_j+1}$. Since $r = t$, and since the distance between v_{i_1} and $v_{i'_j}$ on the outer cycle is no more than $t-2$, it follows from equation (1.1) that $d(v_{i_1}, w_j) = d(v_{i_1+1}, w_j) = \cdots = d(v_{i'_j}, w_j)$, and thus none of the pairs $(v_{i_1}, v_{i_2}), \dots, (v_{i_{j-1}}, v_{i_j})$ can be resolved by w_j. A similar argument shows that none of the pairs $(v_{i_{j+1}}, v_{i_{j+2}}), \dots, (v_{i_{x-p-1}}, v_{i_{x-p}})$ can be resolved by w_j. Therefore, any vertex in $W \backslash W_1$ resolving one of the pairs $(v_{i_1}, v_{i_2}), \dots, (v_{i_{x-p-1}}, v_{i_{x-p}})$ cannot resolve the other, implying that $W \backslash W_1$ consists of at least $x-p-1$ vertices, and so $\sharp(W) \ge x-1$. ∎

Theorem 2.3 *Let $n = 2tk + t$ where t is odd. Then* $\dim(C_n(1, 2, \dots, t)) \ge t+1$.

Proof Let W be a resolving set of the graph $C_n(1, 2, \dots, t)$. Suppose on the contrary that $\sharp(W) = t$. We can assume $v_0 \in W$.

Let us first show that $W \cap \{v_{i-tk}, v_{i+tk}\} \ne \varnothing$ holds for each vertex $v_i \in W$. Suppose on the contrary that there exists a vertex $v_j \in W$ with $W \cap \{v_{j-tk}, v_{j+tk}\} = \varnothing$, since the circulant graph $C_n(1, 2, \dots, t)$ is vertex-transitive, and we may take $j = 0$. Let $p \ge 0$ be such that $v_{n-0}, v_{n-1}, \dots, v_{n-p}$ all belong to W while $v_{n-p-1} \notin W$, and let $q \ge 0$ be such that $v_0, v_1, \dots, v_q$ all belong to W while $v_{q+1} \notin W$. It is easy to see that $p + q \le t-1$. Set $W_1 = \{v_{n-p}, v_{n-p+1}, \dots, v_q\}$. Then there is a vertex $w \in W \backslash W_1$ that resolves v_{n-p-1} and v_{q+1}. If $p + q = t-1$, then W consists of at least $t+1$ vertices, leading to the contradiction. Suppose now that $p + q \le t-2$. One can verify that there are two consecutive vertices v_i and v_{i+1} in the $v_{n-p-1} - v_{q+1}$ path of the outer cycle, which can be resolved by w. By symmetry, we can assume $n - t + 1 \le i \le n-1$.

First, consider the case $n - t + 1 \le i \le n-2$. Note that $\{v_{i+1}, v_{i+2}, \dots, v_n\} \subset W_1$, and that $W \backslash (\{v_{i+1}, v_{i+2}, \dots, v_n\} \cup \{w\})$ can distinguish $\{v_{n-t}, v_{n-t+1}, \dots, v_i\}$, which consists of $i + t + 1 - n$ vertices. It follows from Lemma 2.2 that $W \backslash (\{v_{i+1}, v_{i+2}, \dots, v_n\} \cup \{w\})$ has at least $i + t - n$ vertices, and therefore $\sharp(W) \ge t+1$, a contradiction.

Next, consider the case where $i = n-1$. Since $w \notin \{v_{n-1}, v_0, v_{kt}\}$, and since $r = t$, it follows from equation (1.1) that vertices $v_{n-t}, v_{n-t+1}, \dots, v_{n-1}$ have equal distance to w. Hence, $W \backslash \{v_0, w\}$ can distinguish $\{v_{n-t}, v_{n-t+1}, \dots, v_{n-1}\}$, and applying Lemma 2.2, $W \backslash \{v_0, w\}$ has at least $t-1$ vertices, and therefore W consists of at least $t+1$ vertices, which is a contradiction.

We have already verified that $W \cap \{v_{i-tk}, v_{i+tk}\} \ne \varnothing$ holds for each vertex $v_i \in W$. We now claim that $|W \cap \{v_{i-tk}, v_{i+tk}\}| = 1$ holds for each vertex $v_i \in W$. Suppose on the contrary that there is a vertex $v_j \in W$ with $\{v_{j-tk}, v_{j+tk}\} \subset W$, and we may also take $j = 0$. Then $W \backslash \{v_0, v_{kt}, v_{n-kt}\}$ can distinguish $\{v_{kt+1}, v_{kt+2}, \dots, v_{kt+t-1}\}$,

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and applying Lemma 2.2, $W\backslash\{v_0, v_{kt}, v_{n-kt}\}$ consists of at least $t-2$ vertices, and so $\sharp(W) \geq t+1$, a contradiction.

In conclusion, for each vertex $w \in W$, there exists exactly one vertex, say w_1, in W such that w_1 has distance kt from w on the outer cycle, and we say $\{w, w_1\}$ form a "pair" in W. It is easy to see that these "pairs" in W are pairwise disjoint. Hence, the cardinality of W is even, which contradicts the assumption that $\sharp(W) = t$ is odd. ■

In what follows, we shall discuss the case where $r \in \{2\} \cup \{t+1, t+2, \ldots, 2t+1\}$. The following lemma will be needed in the sequel.

Lemma 2.4 *Suppose that $r \in \{2\} \cup \{t+1, t+2, \ldots, 2t+1\}$ and that $2 \leq x \leq t$. If a vertex set W can distinguish x vertices which come from a clique of $x+1$ consecutive vertices, then the cardinality of W is at least $x-1$.*

Proof Suppose that $v_{i_1}, \ldots, v_{i_x}$ come from a clique of $x+1$ consecutive vertices, where $i_1 < i_2 < \cdots < i_x$, and suppose that W can distinguish them.

We first deal with the case where $r \in \{t+1, t+2, \ldots, 2t+1\}$. Let $V = \{v_{i_1}, \ldots, v_{i_x}\}$, and let W_1 be the intersection of W and V, and p the cardinality of W_1. We can assume $p \leq x-2$, and then assume $V\backslash W_1 = \{v_{j_1}, \ldots, v_{j_{x-p}}\}$, where $j_1 < \cdots < j_{x-p}$. It follows that $W\backslash W_1$ can distinguish $x-p-1$ pairs of vertices $(v_{j_1}, v_{j_2}), \ldots, (v_{j_{x-p-1}}, v_{j_{x-p}})$.

We remark that since $t+1 \leq r \leq 2t+1$, if a vertex w can resolve two consecutive vertices v_i and v_{i+1}, and if $w \neq v_i, v_{i+1}$, then it follows from equation (1.1) that

$$d(w, v_{i-t+1}) = d(w, v_{i-t+2}) = \cdots = d(w, v_i)$$

and

$$d(w, v_{i+1}) = d(w, v_{i+2}) = \cdots = d(w, v_{i+t}).$$

This remark shows that any vertex in $W\backslash W_1$ resolving one of the pairs of vertices $(v_{j_1}, v_{j_2}), \ldots, (v_{j_{x-p-1}}, v_{j_{x-p}})$ cannot resolve the other, implying $W\backslash W_1$ consists of at least $x-p-1$ vertices, and therefore $\sharp(W) \geq x-1$.

Let us turn to the case where $r = 2$. Let $V' = \{v'_{i_1}, \ldots, v'_{i_x}\}$, and let W_2 be the intersection of W and V'. Denote by q the cardinality of W_2. We can assume that $p+q \leq x-2$, and then assume $V\backslash(W_1 \cup W'_2) = \{v_{j_1}, \ldots, v_{j_s}\}$, where $j_1 < \cdots < j_s$ and $s \geq x-p-q$. It follows that $W\backslash(W_1 \cup W_2)$ can distinguish $s-1$ pairs of vertices $(v_{j_1}, v_{j_2}), \ldots, (v_{j_{s-1}}, v_{j_s})$. Similarly, any vertex in $W\backslash(W_1 \cup W_2)$ resolving one of these pairs cannot resolve the other, implying $W\backslash(W_1 \cup W_2)$ consists of at least $s-1$ vertices, and therefore $\sharp(W) \geq x-1$. ■

The authors showed in [2] that $\dim(C_n(1, 2, \ldots, t))$ has a lower bound of $t+1$ if $r \in \{2\} \cup \{t+1, t+2, \ldots, 2t\}$. We provide an alternate proof.

Theorem 2.5 *([2, Theorem 2.7]) Let $n = 2tk + r$ where $r \in \{2\} \cup \{t+1, t+2, \ldots, 2t\}$. Then $\dim(C_n(1, 2, \ldots, t)) \geq t+1$.*

Proof It is sufficient to show that any resolving set W of the graph $C_n(1, 2, \ldots, t)$ has at least $t+1$ vertices. Without loss of generality, we assume $v_0 \in W$.

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Let us first discuss the case where $r \in \{t+1, t+2, \dots, 2t\}$. Let $p \geq 0$ be such that $v_{n-0}, v_{n-1}, \dots, v_{n-p}$ all belong to W while $v_{n-p-1} \notin W$, and let $q \geq 0$ be such that $v_0, v_1, \dots, v_q$ all belong to W while $v_{q+1} \notin W$. We can assume $p + q \leq t - 1$. Set $W_1 = \{v_{n-p}, v_{n-p+1}, \dots, v_q\}$. Then there is a vertex $w \in W \backslash W_1$ that resolves v_{n-p-1} and v_{q+1}, and therefore there exist two consecutive vertices v_i and v_{i+1} in the $v_{n-p-1} - v_{q+1}$ path of the outer cycle which can be resolved by w. By symmetry, assume $0 \leq i \leq q$. Since $r \geq t+1$, it follows from equation (1.1) that $v_{i+1}, v_{i+2}, \dots, v_t$ have equal distance to w. Hence, $W \backslash (W_1 \cup \{w\})$ can distinguish $\{v_{q+1}, \dots, v_{t-p}\}$, which consists of $t - p - q$ consecutive vertices. Applying Lemma 2.4, $W \backslash (W_1 \cup \{w\})$ has at least $t - p - q - 1$ vertices, and thus W has at least $t + 1$ vertices.

The proof for the case where $r = 2$ is analogous to that for the preceding case. We first note that the definitions of p and q are changed, that is, let $p \geq 0$ be such that $v_{n-0}, v_{n-1}, \dots, v_{n-p}$ all belong to the union of W and W' while $v_{n-p-1} \notin W \cup W'$, and $q \geq 0$ such that $v_0, v_1, \dots, v_q$ all belong to the union of W and W' while $v_{q+1} \notin W \cup W'$. Set

$$W_2 = \left(\{v_{n-p}, v_{n-p+1}, \dots, v_q\} \cup \{v'_{n-p}, v'_{n-p+1}, \dots, v'_q\}\right) \cap W,$$

where $\sharp(W_2) \geq p + q + 1$. An entirely similar argument shows that there is a vertex $w \in W \backslash W_2$ that resolves v_{n-p-1} and v_{q+1}, and that $W \backslash (W_2 \cup \{w\})$ has at least $t - p - q - 1$ vertices, implying $\sharp(W) \geq t + 1$. ∎

In [2], the authors have shown that when $r = 2t + 1$, $\dim(C_n(1, 2, \dots, t))$ has a lower bound of $t + 2$. We provide an alternate proof.

Theorem 2.6 *([2, Theorem 2.17]) Let $n = 2tk + 2t + 1$. Then* $\dim(C_n(1, 2, \dots, t)) \geq t + 2$.

Proof It is sufficient to show that any resolving set W for the graph $C_n(1, 2, \dots, t)$ has at least $t + 2$ vertices. Without loss of generality, we assume $v_0 \in W$. The only vertices that can resolve v_{dt} and v_{dt+1} are

$$v_{n-t}, v_{n-2t}, \dots, v_{n-dt} = v_{dt+1}, v_{dt}, v_{dt-t}, \dots, v_t.$$

By symmetry, we assume $v_{n-pt} \in W$, where $p \in \{1, 2, \dots, d\}$. We shall consider two cases.

Case 1 ($p \leq k$): The only vertices that can resolve v_{dt+1} and v_{dt+2} are

$$v_{n+1-t}, v_{n+1-2t}, \dots, v_{n+1-dt} = v_{dt+2}, v_{dt+1}, v_{dt+1-t}, \dots, v_{t+1}.$$

If $v_{qt+1} \in W$ for some $q \in \{1, \dots, d\}$, one can easily verify that $\{v_0, v_{qt+1}, v_{n-pt}\}$ cannot distinguish any pair of vertices in $\{v_1, v_2, \dots, v_t\}$. It follows from Lemma 2.4 that $W \backslash \{v_0, v_{qt+1}, v_{n-pt}\}$ has at least $t - 1$ vertices, which confirms the assertion. If $v_{n+1-qt} \in W$ for some $q \in \{1, \dots, d\}$, it is easy to see that $\{v_0, v_{n+1-qt}, v_{n-pt}\}$ cannot distinguish any pair of vertices in $\{v_{(d-q)t+1}, v_{(d-q)t+2}, \dots, v_{(d-q+1)t}\}$, and according to Lemma 2.4, $W \backslash \{v_0, v_{n+1-qt}, v_{n-pt}\}$ has at least $t - 1$ vertices, and therefore W has at least $t + 2$ vertices.

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Case 2 ($p = d$): The only vertices that can resolve v_{kt+1} and v_{kt+2} are

$$v_{n+1-2t}, \dots, v_{n+1-dt}, v_{n+1-dt-t} = v_{kt+2}, v_{kt+1}, v_{kt+1-t}, \dots, v_1.$$

If $v_{qt+1} \in W$ for some $q \in \{1, 2, \dots, k\}$, one can verify that $\{v_0, v_{qt+1}, v_{n-dt}\}$ cannot distinguish any pair of vertices in $\{v_1, v_2, \dots, v_t\}$. If $v_{n+1-qt} \in W$ for some $q \in \{2, 3, \dots, d\}$, one can verify that $\{v_0, v_{n+1-qt}, v_{n-dt}\}$ cannot distinguish any pair of vertices in $\{v_{(d-q)t+1}, v_{(d-q)t+2}, \dots, v_{(d-q+1)t}\}$. If $v_{kt+2} \in W$, then it is easy to see that $\{v_0, v_{kt+2}, v_{n-dt}\}$ cannot distinguish any pair of vertices in $\{v_{n-(t-1)}, \dots, v_{n-2}, v_{n-1}, v_1\}$, which consists of t vertices coming from a clique of $t+1$ consecutive vertices. If $v_1 \in W$, then $\{v_0, v_1, v_{n-dt}\}$ cannot distinguish any pair of vertices in $\{v_{dt}, v_{dt+2}, v_{dt+3}, \dots, v_{dt+t}\}$. In both cases, it follows quickly from Lemma 2.4 that W has at least $(t-1) + 3 = t + 2$ vertices. The proof is complete. ∎

3 Upper bounds

This section is devoted to the study of upper bounds for $\dim(C_n(1, 2, \dots, t))$. The following three theorems provide a great deal of useful information about this topic.

Theorem 3.1 *([4, Theorem 2.9]) Let $n = 2tk + r$ where $2 \le r \le t + 1$. Then*

$$\dim(C_n(1, 2, \dots, t)) \le t + 1.$$

Theorem 3.2 *([10, Theorem 2.1 and Theorem 2.2]) Let $n = 2tk + t + p$ where t and p are both even, and $0 \le p \le t$. Then*

$$\dim(C_n(1, 2, \dots, t)) \le t + \frac{p}{2}.$$

Theorem 3.3 *([12, Theorem 5]) Let $n = 2tk + t + p$ where t is even, p is odd, and $1 \le p \le t + 1$. Then*

$$\dim(C_n(1, 2, \dots, t)) \le t + \frac{p+1}{2}.$$

Motivated by the work of Vetrík, we provide an upper bound on the metric dimension of $C_n(1, 2, \dots, t)$, where t is odd and $r \ge t + 2$.

Theorem 3.4 *Let $n = 2tk + t + p$ where t is odd, p is even, and $2 \le p \le t + 1$. Then*

$$\dim(C_n(1, 2, \dots, t)) \le t + \frac{p}{2}.$$

Proof Let

$$W_1 = \{v_0, v_2, \dots, v_{t-1}\} \quad \text{and} \quad W_2 = \{v_{kt}, v_{kt+2}, v_{kt+4}, \dots, v_{kt+t+p-3}\},$$

where $\sharp(W_1) = \frac{t+1}{2}$ and $\sharp(W_2) = \frac{t+p-1}{2}$. Let us show that $W = W_1 \cup W_2$ is a resolving set of the graph $C_n(1, 2, \dots, t)$. Divide the vertex set of $C_n(1, 2, \dots, t)$ into four disjoint sets:

$V_1 = \{v_0, v_1, \dots, v_t\}$, $V_2 = \{v_{t+1}, v_{t+2}, \dots, v_{kt-1}, v_{kt}\}$,

$V_3 = \{v_{kt+1}, v_{kt+2}, \dots, v_{kt+t+p-2}, v_{kt+t+p-1}\}$, $V_4 = \{v_{kt+t+p}, v_{kt+t+p+1}, \dots, v_{n-2}, v_{n-1}\}$.

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We claim that any pair of distinct vertices $u \in V_{r_1}$ and $v \in V_{r_2}$ have different metric representations with respect to W. We need only consider the following six cases, since in other cases, it is easy to check that v_0 can resolve u and v.

Case 1 ($r_1 = r_2 = 1$): It suffices to prove that no two vertices in $V_1 \backslash W_1 = \{v_j : j = 1, 3, \ldots, t\}$ have the same metric representation with respect to $W_{21} := \{v_{kt+2}, v_{kt+4}, \ldots, v_{kt+t-1}\}$; W_{21} is obviously a subset of W_2. We observe that for $j = 1, 3, \ldots, t$, $r(v_j | W_{21}) = (k, \ldots, k, k+1, \ldots, k+1)$, of which the first $\frac{j-1}{2}$ entries are equal to k, and the other $\frac{t-j}{2}$ entries are equal to $k+1$, the desired result follows.

Case 2 ($r_1 = r_2 = 2$): For $x = 1, \ldots, k-1$ and $j = 1, 2, \ldots, t$, the metric representation of $v_{tx+j} \in V_2$ with respect to W_1 is

$$r(v_{tx+j} | W_1) = (\underbrace{x+1, \ldots, x+1}_{\lceil \frac{j}{2} \rceil}, \underbrace{x, \ldots, x}_{\frac{t+1}{2} - \lceil \frac{j}{2} \rceil}).$$

Hence, the only vertices in V_2 with the same metric representations with respect to W_1 are the pairs (v_{tx+j-1}, v_{tx+j}), where $j = 2, 4, \ldots, t-1$ and $x = 1, 2, \ldots, k-1$. Since v_{kt+j} belongs to W_2 for each $j \in \{2, 4, \ldots, t-1\}$, and since

$$d(v_{kt+j}, v_{tx+j-1}) = k - x + 1 \quad \text{and} \quad d(v_{kt+j}, v_{tx+j}) = k - x,$$

it follows that W_2 can distinguish all these pairs.

Case 3 ($r_1 = r_2 = 3$): Note that

$$\begin{aligned} r(v_{kt+j} | W_1) &= (\overbrace{k+1, \ldots, k+1}^{\lceil \frac{j}{2} \rceil}, \overbrace{k, \ldots, k}^{\frac{t+1}{2} - \lceil \frac{j}{2} \rceil}) \quad \text{for } j = 1, 2, \ldots, t-1, \\ r(v_{kt+j} | W_1) &= (k+1, \ldots, k+1) \quad \text{for } j = t, t+1, \ldots, t+p-1. \end{aligned}$$

Write $u = v_{kt+j_1}$ and $v = v_{kt+j_2}$. We need only consider the following two subcases, since in other cases, $v_{t-1} \in W_1$ can already resolve u and v.

Case 3.1 ($j_1 < t$, $j_2 < t$): In this case, the only vertices with the same metric representations with respect to W_1 are the pairs (v_{kt+j-1}, v_{kt+j}), where $j = 2, 4, \ldots, t-1$. Since W_2 contains v_{kt+j} for each $j \in \{2, 4, \ldots, t-1\}$, it follows that W_2 can distinguish these pairs.

Case 3.2 ($j_1 \geq t$, $j_2 \geq t$): Recalling the construction of W_2, we need only show that no two vertices in $\{v_{kt+t+j} : j = 0, 2, \ldots, p-2\} \cup \{v_{kt+t+p-1}\}$ have the same metric representation with respect to $W_{22} := \{v_{kt}, v_{kt+2}, \ldots, v_{kt+p-2}\}$; W_{22} is obviously a subset of W_2. We observe that $r(v_{kt+t+j} | W_{22}) = (2, \ldots, 2, 1, \ldots, 1)$, $j = 0, 2, \ldots, p-2$, of which the first $\frac{j}{2}$ entries are equal to 2 and the other $\frac{p-j}{2}$ entries are equal to 1, and that all the distances from $v_{kt+t+p-1}$ to the vertices in W_{22} are 2; the desired result follows.

Case 4 ($r_1 = r_2 = 4$): It is not difficult to see that for $x = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, t-1$, the metric representation of $v_{n-tx+j} \in V_4$ with respect to W_1 is

$$r(v_{n-tx+j} | W_1) = (\underbrace{x, \ldots, x}_{\lfloor \frac{j}{2} \rfloor + 1}, \underbrace{x+1, \ldots, x+1}_{\frac{t-1}{2} - \lfloor \frac{j}{2} \rfloor}).$$

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Thus, the only vertices in V_4 with the same metric representations with respect to W_1 are the pairs $(v_{n-tx+j}, v_{n-tx+j+1})$, where $j = 0, 2, \ldots, t-3$ and $x = 1, 2, \ldots, k$. Since $v_{n-kt-t+j}$ belongs to W_2 for each $j \in \{0, 2, \ldots, t-3\}$, and since

$$d(v_{n-kt-t+j}, v_{n-tx+j}) = k+1-x \quad \text{and} \quad d(v_{n-kt-t+j}, v_{n-tx+j+1}) = k+2-x,$$

it follows that W_2 can distinguish these pairs.

Case 5 ($r_1 = 1$, $r_2 = 4$): The distances from the vertices in V_1 to v_{kt} are at most k, and the distances from the vertices in V_4 to v_{kt} are $k+1$, and therefore v_{kt} can resolve u and v.

Case 6 ($r_1 = 2$, $r_2 = 4$): In this case, it is clear that the only vertices with the same metric representations with respect to W_1 are the pairs (v_{tx+t}, v_{n-tx-1}), where $x = 1, 2, \ldots, k-1$. Since

$$d(v_{kt}, v_{tx+t}) = k-x-1 \quad \text{and} \quad d(v_{kt}, v_{n-tx-1}) = k-x+2,$$

it follows that $v_{kt} \in W_2$ can resolve all these pairs. ■

Theorem 3.5 *Let $n = 2tk + t + p$ where t and p are both odd, and $3 \le p \le t$. Then*

$$\dim(C_n(1, 2, \ldots, t)) \le t + \frac{p-1}{2}.$$

Proof Let

$$W_1 = \{v_0, v_2, \ldots, v_{t-1}\}, W_2 = \{v_{n-(t-1)}, v_{n-(t-3)}, \ldots, v_{n-2}\},$$
$$W_3 = \{v_{kt+1}, v_{kt+3}, \ldots, v_{kt+p-2}\},$$

where $\sharp(W_1) = \frac{t+1}{2}$, $\sharp(W_2) = \frac{t-1}{2}$, and $\sharp(W_3) = \frac{p-1}{2}$. Let us show that $W = W_1 \cup W_2 \cup W_3$ is a resolving set of the graph $C_n(1, 2, \ldots, t)$. As before, divide the vertex set of $C_n(1, 2, \ldots, t)$ into four disjoint sets:

$$V_1 = \{v_0, v_1, \ldots, v_t\}, V_2 = \{v_{t+1}, v_{t+2}, \ldots, v_{kt-1}, v_{kt}\},$$
$$V_3 = \{v_{kt+1}, v_{kt+2}, \ldots, v_{kt+t+p-2}, v_{kt+t+p-1}\}, V_4 = \{v_{kt+t+p}, v_{kt+t+p+1}, \ldots, v_{n-2}, v_{n-1}\}.$$

We claim that any pair of distinct vertices $u \in V_{r_1}$ and $v \in V_{r_2}$ have different metric representations with respect to W, and only consider six cases.

Case 1 ($r_1 = r_2 = 1$): We need only show that no two vertices in $V_1 \backslash W_1 = \{v_j : j = 1, 3, \ldots, t\}$ have the same metric representation with respect to W_2. Observe that for $j = 1, 3, \ldots, t$, $r(v_j|W_2) = (2, \ldots, 2, 1, \ldots, 1)$, of which the first $\frac{j-1}{2}$ entries are equal to 2, and the other $\frac{t-j}{2}$ entries are equal to 1, the desired result follows.

Case 2 ($r_1 = r_2 = 2$): It is easy to verify that, for $x = 1, \ldots, k-1$ and $j = 1, 2, \ldots, t$, the metric representation of $v_{tx+j} \in V_2$ with respect to W_1 is

$$r(v_{tx+j}|W_1) = (\underbrace{x+1, \ldots, x+1}_{\lceil \frac{j}{2} \rceil}, \underbrace{x, \ldots, x}_{\frac{t+1}{2} - \lceil \frac{j}{2} \rceil}).$$

Hence, the only vertices in V_2 with the same metric representations with respect to W_1 are the pairs (v_{tx+j}, v_{tx+j+1}), where $j = 1, 3, \ldots, t-2$ and $x = 1, 2, \ldots, k-1$. Since

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v_{n-t+j} belongs to W_2 for each $j \in \{1, 3, \ldots, t-2\}$, and since

$$d(v_{n-t+j}, v_{tx+j}) = x+1 \quad \text{and} \quad d(v_{n-t+j}, v_{tx+j+1}) = x+2,$$

it follows that W_2 can distinguish these pairs.

Case 3 ($r_1 = r_2 = 3$): The metric representations of the vertices in V_3 with respect to W_1 and W_2 are the following:

$$\begin{aligned}
r(v_{kt+j}|W_1) &= (\overbrace{k+1,\ldots,k+1}^{\lceil \frac{j}{2} \rceil}, \overbrace{k,\ldots,k}^{\frac{t+1}{2}-\lceil \frac{j}{2} \rceil}) \quad \text{for } j = 1,2,\ldots,t-1,\\
r(v_{kt+j}|W_1) &= (k+1,\ldots,k+1) \quad \text{for } j = t, t+1, \ldots, t+p-1,\\
r(v_{kt+j}|W_2) &= (k+1,\ldots,k+1) \quad \text{for } j = 1,2,\ldots,p-1,\\
r(v_{kt+j}|W_2) &= (\underbrace{k,\ldots,k}_{\lceil \frac{j-p}{2} \rceil}, \underbrace{k+1,\ldots,k+1}_{\frac{t-1}{2}-\lceil \frac{j-p}{2} \rceil}) \quad \text{for } j = p, p+1, \ldots, t+p-1.
\end{aligned}$$

Write $u = v_{kt+j_1}$ and $v = v_{kt+j_2}$. There are two subcases to consider.

Case 3.1 ($j_1 < t$, $j_2 < t$): In this case, the only vertices with the same metric representations with respect to W_1 are the pairs (v_{kt+j}, v_{kt+j+1}), where $j = 1, 3, \ldots, t-2$. If $p = t$, then W_3 can already distinguish all the pairs. Suppose now that $p \le t-2$. In view of the definition of W_3, it is sufficient to show that (v_{kt+j}, v_{kt+j+1}) can be distinguished by W_2 for $j = p, p+2, \ldots, t-2$. Noticing that $v_{2kt+j+1}$ belongs to W_2 for each $j \in \{p, p+2, \ldots, t-2\}$, and that

$$d(v_{2kt+j+1}, v_{kt+j}) = k+1 \quad \text{and} \quad d(v_{2kt+j+1}, v_{kt+j+1}) = k,$$

the desired result follows.

Case 3.2 ($j_1 \ge t$, $j_2 \ge t$): In this case, the only vertices with the same metric representations with respect to W_2 are the pairs $(v_{kt+t+j}, v_{kt+t+j+1})$, where $j = 1, 3, \ldots, p-2$. Since v_{kt+j} belongs to W_3 for each $j \in \{1, 3, \ldots, p-2\}$, and since

$$d(v_{kt+t+j}, v_{kt+j}) = 1 \quad \text{and} \quad d(v_{kt+t+j+1}, v_{kt+j}) = 2,$$

it follows that W_3 can distinguish these pairs.

Case 4 ($r_1 = r_2 = 4$): For $x = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, t-1$, the metric representation of $v_{n-tx+j} \in V_4$ with respect to W_1 is

$$r(v_{n-tx+j}|W_1) = (\underbrace{x,\ldots,x}_{\lfloor \frac{j}{2} \rfloor + 1}, \underbrace{x+1,\ldots,x+1}_{\frac{t-1}{2}-\lfloor \frac{j}{2} \rfloor}).$$

Hence, the only vertices in V_4 with the same metric representations with respect to W_1 are the pairs $(v_{n-tx+j-1}, v_{n-tx+j})$, where $j = 1, 3, \ldots, t-2$ and $x = 1, 2, \ldots, k$. Since v_{n-t+j} belongs to W_2 for each $j \in \{1, 3, \ldots, t-2\}$, and since

$$d(v_{n-tx+j}, v_{n-t+j}) = x-1 \quad \text{and} \quad d(v_{n-tx+j-1}, v_{n-t+j}) = x,$$

it follows that W_2 can distinguish all these pairs.

Case 5 ($r_1 = 1$, $r_2 = 4$): In this case, the only vertices with the same metric representations with respect to W_1 are the pairs (v_{n-1}, v_j), where $j = 1, 3, \ldots, t$, which can be resolved by $v_{kt+1} \in W_3$.

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Case 6 ($r_1 = 2$, $r_2 = 4$): In this case, the only vertices with the same metric representations with respect to W_1 are the pairs (v_{tx+t}, v_{n-tx-1}), where $x = 1, 2, \ldots, k-1$. Note that v_{n-2} belongs to W_2, and that

$$d(v_{tx+t}, v_{n-2}) = x + 2 \quad \text{and} \quad d(v_{n-tx-1}, v_{n-2}) = x.$$

Therefore, W_2 can distinguish these pairs. This completes our proof. ■

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