Examples in the Geometry of Cross Ratios

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When P is joined to four points A, B, C, D coplanar with P, a pencil of four lines is formed whose cross ratio is constant if ABCD are collinear. If A, B, C, D are not in a line the cross ratio P(ABCD) has a value which in general varies with the position of P, but which should be known when P is given in position and also A, B, C, D. A simple expression for the cross ratio is given and its utility in locus problems is illustrated by a variety of simple examples, which in several cases furnish methods for constructing a general cubic curve, with or without double point, a trinodal quartic, etc.

§ 1. Let 1, 2, 3, 4 be four points in a plane, P any fifth point. There exists the following relation in the signed areas of the triangles P12, etc.

$$(P12)\ (P34) + (P13)\ (P42) + (P14)\ (P23) = 0 \qquad \text{I};$$

and the cross ratio of the pencil P(1234) is given by

$$P(1234) = (P13)\ (P24) / (P14)\ (P23) \qquad \text{II}.$$

These are easily established. Let $|\,P1\,| = r_1$, $|\,P2\,| = r_2$, etc.; and let (12) denote the signed angle 1P2.

Then $P12 = \frac{1}{2} r_1 r_2 \sin(12)$; $P34 = \frac{1}{2} r_3 r_4 \sin(34)$; etc.

Substitution in I leads to the trigonometrical identity $\sin(12)\ \sin(34) + \sin(13)\ \sin(24) + \sin(14)\ \sin(23) = 0$ and the right side of II reduces to the well-known expression for the cross ratio $\sin(13)\ \sin(24)/\sin(14)\ \sin(23)$.

Cor. 1. $P(1234) = P(1235)\ .\ P(1254)$;

$\qquad = (P1235)\ P(1256)\ P(1264)$; etc.

Cor. 2. $P(1234)\ .\ P(1342)\ P(1423) = -1$.

Cor. 3. $A(BCPQ)\ .\ B(CAPQ)\ C(ABPQ) = +1$.

Cor. 4. Divide in I by (P14) (P23) when it gives rise to $P(1234) + P(1324) = 1$.

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§ 2. Let $P(1234) = \lambda$ it is then easy to establish the following well-known relations either by direct substitution in II or by the aid of *Cor.* 1. and *Cor.* 4. of § (1):

$$P(1234) = P(2143) = P(3412) = P(4321) = \lambda \quad (1)$$
$$P(1243) = 1/\lambda, \quad (2)$$
$$P(1324) = 1 - \lambda, \quad (3)$$
$$P(1342) = 1/(1 - \lambda), \quad (4)$$
$$P(1423) = 1 - 1/\lambda = (\lambda - 1)/\lambda, \quad (5)$$
$$P(1432) = \lambda/(\lambda - 1). \quad (6).$$

§ 3. In the dual problem let 1234 be a transversal to the sides of a quadrilateral ABCD as in Figure 7.

Let a, b, c, d be the perpendicular distances of the vertices of ABCD from the transversal.

Denote by $\overline{13}$ the directed segment from 1 to 3, and by (13) the angle between the sides of the quadrilateral that pass through 1 and 3. Let sin(1234) denote sin(13) sin(24)/sin(14) sin(23).

Then
$$A1 \,.\, A3 \sin(13) = a \,.\, \overline{13} = 2\triangle A13$$
$$C2 \,.\, C4 \sin(24) = c \,.\, \overline{24} = 2\triangle C\,24$$
$$B1 \,.\, B4 \sin(14) = b \,.\, \overline{14} = 2\triangle B\,14$$
$$D2 \,.\, D3 \sin(23) = d \,.\, \overline{23} = 2\triangle D23.$$

Hence
$$\frac{A1}{B1} \cdot \frac{C2}{D2} \cdot \frac{A3}{D3} \cdot \frac{C4}{B4} \times \sin(1234)$$
$$= \frac{ac}{bd}(1234)$$

i.e.
$$\frac{ac}{bd} \sin(1234) = (1234) \quad \left(\because \frac{A1}{B1} = \frac{a}{b}; \text{ etc}\right)$$

Also
$$(A13)(C24)/(B14)(D23) = \frac{ac}{bd}(1234)$$
$$\therefore \frac{A13 \,.\, C24}{B14 \,.\, D23} \sin(1234) = (1234)^2$$

Cor. From these equations a great variety of identities may be deduced. In particular if $(1234) = -1$ we obtain

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$$\frac{(1A3)\ (2C4)}{(1B4)\ (2D3)} \times \frac{\sin(1A3)\ \sin(2C4)}{\sin(1B4)\ \sin(2D3)} = 1.$$

Similarly when ABCD are united into a single point we obtain the formula in areas given in § (1).

§ 4. If P, 1, 2, 3, 4 have co-ordinates (xyz); $(x_1y_1z_1)$; etc., then

$$P\,(1234) = \frac{(x\,y_1\,z_3)\ (x\,y_2\,z_4)}{(x\,y_1\,z_4)\ (x\,y_2\,z_3)}$$

where $(x\,y_1\,z_3)$ is the determinant whose rows are $x\ y\ z$; $x_1\ y_1\ z_1$; $x_3\,y_3\,z_3$.

For non-homogeneous co-ordinates put $z = 1$ throughout.

In the dual theorem if 1234 is the line $lx + my + n = 0$ and the lines through 1, 2, 3, 4 forming the sides of ABCD are $l_1 x + m_1 y + n_1 z = 0$, etc., then

$$(1234) = \frac{(l\,m_1\,n_3)\ (l\,m_2\,n_4)}{(l\,m_1\,n_4)\ (l\,m_2\,n_3)}.$$

For non-homogeneous co-ordinates put $n = 1$ throughout. The identity of form in these two analytical expressions enables us to dispense with the distinction of co-ordinates.

§ 5. If five points 1, 2, 3, 4, 5 are taken and P be any other point in their plane, there are but two independent cross ratios formed by joining P to four of these points.

Let	$P(1234) = \lambda$; - - - - - -	I
	$P(1235) = \mu$ - - - - - -	II
Then	$P(1254) = P(1234) / P(1235) = \lambda/\mu$ - -	III
	$P(1345) = P(1325) / P(1324) = (1-\mu) (/ 1 - \lambda)$	IV
	$P(2345) = P(2315) / P(2314)$	
	$= \lambda(\mu - 1) / \mu(\lambda - 1)$ - - -	V

It therefore follows that if *abcd* are any four of the points 12345 then P(*abcd*) is a rational function of λ and μ linear in either of the variables. The elimination of λ and μ from these relations gives rise to a great variety of identical relations.

Cor. The equations I–V also furnish the values of the cross ratio of any four of the five points 1, 2, 3, 4, 5 for the conic containing all the five points.

These may also be verified for the conic as follows. We have the identity

$$5(1234) \,.\, 3(1245) \,.\, 4(1253) = 1$$

If $\lambda = 5(1234)$

$\mu = 4(1235)$ as on the conic

then $3(1245) \times \lambda \times 1/\mu = 1$

Hence $3(1245) = \mu/\lambda$

or $3(1254) = \lambda/\mu$ as in III.

§ 6. If $P(1234) = \lambda$; $P(1235) = \mu$, then P is thereby uniquely determined. For $P(1234) = \lambda$ represents a conic through 1, 2, 3, 4; and $P(1235) = \mu$ a second conic through 1, 2, 3, 5. The conics cut in 1, 2, 3 and in the unique point P. We may therefore speak of λ and μ as being the co-ordinates of P. Only when $P(1234) = 5(1234)$; and $P(1235) = 4(1235)$ is P indeterminate, being then any point on the conic through 1, 2, 3, 4, 5.

The expression of λ and μ in trilinear co-ordinates may be conveniently found by taking 1, 2, 3 as triangle of reference. Let 4 be the point $(\alpha\,\beta\,\gamma)$, P the point $(x\,y\,z)$.

By the data

$$\lambda = \triangle P13 \cdot \triangle P24 / \triangle P14 \cdot \triangle P23;$$

and on making the calculation we find

$$\lambda = (\gamma/z - \alpha/x)/(\gamma/z - \beta/y) \qquad . \qquad . \qquad . \qquad (1)$$

Similarly if 5 is the point $(\alpha', \beta', \gamma')$

$$\mu = (\gamma'/z - \alpha'/x) / (\gamma'/z - \beta'/y) \qquad . \qquad . \qquad (2)$$

Equations (1) and (2) furnish the relations

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \lambda\,\mu\,(\beta'\gamma - \beta\gamma') - \beta'\gamma\,\mu + \beta\gamma'\lambda$$

$$: \alpha'\gamma\,(\lambda - 1) - \alpha\,\gamma'\,(\mu - 1) : \alpha'\beta\,\lambda - \alpha\beta'\mu \qquad . \qquad . \qquad (3)$$

Cor. The equation to a line being of the form $A/yz + B/zx + C/xy = 0$, it follows that to a straight line in general corresponds a cubic in λ and μ of the second degree at most in either λ or μ. If, however, 4 and 5 are chosen so that $\beta/\beta' = \gamma/\gamma'$, *i.e.*, if they are in a line with 1, then to a straight line corresponds a quadratic equation in λ and μ. For, if $\beta/\beta' = \gamma/\gamma'$

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \text{L} : \text{M} : \text{N}$$

where L, M, N are linear functions of λ and μ; and $\therefore$ to $\Sigma \text{A}/yz = 0$ corresponds

$$\text{AMN} + \text{BNL} + \text{CLM} = 0.$$

Similarly if 2, 3, 4 say are collinear, and also 1, 2, 5 then the equations become

$$\frac{\text{P13}}{\text{P14}} = c\lambda \text{ ; } \frac{\text{P13}}{\text{P23}} = c'\mu \text{ ;}$$

in which $c = 23/24$; $c' = 15/25$. In such a case to a linear equation in x and y corresponds a linear equation in λ and μ and inversely. We shall in general assume that no three points in question are collinear.

§ 7. A number of interesting geometrical theorems may be more easily deduced by taking particular algebraic relations connecting λ and μ. In what follows, ordinary cartesian co-ordinates are used, and by (Pab) is meant the determinant

$$\begin{vmatrix} x & y & 1 \\ x_a & y_a & 1 \\ x_b & y_b & 1 \end{vmatrix}$$

Ex. 1. To $\lambda = c$ a constant corresponds the conic section given by

$$(\text{P13})\ (\text{P24}) = c\ (\text{P14})\ (\text{P23}.)$$

Ex. 2. To the relation

$$\text{A}\lambda + \text{B}\mu = 0$$

corresponds likewise a conic through the points 1, 2, 4, 5 but not in general through 3.

For $\quad \text{P}(1234) / \text{P}(1235) = \text{P}(1254).$

$$\therefore \text{P}(1254) = -\text{B} / \text{A} = \text{constant}$$

Ex. 3. To $\quad \text{A}\lambda + \text{B}\mu + \text{C} = 0$

corresponds $\quad \text{A} \times \text{P}(1234) + \text{B} \times \text{P}(1235) + \text{C} = 0,$

i.e., $\text{A}(\text{P13})(\text{P15})(\text{P24}) + \text{B}(\text{P13})(\text{P14})(\text{P25}) + \text{C}(\text{P14})(\text{P15})(\text{P23}) = 0$

This equation in general represents a cubic curve possessing a double point at 1, and ordinary points at 2, 3, 4, 5.

Now the datum of a double point is equivalent to three conditions, and each ordinary point is given by one condition. There are therefore a twofold infinity of cubics possessing a double point at 1 and through 2, 3, 4, 5. Also the equation in λ and μ contains two arbitrary constants. We have, therefore, the following theorem suggested.

"Take a cubic with double point at 1, and let 2, 3, 4, 5 be any four fixed points on it. Let π_1 denote P(1234), and κ_1 denote P(1235), P being any other point on the cubic. There is a linear relation connecting π_1 and κ_1, and $(\pi_1 \pi_2 \pi_3 \pi_4) = (\kappa_1 \kappa_1 \kappa_3 \kappa_4)$."

For confirmation see Salmon's Higher Plane Curves, § 163.

Particular cases arise when 2 and 3 are the circular points at infinity.

But if $5(1234) = \alpha$, and $4(1235) = \beta$; and if $A\alpha + B\beta + C = 0$, then the curve given by $A\lambda + B\mu + C = 0$ is a degenerate cubic consisting of a conic and a straight line through 1.

Ex. 4. To the relation

$$A\lambda\mu + B\lambda + C\mu + D = 0$$

corresponds

$$\begin{aligned} &A(P13)^2(P24)(P25) + B(P13)(P23)(P15)(P24) \\ &+ C(P13)(P23)(P14)(P25) + D(P23)^2(P14)(P15) = 0 \end{aligned}$$

This equation represents a quartic curve in general, possessing double points at 1, 2, 3, and ordinary points at 4, 5.

There are three arbitrary constants in the $\lambda - \mu$ equation. But only a threefold infinity of quartics are possible possessing nodes at three given points and through other two points. We have therefore the following theorem suggested.

"Take a tri-nodal quartic with nodes at 1, 2, 3. Let 4 and 5 be any two fixed points on it, and P an arbitrary point on the curve. Then $(\pi_1 \pi_2 \pi_3 \pi_4) = (\kappa_1 \kappa_2 \kappa_3 \kappa_4)$."

There are a variety of degenerate cases. For example if ABCD are such that $A\alpha\beta + B\alpha + C\beta + D = 0$, then the quartic reduces to the base conic 12345, and a second conic through 1, 2, 3, (Degenerate cases may also arise should any of the base points be collinear).

Any relation $f(\lambda, \mu) = 0$ will furnish a degenerate curve when $f(\alpha, \beta) = 0$.

Ex. 5. If a bilinear equation is given connecting P(1234) and P(5678) the locus of P is in general a quartic; but in a large variety of cases the curve is of lower order.

(*a*.) If P(1342)/P(1562) = λ, a constant, the locus of P is a cubic

$$(P14)(P32)(P56) - \lambda(P34)(P16)(P52) = 0$$

passing through 1, 2, 3, 4, 5, 6; and through the intersection 7 of 14 and 52; 8 of 32 and 16; 9 of 56 and 34. (Figure 8.)

The nine points thus obtained form nine associated points of a pencil if cubics in triads upon two systems of three lines; and one obtains (*v.* Salmon's Higher Plane Curves) the most general form of the cubic, from which its more elementary properties are generally deduced. It will be noted that if 3, 4; and 1, 2, are given on a fixed cubic, the points 5 and 6 are uniquely determined by the collinearities:—

1 4 7	and	2 3 8
2 7 5		1 8 6.

Also 34 and 56 cut on the cubic.

The same cubic could be obtained by a variety of such equations, *e.g.*, P(5164)/P(5324) = μ, corresponding to

6 5 9	and	1 4 7
9 4 3		7 5 2.

Since all the relations are algebraic, a bilinear equation, λ = μ, connecting λ and μ is suggested for the same cubic.

(*b*.) P(2134)/P(2536) = λ gives

$$(P14)(P26)(P53) - \lambda(P56)(P24)(P13) = 0,$$

a cubic through 1, 2, 3, 4, 5, 6 and through the intersections 7, 8, 9 of 14 and 56; 26 and 13; 53 and 24. (Figure 9.)

These again form nine associated points. They may also be found in any given cubic as follows.

Take the base points 1, 2, 5; and the point 7. Form the Steinerian hexagon

7 1 4
4 2 9
9 5 3
3 1 8
8 2 6
and ∴ 6 5 7

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Then P(2134)/P(2536) = constant for all points P on the cubic. Similiar base points for the same configuration are

$$169 : 237 : 346 : 458 : 789.$$

Thus start with 1, ($\therefore$ 7438 excluded). The other two base points must come from 2569. Take 6, thereby excluding 5728, *i.e.*, excluding in addition 5 and 2, and leaving 9; $\therefore$ 169 as possible base points

Thus

4 1 7
7 6 5
5 9 3
3 1 8
8 6 2
2 9 4;

and P(6137)/P(6932) = μ a constant.

Ex. 6. Consider the cross ratios

$$1(2345);\ 2(3451);\ \ldots\ldots;\ 5(1234).$$

Equate the product of two or more of these to a constant. Fix four of the points, when the fifth traces out a curve which is at most of the fourth degree, and is generally, but not always, unicursal.

(A) *e.g.* Let 1(2345) . 2(3451) . 3(4512) = constant.

(i) Put P for 5

$$\therefore (P23)^2/(P12)(P24) = \text{constant}$$

$\therefore$ the locus of P is a pair of lines.

(ii) Put P for 4. The locus is again a degenerate conic.

(iii) Put P for 3

$$\therefore (P15)(P25)(P23)/(P12)(P24)(P13) = \text{constant}.$$

The locus is a cubic with double point at 2; through 1 and 3; through the intersection of 15 and 24, and of 25 and 13; and tangent to 15 at 1.

(iv) Put P for 2

$$\therefore (P14)^2(P35)^2/(P13)(P15)(P23)(P45) = \text{constant}.$$

The locus is a quartic with double points at 1, 3, 5, through 4, but not through 2; 23 is tangent where 14 again cuts the curve; 45 is tangent at 4.

(v) Put P for 1 when the locus is a pair of lines through 2.

(B) Equate the product of the five ratios to a constant and put P for 5, say.

$$\therefore\ P13 \,.\, P23 \,.\, P24/P12 \,.\, P34 \,.\, P14 = \text{constant}.$$

This is a cubic through 1, 2, 3, 4 : 13 is tangent at 1, 34 is tangent at 3, 42 is tangent at 4, and 21 is tangent at 2. Also 23 and 14 intersect on the curve.

The quadrilateral 1234 is therefore both inscribed and circumscribed to the cubic. A cubic curve with no double point has always a limited number of such quadrilaterals.

Ex. 7.
$$P(1234) \times P(1278) = \text{constant}$$
gives a quartic with double points at 1 and 2.

$$P(1234) \times P(1678) = \text{constant}$$
furnishes a quartic with a double point at 1.

$$P(1234)\ P(5678) = \text{constant}$$
furnishes a quartic which does not in general possess a double point.