Proceedings of the Edinburgh Mathematical Society (2005) **48**, 241–252 ©
DOI:10.1017/S0013091504000811 Printed in the United Kingdom

CRITICAL EXPONENTS FOR A REACTION–DIFFUSION MODEL WITH ABSORPTIONS AND COUPLED BOUNDARY FLUX

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(Received 10 September 2004)

Abstract This paper deals with a reaction–diffusion model with inner absorptions and coupled nonlinear boundary conditions of exponential type. The critical exponents are described via a pair of parameters that satisfy a certain matrix equation containing all the six nonlinear exponents of the system. Whether the solutions blow up or not is determined by the signs of the two parameters. A more precise analysis, depending on the geometry of Ω and the absorption coefficients, is proposed for the critical sign of the parameters.

Keywords: critical exponents; nonlinear absorption; nonlinear boundary flux; blow-up; global boundedness

2000 *Mathematics subject classification:* Primary 35K55; 35B33

1. Introduction

In this paper we consider the following reaction–diffusion model with nonlinear absorptions and coupled nonlinear boundary flux:

$$\left.\begin{aligned} &u_t = \Delta u - a_1 e^{\alpha_1 u}, && v_t = \Delta v - a_2 e^{\beta_1 v}, && (x,t) \in \Omega \times (0,T), \\ &\frac{\partial u}{\partial \eta} = e^{\alpha_2 u + pv}, && \frac{\partial v}{\partial \eta} = e^{qu + \beta_2 v}, && (x,t) \in \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x), && v(x,0) = v_0(x), && x \in \bar{\Omega}, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; $p, q, a_i > 0$, $\alpha_i, \beta_i \geqslant 0$, $i = 1, 2$, are constants; and u_0 and v_0 are non-negative functions satisfying compatibility conditions. Parabolic equations like (1.1) can be used to describe, for example, heat propagations in mixed solid nonlinear media with nonlinear absorptions and nonlinear boundary flux [**1**–**4**, **6**, **9**, **11**]. The nonlinear Neumann boundary values in (1.1), coupling the two heat equations, represent some cross-boundary flux.

The problem of heat equations

$$u_t = \Delta u, \quad v_t = \Delta v \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

coupled via somewhat special nonlinear Neumann boundary conditions

$$\frac{\partial u}{\partial n} = v^p, \quad \frac{\partial u}{\partial n} = u^q \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

was studied by Deng [**5**] and Lin and Xie [**9**], who showed that the solutions globally exist if $pq \leqslant 1$ and may blow up in a finite time if $pq > 1$ with the blow-up rates $O((T-t)^{-(p+1)/2(pq-1)})$ and $O((T-t)^{-(q+1)/2(pq-1)})$. Similarly, the blow-up rate for the corresponding scalar case of (1.2) and (1.3) was shown to be $O((T-t)^{-1/2(p-1)})$ in [**7**].

The system (coupled via a variational boundary flux of exponential type)

$$\left.\begin{array}{lll} u_t = \Delta u, & v_t = \Delta v & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = \mathrm{e}^{pv}, & \dfrac{\partial v}{\partial \eta} = \mathrm{e}^{qu} & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x) & \text{on } \bar{\Omega} \end{array}\right\} \tag{1.4}$$

was studied by Deng [**5**], and has blow-up rates

$$\left.\begin{array}{l} -\dfrac{1}{2q}\log c(T-t) \leqslant \max\limits_{\bar{\Omega}} u(\cdot,t) \leqslant -\dfrac{1}{2q}\log C(T-t), \\ -\dfrac{1}{2p}\log c(T-t) \leqslant \max\limits_{\bar{\Omega}} v(\cdot,t) \leqslant -\dfrac{1}{2p}\log C(T-t) \end{array}\right\} \tag{1.5}$$

for $t \in (0,T)$. This is the special case with $\alpha_i = \beta_i = a_i = 0$, $i = 1,2$, in our system (1.1).

Zhao and Zheng [**12**] studied the following nonlinear parabolic system:

$$\left.\begin{array}{lll} u_t = \Delta u, & v_t = \Delta v & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = \mathrm{e}^{\alpha_2 u + pv}, & \dfrac{\partial v}{\partial \eta} = \mathrm{e}^{qu + \beta_2 v} & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x) & \text{on } \bar{\Omega}. \end{array}\right\} \tag{1.6}$$

The blow-up rates for (1.6) were shown to be

$$\max_{\bar{\Omega}} u(\cdot,t) = O(\log(T-t)^{-\alpha/2}), \qquad \max_{\bar{\Omega}} v(\cdot,t) = O(\log(T-t)^{-\beta/2}) \tag{1.7}$$

as $t \to T$, where $(\alpha,\beta)^T$ is the only positive solution of

$$\begin{pmatrix} \alpha_2 & p \\ q & \beta_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

namely,

$$\alpha = \frac{p - \beta_2}{pq - \alpha_2\beta_2}, \qquad \beta = \frac{q - \alpha_2}{pq - \alpha_2\beta_2}.$$

Clearly, the blow-up rate estimate (1.5) is just the special case of (1.7) with $\alpha_2 = \beta_2 = 0$.

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To describe the critical exponents for (1.1), inspired by [**12**–**14**], we introduce parameters τ_1 and τ_2 satisfying the following matrix equation

$$\begin{pmatrix} \alpha_2 - \frac{1}{2}\alpha_1 & p \\ q & \beta_2 - \frac{1}{2}\beta_1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{1.8}$$

namely,

$$\tau_1 = \frac{p + \frac{1}{2}\beta_1 - \beta_2}{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}, \qquad \tau_2 = \frac{q + \frac{1}{2}\alpha_1 - \alpha_2}{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}. \tag{1.9}$$

Observe that all six exponents p, q, α_i, β_i $(i = 1, 2)$ of (1.1) are included in (1.8). Since for the case of $q = \alpha_2 - \frac{1}{2}\alpha_1$ with $p \neq \beta_2 - \frac{1}{2}\beta_1$, we have

$$\frac{1}{\tau_1} = \frac{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}{p + \frac{1}{2}\beta_1 - \beta_2} = q.$$

It is reasonable to define $1/\tau_1 = 2q$, $1/\tau_2 = 2p$ with both $q = \alpha_2 - \frac{1}{2}\alpha_1$ and $p = \beta_2 - \frac{1}{2}\beta_1$.

Let φ_0 be the first eigenfunction of

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega, \tag{1.10}$$

with the first eigenvalue λ_0, normalized by $\|\varphi_0\|_\infty = \max_{\bar{\Omega}} \varphi_0(\cdot) = 1$. Then $\varphi_0 > 0$ in Ω and

$$c_1 \leqslant \left| \frac{\partial\varphi_0}{\partial\eta} \right|_{\partial\Omega} = \left(-\frac{\partial\varphi_0}{\partial\eta} \right)\bigg|_{\partial\Omega} \leqslant c_2 \tag{1.11}$$

for some constants $c_1, c_2 > 0$. Moreover, there exist positive constants ε_0 and c_3 such that

$$|\nabla\varphi_0| \geqslant \tfrac{1}{2}c_1 \quad \text{for } x \in \Omega_1 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leqslant \varepsilon_0\}, \tag{1.12}$$

$$\varphi_0 \geqslant c_3 \quad \text{for } x \in \Omega_2 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geqslant \varepsilon_0\}. \tag{1.13}$$

Let

$$\max_{\bar{\Omega}} |\nabla\varphi_0| = c_4 \geqslant c_2. \tag{1.14}$$

It is well known that λ_0, ε_0 and c_i $(i = 1, 2, 3, 4)$ depend on the size and shape of Ω.

Since the system (1.1) is uniformly parabolic, the existence and uniqueness of local classical solutions to (1.1) are standard [**8**]. We say that a solution (u, v) of (1.1) blows up at finite time T if

$$\lim_{t \to T^-} \max_{\bar{\Omega}} (|u(\cdot, t)| + |v(\cdot, t)|) = +\infty.$$

The aim of this paper is to establish the critical exponents for (1.1), a simple and precise description for which will be given via parameters $1/\tau_1$ and $1/\tau_2$ defined by (1.8). Whether the solutions blow up or not is determined by the signs of $1/\tau_1$ and $1/\tau_2$. As for the critical sign of the parameters $(1/\tau_1, 1/\tau_2) = (0, 0)$, a further analysis to the geometry of Ω and the absorption coefficients will be proposed for more precise blow-up criteria.

The main results are the following theorems.

Theorem 1.1. *If $1/\tau_1 > 0$ or $1/\tau_2 > 0$, then the solutions of system (1.1) blow up in finite time with large initial data.*

Theorem 1.2. *If $1/\tau_i < 0$, $i = 1, 2$, then the solutions of (1.1) are globally bounded.*

Theorem 1.3. *Assume that $1/\tau_1 = 1/\tau_2 = 0$.*

(i) *If $\alpha_2 > \frac{1}{2}\alpha_1$ and $\beta_2 > \frac{1}{2}\beta_1$, then the solutions of system (1.1) blow up in finite time with large initial data.*

(ii) *If*

$$a_1 \geqslant 2^{\alpha_1}\left(\frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2}\right), \qquad a_2 \geqslant 2^{\beta_1}\left(\frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2}\right) \tag{1.15}$$

with $\alpha_2 < \frac{1}{2}\alpha_1$, $\beta_2 < \frac{1}{2}\beta_1$, then the solutions of (1.1) are globally bounded.

(iii) *If*

$$a_1 \leqslant \min\left\{\frac{c_1^2 M^2}{4\alpha_1}, \frac{\lambda_0 c_3^2 M^2}{\alpha_1}\right\}, \qquad a_2 \leqslant \min\left\{\frac{c_1^2 M^2}{4\beta_1}, \frac{\lambda_0 c_3^2 M^2}{\beta_1}\right\} \tag{1.16}$$

with $\alpha_2 < \frac{1}{2}\alpha_1$, $\beta_2 < \frac{1}{2}\beta_1$, $M = \min\{\alpha_1/(2c_2), \beta_1/(2c_2)\}$, then the solutions of (1.1) blow up in finite time for large initial data.

The main technique employed in this paper relies on finding suitable sub- or supersolutions, which blow up in finite time or remain bounded for all time. We refer to, for example, [**10**] for the idea and techniques of constructing such sub- and supersolutions. For example, observing that $u(x,t)$ and $v(x,t)$ considered here attain their maximum on the boundary, we can seek blowing-up subsolutions of the form $\log(M\varphi_0 + (1-ct)^K)^{-\eta}$, where φ_0 is the normalized first eigenfunction of (1.10), and M, K, η are suitable positive constants to be determined.

This paper is arranged as follows. Theorems 1.1 and 1.2 will be proved in the next two sections for blow-up and global boundedness of solutions, respectively. Section 4 deals with the more interesting critical case of $(1/\tau_1, 1/\tau_2) = (0, 0)$ in Theorem 1.3. A discussion of the critical exponents is given in the last section.

2. Blow-up of solutions

This section deals with blow-up for the solutions of (1.1) in Theorem 1.1.

Proof of Theorem 1.1. Let

$$\underline{u} = \log\frac{A}{[\varphi A^{\alpha_1/2} + (1-ct)^K]^{2/\alpha_1}}, \qquad \underline{v} = \log\frac{B}{[\varphi B^{\beta_1/2} + (1-ct)^L]^{2/\beta_1}}, \tag{2.1}$$

where $\varphi = M\varphi_0$, φ_0 is the normalized first eigenfunction of (1.10) with the first eigenvalue λ_0, and A, B, c, K, L, M are positive constants to be determined.

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For $(x,t) \in \partial\Omega \times (0, 1/c)$, we have

$$\frac{\partial \underline{u}}{\partial \eta} = \frac{2A^{\alpha_1/2}(-\partial\varphi/\partial\eta)}{\alpha_1[\varphi A^{\alpha_1/2} + (1-ct)^K]} \leqslant \frac{2A^{\alpha_1/2}Mc_2}{\alpha_1(1-ct)^K}, \qquad \frac{\partial \underline{v}}{\partial \eta} \leqslant \frac{2B^{\beta_1/2}Mc_2}{\beta_1(1-ct)^L}, \tag{2.2}$$

$$e^{\alpha_2\underline{u}+p\underline{v}} = \frac{A^{\alpha_2}B^p}{(1-ct)^{(2pL/\beta_1)+(2\alpha_2K/\alpha_1)}}, \qquad e^{q\underline{u}+\beta_2\underline{v}} = \frac{A^qB^{\beta_2}}{(1-ct)^{(2\beta_2L/\beta_1)+(2qK/\alpha_1)}}. \tag{2.3}$$

In $\Omega \times (0, 1/c)$, a simple computation shows

$$\begin{aligned}
\underline{u}_t &= \frac{2K(1-ct)^{K-1}c}{\alpha_1[M\varphi_0A^{\alpha_1/2} + (1-ct)^K]}, \qquad \underline{v}_t = \frac{2L(1-ct)^{L-1}c}{\beta_1[M\varphi_0B^{\beta_1/2} + (1-ct)^L]}, \\
\Delta\underline{u} &= \frac{2A^{\alpha_1/2}M\lambda_0\varphi_0}{\alpha_1[M\varphi_0A^{\alpha_1/2} + (1-ct)^K]} + \frac{2A^{\alpha_1}M^2|\nabla\varphi_0|^2}{\alpha_1[M\varphi_0A^{\alpha_1/2} + (1-ct)^K]^2}, \\
\Delta\underline{v} &= \frac{2B^{\beta_1/2}M\lambda_0\varphi_0}{\beta_1[M\varphi_0B^{\beta_1/2} + (1-ct)^L]} + \frac{2B^{\beta_1}M^2|\nabla\varphi_0|^2}{\beta_1[M\varphi_0B^{\beta_1/2} + (1-ct)^L]^2}, \\
a_1e^{\alpha_1\underline{u}} &= \frac{a_1A^{\alpha_1}}{[M\varphi_0A^{\alpha_1/2} + (1-ct)^K]^2}, \qquad a_2e^{\beta_1\underline{v}} = \frac{a_2B^{\beta_1}}{[M\varphi_0B^{\beta_1/2} + (1-ct)^L]^2}.
\end{aligned}$$

Next, we will treat $1/\tau_i > 0$, $i = 1$ or (and) 2, in the following four cases, respectively:

1. $\alpha_2 \leqslant \frac{1}{2}\alpha_1$, $\beta_2 \leqslant \frac{1}{2}\beta_1$.
2. $\alpha_2 > \frac{1}{2}\alpha_1$, $\beta_2 > \frac{1}{2}\beta_1$.
3. $\alpha_2 > \frac{1}{2}\alpha_1$, $\beta_2 \leqslant \frac{1}{2}\beta_1$.
4. $\alpha_2 \leqslant \frac{1}{2}\alpha_1$, $\beta_2 > \frac{1}{2}\beta_1$.

Case 1 ($\boldsymbol{\alpha_2 \leqslant \frac{1}{2}\alpha_1}$, $\boldsymbol{\beta_2 \leqslant \frac{1}{2}\beta_1}$). Observe that $1/\tau_1$ or $1/\tau_2 > 0$ and $\alpha_2 \leqslant \frac{1}{2}\alpha_1$, $\beta_2 \leqslant \frac{1}{2}\beta_1$ imply $pq > (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)$. If, for example, $\beta_2 < \frac{1}{2}\beta_1$, then $(\frac{1}{2}\alpha_1 - \alpha_2)/p < q/(\frac{1}{2}\beta_1 - \beta_2)$. Thus, for any fixed $M > 0$, there exist large constants A, B, K and L such that

$$(2Mc_2/\alpha_1)^{1/p}A^{(\alpha_1/2-\alpha_2)/p} \leqslant B \leqslant [\beta_1/(2Mc_2)]^{1/((\beta_1/2)-\beta_2)}A^{q/((\beta_1/2)-\beta_2)}, \tag{2.4}$$

$$\frac{\beta_1}{2p}\left(\frac{\alpha_1 - 2\alpha_2}{\alpha_1}\right)K < L < \frac{2q}{\alpha_1}\left(\frac{\beta_1}{\beta_1 - 2\beta_2}\right)K. \tag{2.5}$$

It follows from (2.2)–(2.5) that

$$\frac{\partial \underline{u}}{\partial \eta} \leqslant e^{\alpha_2\underline{u}+p\underline{v}}, \quad \frac{\partial \underline{v}}{\partial \eta} \leqslant e^{q\underline{u}+\beta_2\underline{v}} \quad \text{on } \partial\Omega \times (0, 1/c). \tag{2.6}$$

If $\frac{1}{2}\alpha_1 - \alpha_2 = \frac{1}{2}\beta_1 - \beta_2 = 0$, then (2.6) is obviously true because of (2.2), (2.3) for any fixed M provided A and B are large enough.

For $(x,t) \in \Omega_1 \times (0, 1/c)$, we have with (1.12) that

$$\Delta\underline{u} \geqslant \frac{2A^{\alpha_1}M^2|\nabla\varphi_0|^2}{\alpha_1[M\varphi_0A^{\alpha_1/2} + (1-ct)^K]^2} \geqslant \frac{A^{\alpha_1}M^2c_1^2}{2\alpha_1[M\varphi_0A^{\alpha_1/2} + (1-ct)^K]^2},$$

and hence

$$\Delta\underline{u} - a_1 e^{\alpha_1 \underline{u}} - \underline{u}_t \geqslant \frac{a_1 A^{\alpha_1}}{[M\varphi_0 A^{\alpha_1/2} + (1-ct)^K]^2}\left(\frac{M^2 c_1^2}{4\alpha_1 a_1} - 1\right) + \frac{1}{\alpha_1[M\varphi_0 A^{\alpha_1/2} + (1-ct)^K]}\left[\frac{A^{\alpha_1} M^2 c_1^2}{4(MA^{\alpha_1/2}+1)} - 2Kc\right] \geqslant 0, \tag{2.7}$$

provided $M^2 \geqslant 4\alpha_1 a_1/c_1^2$ and $c \leqslant M^2 c_1^2/[8K(MA^{\alpha_1/2}+1)]$.

Similarly, if we take $M^2 \geqslant 4\beta_1 a_2/c_1^2$ and $c \leqslant M^2 c_1^2/[8L(MB^{\beta_1/2}+1)]$, then

$$\Delta\underline{v} - a_2 e^{\beta_1 \underline{v}} - \underline{v}_t \geqslant 0 \quad \text{in } \Omega_1 \times (0, 1/c).$$

For $(x,t) \in \Omega_2 \times (0,1/c)$, due to (1.13), we have

$$\Delta\underline{u} \geqslant \frac{2A^{\alpha_1/2} M\lambda_0\varphi_0}{\alpha_1[M\varphi_0 A^{\alpha_1/2} + (1-ct)^K]} \geqslant \frac{2A^{\alpha_1/2} M\lambda_0 c_3}{\alpha_1[M\varphi_0 A^{\alpha_1/2} + (1-ct)^K]},$$

and hence

$$\Delta\underline{u} - a_1 e^{\alpha_1 \underline{u}} - \underline{u}_t \geqslant \frac{a_1 A^{\alpha_1}}{[M\varphi_0 A^{\alpha_1/2} + (1-ct)^K]^2}\left(\frac{\lambda_0 M^2 c_3^2}{\alpha_1 a_1} - 1\right) + \frac{2Kc}{\alpha_1[M\varphi_0 A^{\alpha_1/2} + (1-ct)^K]}\left(\frac{\lambda_0 M c_3}{2Kc} - 1\right) \geqslant 0, \tag{2.8}$$

whenever $M^2 \geqslant a_1\alpha_1/(\lambda_0 c_3^2)$, $K \geqslant 1$, and $c \leqslant \lambda_0 M c_3/(2K)$.

Similarly, if we take $M^2 \geqslant \beta_1 a_2/(\lambda_0 c_3^2)$, $L \geqslant 1$, and $c \leqslant \lambda_0 M c_3/(2L)$, then

$$\Delta\underline{v} - a_2 e^{\beta_1 \underline{v}} - \underline{v}_t \geqslant 0 \quad \text{in } \Omega_2 \times (0, 1/c).$$

In summary, if

$$M^2 = \max\left\{\frac{4\alpha_1 a_1}{c_1^2}, \frac{4\beta_1 a_2}{c_1^2}, \frac{\alpha_1 a_1}{\lambda_0 c_3^2}, \frac{\beta_1 a_2}{\lambda_0 c_3^2}\right\},$$

A, B, L and K are taken as (2.4), (2.5), and

$$c = \min\left\{\frac{M^2 c_1^2}{8K(MA^{\alpha_1/2}+1)}, \frac{M^2 c_1^2}{8L(MB^{\beta_1/2}+1)}, \frac{\lambda_0 M c_3}{2K}, \frac{\lambda_0 M c_3}{2L}\right\},$$

then $\underline{u}$ and $\underline{v}$ satisfy (2.6) on $\partial\Omega \times (0,1/c)$ and

$$\underline{u}_t \leqslant \Delta\underline{u} - a_1 e^{\alpha_1 \underline{u}}, \quad \underline{v}_t \leqslant \Delta\underline{v} - a_2 e^{\beta_1 \underline{v}} \quad \text{in } \Omega \times (0, 1/c). \tag{2.9}$$

If, moreover, the initial data are sufficiently large that $u_0(x) \geqslant \underline{u}(x,0)$, $v_0(x) \geqslant \underline{v}(x,0)$ on $\bar{\Omega}$, then $(\underline{u}, \underline{v})$ is a blowing-up subsolution of (1.1).

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Case 2 ($\boldsymbol{\alpha_2 > \frac{1}{2}\alpha_1}$, $\boldsymbol{\beta_2 > \frac{1}{2}\beta_1}$). This is a somewhat trivial case. Since $\alpha_2 > \frac{1}{2}\alpha_1$, $\beta_2 > \frac{1}{2}\beta_1$, we clearly have

$$L < \frac{2\beta_2 L}{\beta_1} + \frac{2qK}{\alpha_1}, \tag{2.10}$$

$$K < \frac{2\alpha_2 K}{\alpha_1} + \frac{2pL}{\beta_1}, \tag{2.11}$$

for any positive constants K, L. By using (2.10), (2.11) in (2.2), (2.3), we can get (2.6) on $\partial\Omega \times (0, 1/c)$ immediately provided A and B are large enough. Moreover, we can obtain (2.9) in $\Omega \times (0, 1/c)$ by the same argument as that for Case 1.

Case 3 ($\boldsymbol{\alpha_2 > \frac{1}{2}\alpha_1}$, $\boldsymbol{\beta_2 \leqslant \frac{1}{2}\beta_1}$). Under this assumption, (2.11) is obviously true for any positive constants K, L. Take $K, L, A, B > 1$ such that

$$\left(1 - \frac{2\beta_2}{\beta_1}\right) L \leqslant \frac{2qK}{\alpha_1}, \tag{2.12}$$

$$\left(\frac{2Mc_2}{\alpha_1}\right)^{(\beta_1/2-\beta_2)/p} \leqslant B^{\beta_1/2-\beta_2} \leqslant \frac{\beta_1 A^q}{2Mc_2}. \tag{2.13}$$

Due to (2.11)–(2.13), we can get the boundary inequalities of (2.6) from (2.2), (2.3).

The proof of (2.9) is similar to those in the first two cases.

Case 4 ($\boldsymbol{\alpha_2 \leqslant \frac{1}{2}\alpha_1}$, $\boldsymbol{\beta_2 > \frac{1}{2}\beta_1}$). This is the case parallel to Case 3.

Theorem 1.1 is proved. □

3. Global boundedness of solutions

We will study the global boundedness of solutions with $1/\tau_1, 1/\tau_2 < 0$ to prove Theorem 1.2 in this section.

Proof of Theorem 1.2. The proof will again be based on the comparison principle. We define the time-independent functions

$$\bar{u} = \log \frac{A}{2 - (1-\varphi)^{A^{\alpha_1/2}}}, \qquad \bar{v} = \log \frac{B}{2 - (1-\varphi)^{B^{\beta_1/2}}},$$

where $\varphi = M\varphi_0$, φ_0 is the normalized first eigenfunction of (1.10) with the first eigenvalue λ_0, and

$$M = \min\left\{1, \frac{a_1}{2^{\alpha_1}(\lambda_0 + 2c_4^2)}, \frac{a_2}{2^{\beta_1}(\lambda_0 + 2c_4^2)}\right\}, \tag{3.1}$$

and A, B are positive constants to be determined.

On $\partial\Omega \times (0, T)$, a simple computation shows

$$\frac{\partial \bar{u}}{\partial \eta} = \frac{A^{\alpha_1/2}(1-\varphi)^{A^{\alpha_1/2}-1}(-\partial\varphi/\partial\eta)}{2 - (1-\varphi)^{A^{\alpha_1/2}}} \geqslant A^{\alpha_1/2} M c_1, \qquad \frac{\partial \bar{v}}{\partial \eta} \geqslant B^{\beta_1/2} M c_1, \tag{3.2}$$

$$e^{\alpha_2 \bar{u} + p\bar{v}} = A^{\alpha_2} B^p, \qquad e^{q\bar{u} + \beta_2 \bar{v}} = A^q B^{\beta_2}. \tag{3.3}$$

We first show that $1/\tau_1 < 0$ and $1/\tau_2 < 0$ imply that $\frac{1}{2}\alpha_1 > \alpha_2$ and $\frac{1}{2}\beta_1 > \beta_2$. Note that there should be $(p + \frac{1}{2}\beta_1 - \beta_2)(q + \frac{1}{2}\alpha_1 - \alpha_2) > 0$. If $p < \beta_2 - \frac{1}{2}\beta_1$, $q < \alpha_2 - \frac{1}{2}\alpha_1$, then $pq - (\beta_2 - \frac{1}{2}\beta_1)(\alpha_2 - \frac{1}{2}\alpha_1) < 0$, a contradiction. If $p > \beta_2 - \frac{1}{2}\beta_1$, $q > \alpha_2 - \frac{1}{2}\alpha_1$, then $pq - (\beta_2 - \frac{1}{2}\beta_1)(\alpha_2 - \frac{1}{2}\alpha_1) > 0$ provided $\frac{1}{2}\alpha_1 \leqslant \alpha_2$ or (and) $\frac{1}{2}\beta_1 \leqslant \beta_2$, also a contradiction. Thus we know from $1/\tau_1, 1/\tau_2 < 0$ that $p + \frac{1}{2}\beta_1 - \beta_2 > 0$, $q + \frac{1}{2}\alpha_1 - \alpha_2 > 0$, and $pq - (\frac{1}{2}\beta_1 - \beta_2)(\frac{1}{2}\alpha_1 - \alpha_2) < 0$. Therefore, $q/(\frac{1}{2}\beta_1 - \beta_2) < (\frac{1}{2}\alpha_1 - \alpha_2)/p$. We can choose A, B sufficiently large that

$$(Mc_1)^{-1/((\beta_1/2)-\beta_2)} A^{q/((\beta_1/2)-\beta_2)} < B < (Mc_1)^{1/p} A^{((\alpha_1/2)-\alpha_2)/p}. \tag{3.4}$$

By using (3.2), (3.3) and (3.4), we get

$$\frac{\partial \bar{u}}{\partial \eta} \geqslant \mathrm{e}^{\alpha_2 \bar{u} + p\bar{v}}, \quad \frac{\partial \bar{v}}{\partial \eta} \geqslant \mathrm{e}^{q\bar{u} + \beta_2 \bar{v}} \quad \text{on } \partial\Omega \times (0, T). \tag{3.5}$$

In $\Omega \times (0, T)$, by (1.14) we have

$$\begin{aligned}\Delta \bar{u} &\leqslant \frac{A^{\alpha_1/2}\lambda_0 \varphi (1-\varphi)^{A^{\alpha_1/2}-1}}{2-(1-\varphi)^{A^{\alpha_1/2}}} + \frac{A^{\alpha_1} M^2 c_4^2 (1-\varphi)^{2(A^{\alpha_1/2}-1)}}{[2-(1-\varphi)^{A^{\alpha_1/2}}]^2} \\ &\qquad + \frac{A^{\alpha_1/2} M^2 c_4^2 (A^{\alpha_1/2}-1)(1-\varphi)^{A^{\alpha_1/2}-2}}{2-(1-\varphi)^{A^{\alpha_1/2}}} \\ &\leqslant A^{\alpha_1/2}\lambda_0 M + 2A^{\alpha_1} M^2 c_4^2.\end{aligned}$$

On the other hand, $\bar{u}_t = \bar{v}_t = 0$ and

$$a_1 \mathrm{e}^{\alpha_1 \bar{u}} = \frac{a_1 A^{\alpha_1}}{[2-(1-\varphi)^{A^{\alpha_1/2}}]^{\alpha_1}} \geqslant \frac{a_1 A^{\alpha_1}}{2^{\alpha_1}}.$$

By using (3.1), we have in $\Omega \times (0, T)$ that

$$\begin{aligned}\Delta \bar{u} - a_1 \mathrm{e}^{\alpha_1 \bar{u}} - \bar{u}_t &\leqslant A^{\alpha_1/2}\lambda_0 M + 2A^{\alpha_1} M^2 c_4^2 - \frac{a_1 A^{\alpha_1}}{2^{\alpha_1}} \\ &\leqslant \frac{a_1 A^{\alpha_1}}{2^{\alpha_1}} \left[\frac{2^{\alpha_1} M(\lambda_0 + 2c_4^2)}{a_1} - 1 \right] \\ &\leqslant 0.\end{aligned}$$

Similarly, we can get in $\Omega \times (0, T)$ that

$$\Delta \bar{v} - a_2 \mathrm{e}^{\beta_1 \bar{v}} - \bar{v}_t \leqslant 0.$$

Clearly, $\bar{u}(x, 0) \geqslant u_0(x)$ and $\bar{v}(x, 0) \geqslant v_0(x)$ on $\bar{\Omega}$ with A, B sufficiently large. We have shown that $(\bar{u}, \bar{v})$ is a time-independent supersolution of (1.1), and hence Theorem 1.2 is proved. □

https://doi.org/10.1017/S0013091504000811 Published online by Cambridge University Press

4. Critical case

Now let us consider the more interesting critical case of $(1/\tau_1, 1/\tau_2) = (0,0)$.

Proof of Theorem 1.3. It is easy to see that if $\alpha_2 > \frac{1}{2}\alpha_1$, $\beta_2 > \frac{1}{2}\beta_1$, then the argument for Case 2 in the proof of Theorem 1.1 is also valid for $(1/\tau_1, 1/\tau_2) = (0,0)$. We omit the proof for (i). We observe that $(1/\tau_1, 1/\tau_2) = (0,0)$ is equivalent to $pq = (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)$.

For (ii), we define the time-independent functions

$$\bar{u} = \log \frac{A}{2 - (\varphi + 1)^{-A^{\alpha_1/2}}}, \qquad \bar{v} = \log \frac{B}{2 - (\varphi + 1)^{-B^{\beta_1/2}}},$$

where $\varphi = M\varphi_0$, and A, B, M are positive constants to be determined.

We have for $(x,t) \in \partial\Omega \times (0,T)$ with (1.11) that

$$\frac{\partial \bar{u}}{\partial \eta} = \frac{A^{\alpha_1/2}(-\partial\varphi/\partial\eta)}{[2 - (\varphi+1)^{-A^{\alpha_1/2}}](\varphi+1)^{A^{\alpha_1/2}+1}} \geqslant A^{\alpha_1/2} M c_1, \qquad \frac{\partial \bar{v}}{\partial \eta} \geqslant B^{\beta_1/2} M c_1,$$
$$\mathrm{e}^{\alpha_2 \bar{u} + p\bar{v}} = A^{\alpha_2} B^p, \qquad \mathrm{e}^{q\bar{u} + \beta_2 \bar{v}} = A^q B^{\beta_2}.$$

Since $pq = (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)$, we can choose $A, B > 1$ satisfying $A^{q/((\beta_1/2)-\beta_2)} = B = A^{((\alpha_1/2)-\alpha_2)/p}$. Let $M = 1/c_1$. It is easy to see that (3.5) holds.

For $(x,t) \in \Omega \times (0,T)$, we have by (1.14) that

$$\begin{aligned}\Delta \bar{u} &\leqslant \frac{A^{\alpha_1/2}\lambda_0 \varphi(\varphi+1)^{-(A^{\alpha_1/2}+1)}}{2 - (\varphi+1)^{-A^{\alpha_1/2}}} + \frac{A^{\alpha_1}(\varphi+1)^{-(2A^{\alpha_1/2}+2)} M^2 c_4^2}{[2 - (\varphi+1)^{-A^{\alpha_1/2}}]^2} \\ &\qquad + \frac{A^{\alpha_1/2}(A^{\alpha_1/2}+1)(\varphi+1)^{-(A^{\alpha_1/2}+2)} M^2 c_4^2}{2 - (\varphi+1)^{-A^{\alpha_1/2}}} \\ &\leqslant A^{\alpha_1/2}\lambda_0 M + A^{\alpha_1} M^2 c_4^2 + A^{\alpha_1/2}(A^{\alpha_1/2}+1) M^2 c_4^2\end{aligned}$$

and

$$a_1 \mathrm{e}^{\alpha_1 \bar{u}} = \frac{a_1 A^{\alpha_1}}{[2 - (\varphi+1)^{-A^{\alpha_1/2}}]^{\alpha_1}} \geqslant \frac{a_1 A^{\alpha_1}}{2^{\alpha_1}}.$$

Because of (1.15) with $M = 1/c_1$ and $A > 1$, we obtain

$$\begin{aligned}\Delta \bar{u} - a_1 \mathrm{e}^{\alpha_1 \bar{u}} - \bar{u}_t &\leqslant \frac{A^{\alpha_1}}{2^{\alpha_1}} \left[2^{\alpha_1} \left(\frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2} \right) - a_1 \right] \\ &\leqslant 0,\end{aligned}$$

and, similarly,

$$\Delta \bar{v} - a_2 \mathrm{e}^{\beta_1 \bar{v}} - \bar{v}_t \leqslant 0 \quad \text{in } \Omega \times (0,T).$$

In addition, we can choose A and B sufficiently large that $\bar{u}(x,0) \geqslant u_0(x)$, $\bar{v}(x,0) \geqslant v_0(x)$ on $\bar{\Omega}$. Thus, $(\bar{u}, \bar{v})$ is a time-independent supersolution of (1.1), which implies the global boundedness of solutions to (1.1).

https://doi.org/10.1017/S0013091504000811 Published online by Cambridge University Press

For (iii), we define a pair of functions as in (2.1):

$$\underline{u} = \log \frac{A}{[\varphi A^{\alpha_1/2} + (1-ct)^K]^{2/\alpha_1}}, \qquad \underline{v} = \log \frac{B}{[\varphi B^{\beta_1/2} + (1-ct)^L]^{2/\beta_1}}.$$

Letting $M = \min\{\alpha_1/(2c_2), \beta_1/(2c_2)\}$ with $(\frac{1}{2}\alpha_1 - \alpha_2)/p = q/(\frac{1}{2}\beta_1 - \beta_2)$, we can choose constants $A, B, K, L > 1$ satisfying (2.4) and

$$\frac{\beta_1}{2p}\left(\frac{\alpha_1 - 2\alpha_2}{\alpha_1}\right)K = L = \frac{2q}{\alpha_1}\left(\frac{\beta_1}{\beta_1 - 2\beta_2}\right)K. \tag{4.1}$$

By using (4.1) with (2.4) in (2.2), (2.3), we obtain the boundary inequalities in (2.6).

It is easy to see that for M chosen above, the inequality (2.7) holds in $\Omega_1 \times (0, 1/c)$ provided a_1 is small enough that $a_1 \leqslant c_1^2 M^2/(4\alpha_1)$. Similarly, (2.8) is true in $\Omega_2 \times (0, 1/c)$ with $a_1 \leqslant \lambda_0 c_3^2 M^2/\alpha_1$. The corresponding discussion is valid for v with small a_2 in (1.16). So, the inequalities in (2.9) hold in $\Omega \times (0, 1/c)$ under the assumption (1.16). If we have, in addition, the initial data sufficiently large that $u_0(x) \geqslant \underline{u}(x, 0)$, $v_0(x) \geqslant \underline{v}(x, 0)$ on $\bar{\Omega}$, then $(\underline{u}, \underline{v})$ is a blowing-up subsolution of system (1.1). The proof of Theorem 1.3 is complete. □

5. Discussion

Finally, we discuss the main results of this paper. We have studied the interactions among multi-nonlinearities in the reaction–diffusion system (1.1). One can find from Theorems 1.1 and 1.2 that either small exponents α_1, β_1 in the absorption terms, large exponents α_2, β_2 in the boundary flux, or large coupling exponents p, q benefit the occurrence of blow-up. In addition, Theorem 1.3 says that small coefficients of absorption and small domain (i.e. large λ_0) benefit blowing up as well in the balance case between absorption and boundary flux.

The procedures to prove the main results of this paper are constructive. Due to the fact that the introduced supersolutions are time independent, we get not only global existence, but also global boundedness for the system. The blow-up results come from constructed blowing-up subsolutions. It is easy to see from the forms of the subsolutions that the blow-up could be simultaneous for each blow-up case provided both components of the initial data are large enough.

The two parameters τ_1, τ_2 from the introduced algebraic system (1.8) seem quite convenient to describe the critical exponents of (1.1). As shown through Theorems 1.1–1.3, the critical exponents are just $(1/\tau_1, 1/\tau_2) = (0, 0)$, namely, the solutions of (1.1) are globally bounded if both $1/\tau_1$ and $1/\tau_2$ are negative, and will blow up in finite time if at least one of $1/\tau_1$ and $1/\tau_2$ is positive with large initial data. For the critical sign $(1/\tau_1, 1/\tau_2) = (0, 0)$, further consideration for the absorption coefficients and the geometry of Ω should be added to determine whether or not the solutions blow-up. So, it is reasonable to call the matrix equation (1.8) the *characteristic algebraic system* of (1.1).

We point out that the classification for all six of the nonlinear exponents p, q, α_i, β_i $(i = 1, 2)$ is complete. We summarize this classification in Table 1.

Table 1. *Critical exponents*

$\alpha_2 - \frac{1}{2}\alpha_1$	$\beta_2 - \frac{1}{2}\beta_1$	$p + \frac{1}{2}\beta_1 - \beta_2$	$q + \frac{1}{2}\alpha_1 - \alpha_2$	$pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)$	$1/\tau_1$	$1/\tau_2$
$+$	$+$	$+$	$+$	$+$	$+$	$+$
			0	$+$	$+$	$+\infty$
			$-$	$+$	$+$	$-$
				0	0	0
				$-$	$-$	$+$
		0	$+$	$+$	$+\infty$	$+$
			0	0	$+$	$+$
			$-$	$-$	$-\infty$	$+$
		$-$	$+$	$+$	$-$	$+$
				0	0	0
				$-$	$+$	$-$
			0	$-$	$+$	$-\infty$
			$-$	$-$	$+$	$+$
$+$	$-$ or 0	$+$	$+$	$+$	$+$	$+$
			0	$+$	$+$	$+\infty$
			$-$	$+$	$+$	$-$
$-$ or 0	$+$	$+$	$+$	$+$	$+$	$+$
		0		$+$	$+\infty$	$+$
		$-$		$+$	$-$	$+$
$-$ or 0	$-$ or 0	$+$	$+$	$+$	$+$	$+$
				0	0	0
				$-$	$-$	$-$

Observe the columns of $1/\tau_1$, $1/\tau_2$ in the table. We can find that the last row with negative $1/\tau_1$, $1/\tau_2$ corresponds to the global-boundedness situation and that finite blow-up may occur whenever there is any positive sign of $1/\tau_1$ or $1/\tau_2$ in the table. It is more interesting to consider the critical sign $(1/\tau_1, 1/\tau_2) = (0, 0)$. There are two subcases for it: $\alpha_2 - \frac{1}{2}\alpha_1$, $\beta_2 - \frac{1}{2}\beta_1 > 0$ and $\alpha_2 - \frac{1}{2}\alpha_1$, $\beta_2 - \frac{1}{2}\beta_1 \leqslant 0$. The first one is somewhat trivial for blowing up because the boundary flux dominates the absorption even without help from the other component. As for the second subcase, we have to take into account the influence of the absorption coefficients and the geometry of Ω on the behaviour of solutions. That is considered in Theorem 1.3.

Acknowledgements. This work was supported by the National Natural Science Foundation of China.

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