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Continued Fractions Associated with $SL_3(\mathbf{Z})$ and Units in Complex Cubic Fields

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Abstract. Continued fractions associated with $GL_3(\mathbf{Z})$ are introduced and applied to find fundamental units in a two-parameter family of complex cubic fields.

1 Introduction

Denote by $\mathcal{P}$ the symmetric space $SL_3(\mathbf{R})/SO_3(\mathbf{R})$ which can be identified with the set of definite quadratic forms in three real variables with the leading coefficient 1 (see *e.g.* [9] or [18]). In [21] and [22], a continued fraction algorithm associated with a discrete group acting in a hyperbolic space was defined. The purpose of this work is to extend this definition to the case of the group $\Gamma = GL_3(\mathbf{Z})/\{\pm 1\}$ acting in $\mathcal{P}$ and apply the algorithm to find a fundamental unit of a complex cubic field.

In Section 2, the notion of the height of a point in $\mathcal{P}$ is introduced. The set $K(w)$ in $\mathcal{P}$ is defined so that, for every point $A \in \mathcal{P}$, the points in the Γ-orbit of A with the largest height belong to $K(w)$. The images $K(gw)$ of $K(w)$, $g \in \Gamma$, under the action of Γ form the *K-tessellation* of $\mathcal{P}$.

Assume that $g \in GL_3(\mathbf{R})$ has only one real eigenvalue. The set of points $L_P \in \mathcal{P}$ fixed by g will be called the *axis* of g. L_P is a geodesic in $\mathcal{P}$ (see *e.g.* [9]). The intervals $R(u) = L_P \cap K(u) \neq \varnothing$ form a tessellation of L_P. The corresponding vectors $u \in \mathbf{Z}^3$ are called the *convergents* of L_P. Let $a_1, a_2, \overline{a}_2$ be the eigenvectors of g. In Section 3, it is shown that if u is a convergent of L_P then $|(a_1, u)(a_2, u)^2 / \det(a_1, a_2, \overline{a}_2)|$ is small (Theorem 4).

In Section 4, Algorithm I is defined. It is similar to Voronoi's algorithm (see [19] or *e.g.* [26]) but it is not the same (see Section 6, Example 2). Algorithm I can be used to find all the convergents of the axis L_P of $g \in GL_3(\mathbf{R})$ which has only one real eigenvalue. It can be considered as an extension to group Γ of the algorithm which is introduced in [21] and [22]. If $g \in \Gamma$ then there are only finitely many intervals $R(u)$ which are not congruent modulo the action of Γ. Let Γ_L denote the torsion free subgroup of the stabilizer of L_P in Γ. The union of non-congruent intervals $R(u)$ form a fundamental domain of Γ_L in L_P. Thus, for $g \in \Gamma$, the continued fraction expansion is periodic (Theorem 7). Review of the multi-dimensional continued fraction algorithms and their properties known by 1980 can be found in [3].

In Section 5, Diophantine approximation properties of the convergents of the axes of g and g^T are discussed.

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Let ϵ be an eigenvalue of g. As explained in Section 6, the problem of finding a unit ϵ_1 in the ring of integers $\mathbf{Z}_F$ of the field $F = \mathbf{Q}(\epsilon)$ such that $\mathbf{Z}_F^{\times}/\{\pm 1\} = \langle \epsilon_1 \rangle$ is equivalent to the problem of finding a generator of Γ_L provided the characteristic polynomial of g is irreducible. In [25], systems of fundamental units of families of some totally real fields and quadric fields with signature $(2, 1)$ are found. In [24], Algorithm I associated with Bianchi groups (see [21] or [22]) is used to find fundamental units in families of totally complex quadric fields.

In Example 1, Algorithm I is applied to the well known family of real quadratic fields $\mathbf{Q}(\sqrt{t^2+4})$ with period length $p = 1$. In Example 2, we consider two families of complex cubic fields with period length of the corresponding continued fraction $p = 1$. In [26, p. 254], H. Williams applies Voronoi's algorithm to the same families of fields. He shows that $p = 1$ for one family and $p = 2$ for the other. It follows that Algorithm I introduced in Section 4 does not coincides with Voronoi's algorithm. The following new result is proved in Example 3.

Theorem 1 *Let $f(x) = x^3 - tx^2 - ux - 1$ where t and u are integers such that $t > u(u+1)/2$ if u is odd and $t \geq u(u+2)/2$ if u is even. Assume that $f(x)$ has only one real root ϵ. Let $F = \mathbf{Q}(\epsilon)$. Assume that the discriminant of $f(x)$ is square free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of the ring of integers $\mathbf{Z}_F$ of F and $\mathbf{Z}_F^{\times}/\{\pm 1\} = \langle \epsilon \rangle$.*

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2 Fundamental Domains and K-Tessellation

Let V_3 be the vector space of symmetric 3×3 real matrices. The dimension of V_3 is 6. The action of $g \in G = \mathrm{GL}_3(\mathbf{R})$ on $X \in V_3$ is given by

$$X \longmapsto X[g] = g^T X g.$$

For a subset S of V_3, denote $S[g] = \{X[g] \in V_3 : X \in S\}$.

The one-dimensional subspaces of V_3 form the the five-dimensional real projective space V, so that, for any fixed nonzero $X \in V_3$, all the vectors kX, $0 \neq k \in \mathbf{R}$, represent one point in V. Denote by $\mathcal{P} \subset V$ the set of (positive) definite elements of V and by C the boundary of $\mathcal{P}$ (C can be identified with non-negative elements of V of rank less than 3). The group G preserves both $\mathcal{P}$ and C as does its arithmetic subgroup $\Gamma = \mathrm{GL}_3(\mathbf{Z})$.

The space V_3 (and V) can be also identified with the set of quadratic forms $A[x] = x^T A x$, $A \in V_3$, $x \in \mathbf{R}^3$. With each point $a = (a_1, a_2, a_3)^T \in \mathbf{R}^3$, we associate the matrix $A = aa^T \in C$ and quadratic form

$$A[x] = (a, x)^2 = (a_1 x_1 + a_2 x_2 + a_3 x_3)^2 \tag{1}$$

of rank 1. For $g \in G$, we have $(ga, x) = a^T g^T x = (a, g^T x)$.

Denote $w = (1, 0, 0)^T$ and $W = ww^T$. Then $(w, x)^2 = x_1^2$ and $W[g] = U = uu^T \in C$ where $u = g^T w$.

Denote the stabilizer of W in G (Γ) by G_∞ (Γ_∞). Then

$$G_\infty = \{g \in G : gw = w\} = \{g \in G : g_1 = w\}$$

where g_1 is the first column of g. Thus, $g \in G_\infty$ iff $W[g^T] = W$.

We shall say that $A = (a_{ij}) \in V$ is *w-extremal* if $|A[x]| \geq |A[w]| = a_{11}^2$ for any $x \in \mathbf{Z}^n/(0,0,0)$. Let $\mathcal{A}_3 = \{X \in V : X[w] \neq 0\}$. The elements of $\mathcal{A}_3$ will be normalized so that $X[w] = 1$. Evidently, $\mathcal{P} \subset \mathcal{A}_3$. For $X \in V$, we shall say that

$$\mathrm{ht}(X) = |\det(X)|^{1/3}/|X[w]|$$

is the *height* of X and, for a subset S of V, we define the *height* of S as

$$\mathrm{ht}(S) = \max \mathrm{ht}(X), \quad X \in S.$$

Thus, if $X \in \mathcal{A}_3$ then $\mathrm{ht}(X) = |\det(X)|^{1/3}$. For a fixed $g \in \Gamma$, the set

$$p(g) = \{X \in \mathcal{A}_3 : |X[gw]| < 1\}$$

is called the *g-strip* (*cf.* [23], [20] where this definition is introduced for $\Gamma = \mathrm{GL}_2(\mathbf{Z})$). It is clear that $p(gh) = p(g)$ and $\mathrm{ht}(X[h]) = \mathrm{ht}(X)$ for any $h \in \Gamma_\infty$. The set

$$L^+(g) = \{X \in \mathcal{A}_3 : X[gw] = 1\}$$

is the boundary of the g-strip $p(g)$ which cuts $\mathcal{P}$. The set $\mathcal{R}_w$ of all w-extremal points of V will be called the w-reduction region of Γ. We denote

$$K(w) = \mathcal{P} \cap \mathcal{R}_w.$$

(In the notation of [2, p. 148], $K(w)$ is the dual core of $K_{p\,\mathrm{erf}}$.) Note that $K(w) \subset \mathcal{A}_3$ is bounded by the planes $L^+(g)$. By Margulis' theorem [15], all the points of $\mathcal{R}_w - \mathcal{P}$ are rational.

Let D be any of the fundamental domains of Γ obtained by Minkowski, Korkine and Zolotarev (see *e.g.* [17, p. 13]), or Grenier [11]. For $X \in D$, $X[w] = \inf X[gw]$, $g \in \Gamma$, in any of these cases. Hence $\bigcup D[g] = K(w)$, the union being taken over all $g \in \Gamma_\infty$. Note that the fundamental domain described in [11] coincides with the domain found by Korkine and Zolotarev in 1873 (see [13] or [17]). In Section 6, to prove that a point $X \in \mathcal{P}$ is extremal we shall show that, for some $h \in \Gamma_\infty$, $X[h]$ is Minkowski reduced.

For $g \in \Gamma$, let

$$K(gw) = \{X \in \mathcal{P} : X[g] \in K(w)\}.$$

If $X \in K(w)$, then $X[h] \in K(w)$ for any $h \in \Gamma_\infty$. Hence if $X \in K(gw)$ then $X[gh] \in K(w)$ for any $h \in \Gamma_\infty$. Thus, the sets $K(gw)$ are parameterized by the classes Γ/Γ_∞ or by primitive vectors $u \in \mathbf{Z}^3/(0,0,0)$ so that $\pm u$ represent the same $K(u)$.

The sets $K(gw)$, $g \in \Gamma/\Gamma_\infty$, form a tessellation of $\mathcal{P}$ which will be called the *K-tessellation*. All the vertices of $K(w)$ are congruent to $v = x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_3x_1$ which is called a *perfect* form (see *e.g.* [2] or [17]). Thus, the Hermite's constant $\gamma_3 = 1/\inf \mathrm{ht}(X) = 1/\mathrm{ht}(v) = 2^{-1/3}$. Here the infimum is taken over all $X \in K(w)$.

3 Axes of Irreducible Elements of Γ

Given $a_1, \alpha, \beta \in \mathbf{R}^3$. Let $P = (a_1, a_2, \overline{a_2})$ be the matrix with columns $a_1, a_2, \overline{a_2}$ where $a_2 = \alpha + i\beta$. Denote $A_1 = a_1 a_1^T$ and $A_2 = \alpha\alpha^T + \beta\beta^T$. Let L_P be the interval in $\mathcal{P}$ with endpoints $A_1, A_2 \in C$. The stabilizer of L_P in G consists of $g = PHP^{-1}$, $H = \operatorname{diag}(\lambda_1, \lambda_2, \overline{\lambda_2})$, $\lambda_1 \in \mathbf{R}$, $\overline{\lambda_2} \neq \lambda_2 \in \mathbf{C}$, so that $ga_i = \lambda_i a_i$, where a_i is the fixed eigenvector of g corresponding to its eigenvalue λ_i. Assume that $(a_i, w) \neq 0$, $i = 1, 2$. Then we can choose a_i so that

$$(a_i, w) = 1, \quad i = 1, 2. \tag{2}$$

Assume that (2) holds. The geodesic L_P in $\mathcal{P}$ fixed by g will be called the *axis* of g. It can be identified with the interval $q = \mu A_1 + (1-\mu)A_2$ or with the set of quadratic forms in $\mathcal{A}_3$:

$$q[x] = \mu(x, a_1)^2 + (1-\mu)|(x, a_2)|^2, \quad 0 \leq \mu \leq 1. \tag{3}$$

Since $\det P = 2i \det(a_1, \beta, \alpha)$ and $q[x] = \mu(x, a_1)^2 + (1-\mu)\big((x,\alpha)^2 + (x,\beta)^2\big)$, we have

$$\det q = -\mu(1-\mu)^2(\det P)^2/4.$$

Hence, $|\det q| \leq |\det P|^2/27$ where the equality is attained when $\mu = 1/3$. It follows that, for any L_P, $\operatorname{ht}(X) = |\det(X)|^{1/3} \to 0$ as X approaches the boundary of L_P, and the point

$$q_m[x] = \frac{1}{3}(x, a_1)^2 + \frac{2}{3}|(x, a_2)|^2$$

is the *summit* of L_P that is $\det(q_m) = \max \det(q)$, the maximum being taken over all $q \in L_P$ and, since $q_m \in \mathcal{A}_3$, $\operatorname{ht}(L_P) = \operatorname{ht}(q_m) = (\det q_m)^{1/3} = |\det P|^{2/3}/3$. It is clear that if $R = L_P \cap K(w) \neq \varnothing$ then $q_m \in R$. Note that $3q_m[x]$ is the form size (M_x) from [5, p. 169].

Let $N_P(x) = (x, a_1)|(x, a_2)|^2$ where $(x, a_i) = x^T a_i$. Define

$$\nu(L_P) = \inf \left| \frac{N_P(gw)}{\det P} \right| \tag{4}$$

where the infimum is taken over all $g \in \Gamma$. Evidently $\nu(L_P) = \nu(L_{MP}[h])$ for any $h \in \Gamma$ and $M = \operatorname{diag}(\mu_1, \mu_2, \overline{\mu_2})$, $\mu_1\mu_2 \neq 0$. The projective invariant $\nu(L_P)$ is well known in Geometry of Numbers (see *e.g.* [4]). Since $\operatorname{ht}(L_P) = |\det P|^{2/3}/3$ when (2) hold we have obtained the following.

Lemma 2 *Let L_P be the geodesic in $\mathcal{P}$ fixed by $g \in G$ and defined by (3) where $ga_i = \lambda_i a_i$. Let $P = (a_1, a_2, \overline{a_2})$ be the matrix with columns $a_1, a_2, \overline{a_2}$. Then*

$$\operatorname{ht}(L_P) = \frac{1}{3}\left| \frac{\det P}{N_P(w)} \right|^{2/3}$$

and

$$\nu(L_P) = \inf\big(3\operatorname{ht}(L_P[h])\big)^{-3/2}, \quad h \in \Gamma. \qquad \blacksquare$$

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Assume that $L_P \cap K(w) = \varnothing$. Let q_m be the summit of L_P. Since $q_m \notin K(w)$ there is $g \in \Gamma$ such that $\mathrm{ht}(L_P[g]) \geq \mathrm{ht}(q_m[g]) > \mathrm{ht}(q_m) = \mathrm{ht}(L_P)$. We have obtained the following.

Lemma 3 *Let L_P be the totally geodesic manifold fixed by $g \in G$ and defined by (3) where $ga_i = \lambda_i a_i$. Then*

$$\nu(L_P) = \inf\bigl(3\,\mathrm{ht}(L_P[h_i])\bigr)^{-3/2}, \quad L_P \cap K(h_i w) \neq \varnothing.$$

Thus, $\nu(L_P) < (\gamma_3/3)^{3/2} = \sqrt{2/27} = 0.2722.$ ■

It was shown by Davenport (see [8]) that $\sup \nu(L_P) = 1/\sqrt{23} = 0.2085$ where the equality holds only if $ga_i = (1, \alpha_i, \alpha_i^2)$, $i = 1, 2, 3$, for some $g \in \Gamma$. Here α_i are the roots of $x^3 - x - 1 = 0$.

Assume that $L_P \cap K(gw) \neq \varnothing$ where $g \in \Gamma$. Since $L_P[g] \cap K(w) \neq \varnothing$, by Lemma 1,

$$\mathrm{ht}(L_P[g]) = \mathrm{ht}(L_{g^T P}) = \frac{1}{3}\left|\frac{\det P}{N_{g^T P}(w)}\right|^{2/3} > 2^{-1/3}.$$

But $N_{g^T P}(x) = (x, g^T a_1)|(x, g^T a_2)|^2 = (gx, a_1)|(gx, a_2)|^2$. Hence $N_{g^T P}(w) = N_P(gw)$.

The vector $gw \in \mathbf{Z}^3$ such that $L_P \cap K(gw) \neq \varnothing$ will be called a *convergent* of L_P. We have proved the following.

Theorem 4 *If vector u is a convergent of L_P, that is if $L_P \cap K(u) \neq \varnothing$, then*

$$|N_P(u)| < \sqrt{\frac{2}{27}}\,|\det P|.$$

Hence if L_P cuts infinitely many sets $K(u)$ then this inequality has infinitely many solutions in $u \in \mathbf{Z}^3$. ■

We shall say that the intervals $R(u), R(u') \subset L_P$ are *neighbors* if $\overline{R}(u) \cap \overline{R}(u') \neq \varnothing$ in which case the convergents u and u_i are *neighbors*. The following lemma can be used to find the endpoints of $R = L_P \cap K(w) \neq \varnothing$.

Lemma 5 *Let L_P be the axis of $g \in G$. Assume that $R = L_P \cap K(w) \neq \varnothing$. Let $R' = L_P \cap K(u')$ be a neighbor of R and $\overline{R} \cap \overline{R}' = X$. Then the point $X \in L^+(u')$ and $X = L_P \cap L^+(u')$.*

Proof Assume that $K(w)$ and $K(gw)$ have a common face and that $X \in \overline{K}(w) \cap \overline{K}(gw)$. By the definition of $K(gw)$, $X[g] \in \overline{K}(w)$. Hence $X[w] = X[gw] = 1$ and $X \in L^+(gw)$. Thus, the common face of $K(w)$ and $K(gw)$ lies in $L^+(gw)$. ■

4 Continued Fractions

The axis of $h \in G$ is a geodesic $L = L_P$ in $\mathcal{P}$. It can be identified with the interval $X(\mu) = \mu(x, a_1)^2 + (1-\mu)|(x, a_2)|^2$, $0 < \mu < 1$, where a_1 and a_2 are eigenvectors of h corresponding to its real and complex eigenvalues respectively. Denote

$$R_i = [X_i, X_{i+1}] = L \cap K(u_i), \quad X_i = X(\mu_i), \quad u_i = g_i w, \quad g_i \in \Gamma. \tag{5}$$

The intervals R_i form a tessellation of $L = L_P$. We say that this tessellation is *periodic* if there are only finitely many non-congruent R_i's modulo the action of the stabilizer Γ_L of L in Γ. In that case, the union of all non-congruent R_i's is a fundamental domain of Γ_L in L and $\mathrm{vol}(L/\Gamma_L) < \infty$. The number of non-congruent R_i's in the tessellation of L will be called the *period length.*

The (continued fraction) Algorithm I can be used to find the sequence $\{g_i\} \subset \Gamma$ such that $L \cap K(u_i) \neq \varnothing$ and the sequence of convergents $u_i = g_i w$ of L explicitly. The corresponding shift operator is defined on the sequences

$$\ldots, R_{-1}, R_0, R_1, R_2, \ldots, R_i, \ldots \tag{6}$$

and

$$\ldots, u_{-1}, w, u_1, u_2, \ldots, u_i, \ldots .$$

Let D be the Minkowski fundamental domain of $\Gamma' = \Gamma/\{\mathrm{diag}(1, \pm 1, \pm 1)\}$, that is

$$A = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & a & x_3 \\ x_2 & x_3 & b \end{pmatrix} \tag{7}$$

belongs to D iff $1 \le a \le b$, $0 \le |x_1|, |x_2| \le 1/2$, $0 \le |x_3| \le a/2$, $2(|x_1|+|x_2|+|x_3|) \le 1 + a$ [6, pp. 396–397]. Recall that the *floor* of D consists of the faces of D which do not pass through w. Denote

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 6 *A point A in (7) belongs to the floor of the fundamental domain D of Γ' is Minkowski reduced if and only if*

$$1 = a \le b, \quad 0 \le |x_1|, |x_2|, |x_3| \le 1/2, \quad |x_1| + |x_2| + |x_3| \le 1.$$

Hence the floor of D lies in the boundary $L^+(S_1)$ of the strip $p(S_1)$.

Proof Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbf{Z}^3$. It follows that $A[e_2] = a \leq A[e_3] = b$, $A[e_2 \pm e_3] = a \pm 2x_3 + b \geq b \geq a$ (similarly, $A[e_i \pm e_j] \geq a$, $i \neq j$), and $A[e_1 \pm e_2 \pm e_3] = 1 \pm 2x_1 \pm 2x_2 + a \pm 2x_3 + b \geq b \geq a$. Thus, if A is a boundary point then $A[e_2] = a = 1$. ∎

By Lemma 6, the floor of D consists of one face ϕ of D which lies in $L^+(S_1)$. It is clear that

$$\phi = \overline{D} \cap \overline{D}[S_1].$$

Since for any orbit $z[\Gamma]$ of $z \in \mathcal{P}$, a point of the largest height in the orbit belongs to the fundamental domain D, we can confine ourself to the geodesics which pass through D.

We now introduce the natural *orientation* of a geodesic $L' = L[g]$ (from $\mu = 0$ to $\mu = 1$ so that $\mu_i \to 0$ as $i \to \infty$). The partition of L' into intervals R'_i is defined by (5). It is clear that this partition is invariant under the action of $g \in \Gamma$, that is $R'_i = R_i[g]$ for all i.

We shall say that a geodesic L' is *reduced* if it passes through D and the initial point of $R' = L' \cap K(w)$ lies in ϕ, the floor of D.

Algorithm I

Step 0 If L does not cut $K(w)$ take a point $X \in L$ and find $h \in \Gamma$ such that $X[h] \in K(w)$. (Any of the reduction algorithms (see *e.g.* [6] for references) can be used to find such an h.) Then $L[h]$ cuts $K(w)$. Thus we can assume that $[X', X''] = L \cap K(w)$. Suppose that $X' \in \phi[U_0]$, $U_0 \in \Gamma_\infty$. Denote $L'_0 = L[U_0^{-1}]$. Clearly, L'_0 cuts the floor ϕ of D and it is not reduced.

Step 1 Let $X_1 \in \phi$ be the point of intersection of L'_0 with the floor of D. Denote $g_0 = T_0 = S_1 U_0$, and $L_1 = L'_0[S_1]$. Then $L = L_1[g_0]$ where L_1 is reduced.

Assume that the elements $T_1, \dots, T_{i-1}$ in Γ are determined. Let $g_k = T_k g_{k-1}$ and $L_k = L_{k+1}[T_k]$, $k = 1, \dots, i-1$. Then $L = L_i[T_{i-1} \cdots T_0]$.

Step $i+1$ Let $R_i = [X_i, X_{i+1}] = L_i \cap K(w)$. Let $L'_i = L_i[U_i^{-1}]$ where $U_i \in \Gamma_\infty$ is determined so that $X_{i+1}[U_i^{-1}]$ lies in the floor ϕ of D. Denote $T_i = S_1 U_i$, and $L_{i+1} = L'_i[S_1]$. Then

$$g_i := T_i g_{i-1}$$

and

$$L_i = L_{i+1}[T_i], \quad L = L_{i+1}[T_i \cdots T_0].$$

It is clear that Algorithm I enumerates $g_i \in \Gamma$ in the same order as L passes through the sets $K(g_i w)$, and that there is a 1-1 correspondence between the intervals R_i of L and $T_i \in \Gamma$ as defined by Algorithm I. The corresponding convergents $u_i = g_i w$ satisfy the relation $u_i = T_i u_{i-1}$.

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Remark Denote

$$S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In the Voronoi continued fraction algorithm [19], $T_i = S_2 U_i$ where $U_i \in \Gamma_\infty$. Since $S_1 = S_2 \tau$ where $\tau \in \Gamma_\infty$, in Algorithm I, T_i can be also written in this form. But these two algorithms do not coincide (see Section 6, Example 2).

Let L be the axis of an irreducible element $h \in \Gamma$ with only one real eigenvalue. Let L^o be the a fundamental domain of the cyclic group generated by h on L chosen so that it consists of whole intervals $R_1, \dots, R_p$. Note that $R_{i+p} = h(R_i)$ and $L_{i+p} = L_i$ for all i. Thus the sequence T_i, as generated by Algorithm I, is also periodic, $T_{i+p} = T_i$ for all i, and $h = T_p \cdots T_1$. We have the following.

Theorem 7 *The sequence of intervals (6) of a geodesic L is periodic if and only if L is the axis of an irreducible element in Γ. (If $R_{i+p} = R_i$ and $h = T_p \cdots T_1$, then $L[h] = L$.)*

Suppose that $L = L_0$ is reduced. There are only finitely many reduced geodesics $L_1, \dots, L_p = L_0$ in the Γ-orbit of L and Algorithm I can be used to find all of them. Also,

$$\nu(L) = \inf\bigl(3\,\mathrm{ht}(L_i)\bigr)^{-3/2}, \quad 1 \le i \le p,$$

where $L = L_{i+1}[T_i \cdots T_0]$ and the sequence T_i is generated by Algorithm I.

In particular, if the fundamental domain of $\mathrm{Stab}(L, \Gamma)$ *on L belongs to $K(w)$ (in which case $p = 1$), then $\nu(L) = \bigl(3\,\mathrm{ht}(L)\bigr)^{-3/2}$.*

If the tessellation of a geodesic L is periodic and $R(u_1) \cup \cdots \cup R(u_p) = D_L$, a fundamental domain of $\Gamma_L = \langle h \rangle$, then the set of all convergents of L_P is $\{h^n u_i, i = 1, \dots, p, n \in \mathbf{Z}\}$.

5 Diophantine Approximations

Let vectors $b_j \in \mathbf{C}^3$ be defined by

$$(b_j, a_i) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

Here $a_3 = \overline{a_2}$, $b_3 = \overline{b_2}$ and $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \ne j$. Thus, $(P^T)^{-1} = P^* = (b_1, b_2, \overline{b_2})$ and, assuming $(b_k, w) \ne 0$, $k = 1, 2$, the axis $L^* = L_{P^T}$ of $g^* = P^* H^{-1} P^T$ can be identified with the set of positive definite quadratic forms $q^*[x] = \mu(x, b_1')^2 + (1 - \mu)|(x, b_2')|^2$, $0 < \mu < 1$, where $b_k' = b_k/(b_k, w)$, $k = 1, 2$. The rank of the quadratic form $A_2[x] = |(x, a_2)|^2$ is two and $A_2[b_1] = 0$ since $(b_1, a_2) = 0$. It is easily seen that L^* can be described as follows.

Lemma 8 *Let L_P be the axis of $g \in G$. Then*

$$L^* = \{q \in \mathcal{P} : q^{-1} \in L_P\}$$

is the axis of $g^* = (g^T)^{-1}$. ∎

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Lemma 9 *Let $R_i = L_P \cap K(u_i)$, $u_i = g_i w$, $g_i \in \Gamma$. If $R_i \to A_1$ then $(u_i, a_1) \to 0$, and if $R_i \to A_2$ then $u_i/(u_i, w) \to b_1'$.*

Let $R_i^* = L^* \cap K(v_i)$, $v_i = h_i w$, $h_i \in \Gamma$. Similarly, if $R_i^* \to B_1$ then $(v_i, b_1) \to 0$, and if $R_i^* \to B_2$ then $v_i/(v_i, w) \to a_1$.

Here $B_1[x] = (x, b_1)^2$ and $B_2[x] = |(x, b_2)|^2$.

Proof Let $X_i \in R_i$. Then $X_i' = X_i[g_i] \in K(w)$. Hence $\mathrm{ht}(X_i') \geq 2^{-1/3}$. Since $\mathrm{ht}(X_i) = |\det(X_i)|^{1/3}$ and $\mathrm{ht}(X_i') = |\det(X_i)|^{1/3}/X_i[u_i]$, we have

$$X_i[u_i] = \mathrm{ht}(X_i)/\,\mathrm{ht}(X_i') \leq 2^{1/3}\,\mathrm{ht}(X_i).$$

Since $X_i \to A_1$, $\mathrm{ht}(X_i) \to 0$ and therefore $X_i[u_i] \to 0$ and $A_1[u_i] = (a_1, u_i)^2 \to 0$ as required.

Similarly the other cases can be considered. ∎

In Lemma 9, when L_P is the axis of a primitive $h \in \Gamma$, the rate of convergence in $(u_i, a_1) \to 0$ and $(v_i, b_1) \to 0$ can be specified. Assume that $ha_k = \lambda_k a_k$, $k = 1, 2$, where $\lambda_1 \in \mathbf{R}$, $\lambda_2 = \overline{\lambda}_3 \in \mathbf{C}$ and $|\lambda_1| < 1 < |\lambda_2|$. Then the sequence (6) is periodic. Let $R_1, \dots, R_p$ be a period of this sequence. Let the corresponding convergents be u_i, $i = 1, \dots, p$. Then the convergents of L_P are $u_{i+np} = (h^T)^n u_i$ for any integer n and $1 \leq i \leq p$. Similarly, by Lemma 6, if $R_1^*, \dots, R_s^*$ is a period for L^* and the corresponding convergents are v_j, $j = 1, \dots, s$, then the convergents of L^* are $v_{j+ns} = h^n v_j$ for any integer n and $1 \leq j \leq s$. Hence $(a_1, u_{i+np}) = \big(a_1, (h^T)^n u_i\big) = (h^n a_1, u_i) = \lambda_1^n (a_1, u_i) \to 0$, $1 \leq i \leq p$, and $(b_1, v_{j+ns}) = (b_1, h^n v_j) = \big((h^T)^n b_1, v_j\big) = \lambda_1^n (b_1, v_j) \to 0$, $1 \leq j \leq s$, as $n \to \infty$.

Let L be a 1-flat defined by $X(\mu) = \mu(x, a_1)^2 + (1-\mu)|(x, a_2)|^2$ where $x = (x_1, x_2, x_3) \in \mathbf{Z}^3$, and let $a_1 = (1, \alpha, \beta)$ and a_2 be the eigenvectors of $g \in G$ corresponding to the real and complex eigenvalues of g respectively so that $L[g^T] = L$. Assume that $x_1 + \alpha x_2 + \beta x_3 \neq 0$ for any $(x_1, x_2, x_3) \in \mathbf{Z}^3/(0,0,0)$. The main property of the constant $\nu(L)$ is that the inequality

$$|N_P(x)| = |(x, a_1)(x, a_2)^2| < k|\det P|$$

or

$$|(x, a_1)| = |x_1 + \alpha x_2 + \beta x_3| < k\frac{|\det P|}{A_2[x]}, \tag{8}$$

has infinitely many solutions $x \in \mathbf{Z}^3$ for $k \geq \nu(L)$ and only a finite number of solutions if $k < \nu(L)$. Here $A_2[x] = |(x, a_2)|^2$ is a quadratic form of rank two.

Let $b_1' = (1, \alpha_1, \beta_1)$, $a_2 = (1, \gamma_1 + i\delta_1, \gamma_2 + i\delta_2)$, and $(a_2, b_1') = 0$. Then $\alpha_1 = -\delta_2/\Delta$, $\beta_1 = \delta_1/\Delta$ where $\Delta = \gamma_1\delta_2 - \gamma_2\delta_1$. For $y = (0, y_2, y_3)$, $q_2(y_2, y_3) = |(y, a_2)|^2$ is a positive definite binary quadratic form with $\det(q_2) = -\Delta^2$. If $(x, a_2) \to 0$ then

$$|(x, a_2)|^2 = |(x - x_1 b_1', a_2)|^2 = x_1^2 q_2\left(\frac{x_2}{x_1} - \alpha_1, \frac{x_3}{x_1} - \beta_1\right) \to 0$$

and

$$\frac{x_2}{x_1} - \alpha_1 \longrightarrow 0, \quad \frac{x_3}{x_1} - \beta_1 \longrightarrow 0,$$

since $x - x_1 b_1' = (0, x_2 - x_1\alpha_1, x_3 - x_1\beta_1)$. It follows that $(x, a_1) = x_1(1 + \frac{x_2}{x_1}\alpha + \frac{x_3}{x_1}\beta) \approx x_1(1 + \alpha\alpha_1 + \beta\beta_1) = x_1(b_1', a_1)$. Since $\det P = (-2i)(\Delta - \delta_2\alpha + \delta_1\beta) = -2i\Delta(b_1', a_1)$, the inequality $|N_P(x)| = |(x, a_1)(x, a_2)^2| < k|\det P|$ implies

$$q_2\left(\frac{x_2}{x_1} - \alpha_1, \frac{x_3}{x_1} - \beta_1\right) < \frac{2k}{|x_1|^3}\sqrt{|\det q_2|}. \tag{9}$$

In [7], it is shown that, for any irrational α_1, β_1, this inequality has infinitely many solutions in $x \in \mathbf{Z}^3$ if $k \geq 1/\sqrt{23}$ and that this constant is exact in the case when $q_2(y_2, y_3) = y_2^2 + y_3^2$.

As $x_1 + \alpha x_2 + \beta x_3 \to 0$, we have $x_1 \approx -\alpha x_2 - \beta x_3$ and $A_2[x] \approx q_1(x) = |(x, a_2 - a_1)|^2$. Here q_1 is a binary quadratic form in x_2 and x_3 with $\det(q_1) = -(\Delta - \delta_2\alpha + \delta_1\beta)^2 = -|\det P|^2/4$. Hence $|\det P| = 2\sqrt{|\det q_1|}$ and the inequality (8) can be rewritten as

$$|x_1 + \alpha x_2 + \beta x_3| < 2k\frac{\sqrt{|\det q_1|}}{q_1(x)}. \tag{10}$$

By Theorem 4 and Lemma 9, the inequality (9) holds with $k = \sqrt{2/27}$ for almost all $x = u_i$ such that $R_i \to A_2$, and (10) holds with the same constant for almost all $x = u_i$ such that $R_i \to A_1$.

In general, if we replace $q_1(x)$ by another binary positive quadratic form then $\det P$ and $\nu(L)$ can be changed. Thus, to compare diophantine approximation properties of different vectors $(1, \alpha, \beta)$ we have to fix the form $q_1(x)$. Choose $q_1(x) = x_2^2 + x_3^2$. As mentioned above, for this particular $q_1(x)$, Davenport and Mahler [7] proved that $\sup \nu(L) = 1/\sqrt{23}$ and that the supremum is attained when $a_1 = (1, \phi, \phi^2)$ where ϕ is the real root of the equation $t^3 - t - 1 = 0$. Since $\nu(L) = 1/\sqrt{23}$ when $a_2 = (1, \theta, \theta^2)$, θ being a complex root of $t^3 - t - 1 = 0$, the inequality (10) also holds for the same a_1 with the constant $k = 1/\sqrt{23}$ and $q_1(x) = |(x, a_2 - a_1)|^2$.

Note that the isotropic vector b_1 of the quadratic form $|(x, a_2)|^2$ is used in [1] to find the Voronoi-algorithm expansion for units in two families of complex cubic fields with period length going to infinity introduced by Levesque and Rhim [14]. In [12], the same approach is applied to a similar family of fields.

6 Units in Complex Cubic Fields

As in [5], we denote by $\mathbf{Z}_F$ the ring of integers of an algebraic number field F. A $\mathbf{Z}$-basis of the free $\mathbf{Z}$-module $\mathbf{Z}_F$ will be called a *basis* of $\mathbf{Z}_F$. If, for some $\delta \in \mathbf{Z}_F$, numbers $1, \delta, \delta^2, \dots, \delta^{n-1}$, $n = \deg(F)$, form a basis of $\mathbf{Z}_F$, it is called a *power basis* (*cf.* [16, p. 64]).

Let F be a complex cubic field. Let $\{1, \omega_2, \omega_3\}$ be a basis of $\mathbf{Z}_F$. Denote $a_1 = (1, \omega_2, \omega_3)^T$. Let ϵ_1 be a unit in $\mathbf{Z}_F$. Then $\epsilon_1 a_1 = E a_1$ where $E \in \Gamma$. Hence ϵ_1 is an eigenvalue of E and a_1 is the eigenvector of E corresponding to ϵ_1. Assume that ϵ_1 is real. Let σ_i be the three distinct embeddings of F in $\mathbf{C}$. Let $a_2 = \sigma_2(a_1) = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}^3$, $a_3 = \overline{a_2}$ and $\epsilon_i = \sigma_i(\epsilon_1)$. Then $\epsilon_i a_i = Ea_i$ and the axis L of E is the interval $X(\mu) = \mu A_1 + (1-\mu)A_2$, $0 \le \mu \le 1$, where $A_1 = a_1 a_1^T$ and $A_2 = \alpha\alpha^T + \beta\beta^T$. On the other hand, if the characteristic polynomial of $h \in \Gamma$ is irreducible and it has only one real eigenvalue ϵ, then ϵ is a unit in $\mathbf{Z}_F$, the maximal order in $F = \mathbf{Q}(\epsilon)$. Thus, the problem of finding a generator of $\mathbf{Z}_F^\times/\{\pm 1\}$, which is an infinite cyclic group, is equivalent to the problem of finding a generator of the stabilizer of the axis of $h \in \Gamma$.

Assume that $\mathbf{Z}_F$ has the power basis $\{1, \delta, \delta^2\}$ where $p(\delta) = \delta^3 + c_2\delta^2 + c_1\delta + c_0 = 0$, $c_0, c_1, c_2 \in \mathbf{Z}$. Let $Ca_1 = \delta a_1$. Then

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 \end{bmatrix}$$

is said to be the *companion matrix* of $p(x)$.

Theorem 10 *Let L be the axis of an irreducible element $h \in \Gamma$ with only one real eigenvalue ϵ. Assume that the discriminant of the characteristic polynomial of h is square free. Then $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon \rangle$. Here $\mathbf{Z}_F$ is the maximal order of the complex cubic field $F = \mathbf{Q}(\epsilon)$.*

Proof By assumption, $\mathbf{Z}_F$ has basis $\{1, \epsilon, \epsilon^2\}$. Thus, any $\gamma \in \mathbf{Z}_F$ can be uniquely represented as $\gamma = p(\epsilon) = c_0 + c_1\epsilon + c_2\epsilon^2$, $c_k \in \mathbf{Z}$. Let C be the companion matrix of the characteristic polynomial of h. Then $\gamma = p(\epsilon)$ can be represented by $p(C)$ in the algebra of 3×3 matrices over $\mathbf{Z}$ (*cf.* [5, p. 160]). Assume that $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon_0 \rangle$. Then $\epsilon = \epsilon_0^n$ for some $n \in \mathbf{Z}$. Let $\epsilon_0 = a_0 + a_1\epsilon + a_2\epsilon^2$, $a_k \in \mathbf{Z}$. Then $h_0 = a_0 I + a_1 C + a_2 C^2 \in \Gamma$ and $h = h_0^n$. Since h is irreducible, $n = 1$ or -1. ■

Example 1 Let $\Gamma = \mathrm{GL}_2(\mathbf{Z})$. Then $\mathcal{P}$ is the Klein model of the hyperbolic plane. Denote

$$U = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $t \in \mathbf{Z}$. The characteristic polynomial of $E_0^T = SU$ is $f(x) = x^2 - tx - 1$. Note that $E_0 = E_0^T$. Let L be the axis of E_0. Then I, the identity matrix, is the intersection of L with $L^+(E)$ and the interval $\big[I, I[E]\big)$ is a fundamental domain of Γ_L on L. Let ϵ be an eigenvalue of E_0. Assume that $t^2 + 4$ or $t^2/4 + 1$ is a square free integer. Then $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon \rangle$. Here $\mathbf{Z}_F$ is the maximal order of the field $F = \mathbf{Q}(\epsilon)$. The period length of the corresponding continued fraction $p = 1$. Many other examples related to this algorithm can be found in [21] and [22].

Example 2 (cf. [26, p. 254]) Let t be a positive integer, $\delta = (t^3 + \eta)^{1/3}$, $\eta = \pm 1$, and $f(x) = x^3 - \delta^3$. Let C be the companion matrix of $f(x)$. Let $F = \mathbf{Q}(\delta)$. Assume that $\{1, \delta, \delta^2\}$ is a basis of $\mathbf{Z}_F$. Since $\epsilon = \delta - t \in \mathbf{Z}_F^\times$, the matrix $E = C - tI \in \Gamma$. Let

$a_1 = (1, \delta, \delta^2)^T$, $a_2 = \big(1, \delta\rho, (\delta\rho)^2\big)^T = a_{2R} + ia_{2I}$ where $\rho = (-1 + \sqrt{-3})/2$ and $a_{2R}, a_{2I} \in \mathbf{R}^3$. Then $Ea_1 = \epsilon a_1$ and the interval L with equation $X(\mu) = \mu a_1 a_1^T + (1-\mu)(a_{2R}a_{2R}^T + a_{2I}a_{2I}^T)$, $0 < \mu < 1$, is the axis of E. The point of intersection of L with $L^+(E^T)$ is $B_0 = X(\mu_0)$, $\mu_0 = 1 - (1-\epsilon^2)/(3t\delta)$.

First let t be even. Denote

$$h = \begin{bmatrix} 1 & -t & -t^2/2 \\ 0 & 1 & -t/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $B_0[h] = X_0 = (x_{ij})$ is Minkowski reduced and $x_{11} = x_{22} = 1$, $x_{33} \sim \frac{3}{4}t^2$, $x_{12} \sim -1/(2t)$, $x_{13} \sim -1/4$, $x_{23} \sim -\eta/(6t^2)$ as $t \to \infty$. Therefore B_0 and X_0 are extremal. Denote $B_1 = B_0[E^T]$. The interval $[B_0, B_1)$ is a fundamental domain of Γ_L on L. Hence $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon \rangle$.

Now let t be odd. Denote

$$h = \begin{bmatrix} 1 & -t & -(t^2-t)/2 \\ 0 & 1 & -(t-\eta)/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $B_0[h] = X_0 = (x_{ij})$ is Minkowski reduced and $x_{11} = x_{22} = 1$, $x_{33} \sim \frac{3}{4}t^2 + \frac{1}{4}$, $x_{12} \sim -1/(2t)$, $x_{13} \sim -1/4$, $x_{23} \sim \eta\big(1/2 - 1/(6t^2)\big)$ as $t \to \infty$. Therefore B_0 and X_0 are extremal. Denote $B_1 = B_0[E^T]$. The interval $[B_0, B_1)$ is a fundamental domain of Γ_L on L. Hence $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon \rangle$.

Denote $E_0 = h^T E h^*$. Let $X_1 = B_1[h]$. Let

$$U = \begin{bmatrix} 1 & -3t/2 & -3t^2/4 \\ 0 & 0 & \eta \\ 0 & 1 & -3t/2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & -3t/2 - \eta/2 & -3t^2/4 - 1/4 \\ 0 & 0 & \eta \\ 0 & 1 & -3t/2 + \eta/2 \end{bmatrix}$$

for d even or odd respectively. Then $E_0^T = S_1 U$ and $X_1 = X_0[E_0^T]$. Thus, $L_0 = L[h]$ is reduced, the interval $[X_0, X_1] = L_0 \cap K(w)$, and the period length of the corresponding continued fraction $p = 1$. Note that, for the Voronoi continued fraction, $p = 1$ if $\eta = 1$ and $p = 2$ if $\eta = -1$ (see [26, p. 254]). Thus, Algorithm I does not coincides with Voronoi's algorithm.

Example 3 Let $f(x) = x^3 - tx^2 - ux - 1$. The discriminant of $f(x)$ is $D_L = -27 - 18ut + u^2t^2 + 4u^3 - 4t^3$. (Note that the particular case of $u = 0$ is considered in [10, p. 202].) Assume that $f(x)$ has only one real root ϵ. The other two roots of f are $\epsilon_{1,2} = \Big(t - \epsilon \pm \big((t+\epsilon)^2 - 4\epsilon^2 + 4u\big)^{1/2}\Big)/2$. Let $F = \mathbf{Q}(\epsilon)$. Assume that D_L is square free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of $\mathbf{Z}_F$.

Let E be the companion matrix of $f(x)$. Let L be the axis of E. Denote by B_0 the intersection of L with $L^+(E)$. Let $Ea = \epsilon a$, $Ea_i = \epsilon_i a_i$, $i = 1, 2$. Then the equation of L is $X(\mu) = \mu a_1 a_1^T + (1-\mu)(a_{2R}a_{2R}^T + a_{2I}a_{2I}^T)$, $0 < \mu < 1$, and $B_0 = X\big(1/(\epsilon^2+\epsilon+1)\big)$. Let $\nu = [u/2]$,

$$h = \begin{bmatrix} 1 & 0 & -\nu \\ 0 & 1 & 1-t \\ 0 & 0 & 1 \end{bmatrix}$$

and $X_0 = (x_{ij}) = B_0[h]$.

Let first $u = 2v$. Then

$$x_{11} = x_{22} = 1,$$
$$x_{12} = -(v-1)/\epsilon - \mu(3\epsilon - 2v + 3)/(2\epsilon),$$
$$x_{13} = -(v-1/2)/\epsilon - \mu(2v\epsilon - 3\epsilon + 2v)/(2\epsilon),$$
$$x_{23} = (v^2+v)/\epsilon + \mu\big((v+4)\epsilon^2 - (2v^2 - v - 1)\epsilon + 1\big)/(2\epsilon^2),$$
$$x_{33} = t - v^2 + v + 1 + (2v^2 + 3v - 1)/\epsilon + \mu(2v^2 + v + 6) - \mu\big((2v^2 - 4v - 3)\epsilon - 3\big)/\epsilon^2.$$

Let now $u = 2v + 1$. Then

$$x_{11} = x_{22} = 1,$$
$$x_{12} = -(v-1/2)/\epsilon - \mu(3\epsilon - 2v + 2)/(2\epsilon),$$
$$x_{13} = 1/2 - v/\epsilon + \mu(2v\epsilon - 2\epsilon + 2v - 1)/(2\epsilon),$$
$$x_{23} = (2v^2 + 3v + 2)/(2\epsilon) + \mu\big((v+3)\epsilon^2 - 2v\epsilon + 1\big)/(2\epsilon^2),$$
$$x_{33} = t - v^2 - v - 1 + (2v^2 + 4v + 1)/\epsilon + \mu(2v^2 + 4v + 6) - \mu\big((2v^2 - 3v - 5)\epsilon - 3\big)/\epsilon^2.$$

Assume that $t \geq 2v^2 + 2v$ for $u = 2v$, and $t \geq 2v^2 + 3v + 2$ for $u = 2v + 1$. Then $B_1 = B_0[E^T]$ is Minkowski reduced (see *e.g.* [6, p. 397]). Hence B_0 is extremal and the interval $[B_0, B_1] = L \cap K(w)$ is a fundamental domain of Γ_L on L. Thus, $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon \rangle$.

Let $E_0 = h^T E h^*$. As in the preceding example, $L_0 = L[h]$ is reduced, E_0^T fixes L_0, and $S_1 E_0^T = U \in \Gamma_\infty$. Thus, $p = 1$.

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