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LETTERS TO THE EDITOR

ON THE DEPARTURE PROCESSES OF $M/M/1/N$ AND $GI/G/1/N$ QUEUES

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Abstract

The purpose of this note is to point out the connection *between* the invariance property of $M/M/1$ and $GI/G/1$ queues (which has been reported in several papers) *and* the interchangeability and reversibility properties of tandem queues. This enables us to gain new insights for both problems and obtain stronger invariance results for $M/M/1$, $GI/G/1$, as well as loss systems $M/M/1/N$, $GI/G/1/N$ and tandem systems.

$M/M/1$; $GI/G/1$; DEPARTURE PROCESSES; INTERCHANGEABILITY; REVERSIBILITY; TANDEM QUEUES

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1. Introduction

Ali (1990) considers an $M/M/1$ queue and proves that if the system is initially empty then the Laplace–Stieltjes transform of the expectation of the number of departures in the interval $(0, t]$ is invariant with respect to an interchange of the arrival and service rates, but that this is not so if the system does not start empty. He also mentions an invariance result for the $GI/G/1$ (which he proved in an earlier paper, Ali (1970)). However, he does not seem to be aware that there are simple connections between the invariances with which he is concerned and properties of *reversibility* (due to Muth (1979)) and *interchangeability* (due to Weber (1979)) for tandem queues. In this note we point out this fact and obtain stronger invariance results. Moreover, in an exactly similar way, we obtain invariance results linking $M/M/1/N$ and $GI/G/1/N$ queues with two-station tandem systems.

Other papers that have reported the invariance property include Takács (1962), Greenberg and Greenberg (1966), Ali (1970), and Hubbard et al. (1986). Some references to interchangeability and reversibility in tandem queues are, to list a few which will be used in this note, Weber (1979), (1992), Chao and Pinedo (1991), Chao et al. (1989), and Muth (1979).

2. $M/M/1$ and $M/M/1/N$ queues

For an $M/M/1$ queueing system with arrival rate λ and service rate μ, the time between successive arrivals to the system is exponentially distributed with rate λ. It is easy to see that, as noted on p. 694 of Weber (1979), this arrival process can be regarded as the departure process from a single exponential server station with service rate λ, and an infinite number of customers in front of the first station. Therefore, the departure process of the original $M/M/1$

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queue is the same as that of a two-station tandem system with exponential service times at two stations (with service rates λ and μ respectively), infinite buffers between the two stations, and infinite customers queued at the first station.

The interchangeability result of Weber (1979) states that, for tandem queues with infinite buffers between stations, single exponential server at each station, and arbitrary arrival process at the first station, the departure process is stochastically the same after the order of any two stations is interchanged. This result, combined with the equivalent tandem queueing model above, immediately leads us to the conclusion that the departure process from the $M/M/1$ queue is stochastically the same after the arrival rate λ and service rate μ are interchanged. In particular, the number of departures from the system in $(0, t]$ is (stochastically) symmetric with respect to the arrival and service rates.

In fact, the analysis above has shown an invariance result for a more general system. Consider a tandem system with K exponential stations in series and infinite buffer size at each station, denoted $M/M/1 \rightarrow /M/1 \rightarrow \cdots \rightarrow /M/1$. The service rates at the K stations are $\mu_1, \mu_2, \cdots, \mu_K$ respectively and the arrival process is Poisson with rate λ. An application of the interchangeability shows the following result.

Theorem 1. The departure process from the tandem system described above is stochastically the same after the arrival rate and the service rate of any station are interchanged.

Now we consider an $M/M/1/N$ system in which there is a buffer of size N (which might be infinite). For this system two types of blocking may be studied. In the first type arrivals are lost when the buffer is full. In the second type an arrival is held outside the system when the buffer is full and released to the system as soon as there is space available; while a customer is held outside the system, any further customers that arrive are lost.

It is not hard to see that this finite buffer system $M/M/1/N$ is equivalent to a two-station tandem system (with service rates λ and μ respectively) with a buffer of size N between the two stations, and an infinite number of customers queued at the first station. The two types of blocking described above are precisely the communication and manufacturing blocking mechanisms in the equivalent tandem model (see Chao et al. (1989) for definitions of these two blocking mechanisms). Applying the interchangeability result for finite-buffer tandem systems (see Chao et al. (1989), Weber (1992)), we obtain the following result.

Theorem 2. For the $M/M/1/N$ queueing system with either type of blocking, the departure process is stochastically the same after the arrival and service rates are interchanged. In particular, the expected number of departures in $(0, t]$ is a symmetric function with respect to the arrival and service rates.

The next Markovian system for which we shall obtain a similar result is a two-station tandem system with no buffer in front of either station, and a Poisson arrival process of rate λ at the first station, denoted by $M/M/1/1 \rightarrow /M/1/1$. Arrivals are lost when the first station is busy and there is blocking at the second station, which we shall assume to be communication blocking.

Theorem 3. For the tandem system described above, if the system is initially empty, the departure epochs of customers from the second station have the same distribution if the arrival rate and the service rate at the second station are interchanged. Moreover, if $\mu_1 \geqq \min\{\lambda, \mu_2\}$ the departure process is stochastically the same.

Proof. Consider a three-station tandem system with no buffer between successive stations, and an infinite number of customers in front of the first station, denoted by $/M/1 \rightarrow /M/1/1 \rightarrow /M/1/1$. The service rates at the three stations are λ, μ_1 and μ_2 respectively. We assume communication blocking at the second and third stations.

The reversibility result of Muth (1979) shows that for a K-station tandem system with a finite buffer between stations, an infinite number of customers queued at the first station, and a single server at each station, the departure epoch of the nth customer from the system has the same distribution after the service order is reversed. That is, if originally all the customers go through the system in the service (station) order of $1, 2, \cdots, K$, then the distributions of the departure epochs of all the customers from the last station remain the same when the service order is changed to $K, K-1, \cdots, 1$. In particular, if $K=3$, there is no buffer between stations, and exponential service time at each station, it is shown in Chao and Pinedo (1991) that if the service rate at the second station is not strictly the smallest, even when there is an arbitrary arrival process at the first station, the departure process is stochastically the same after the service order is reversed.

It is clear that the departure process from the original two-station tandem system $M/M/1/1 \to /M/1/1$ is the same as that of the three-station tandem system $/M/1 \to /M/1/1 \to /M/1/1$. A direct application of the reversibility result of Muth (1979) proves the first part of the theorem. For the second part (though we believe that the condition $\mu_1 \geqq \min\{\lambda, \mu_2\}$ is not needed for the result to hold, see a conjecture in Chao and Pinedo (1991)), just apply the result shown in Chao and Pinedo (1991).

Remark 4. Departure epochs having the same distribution does not imply, in general, identical law of the departure processes in reversibility of tandem queues. Counterexamples may be found in Tembe and Wolff (1974).

3. $GI/G/1$ and $GI/G/1/N$ queues

Ali (1970), (1990) also discussed the non-symmetric structure in $M/M/1$ and $GI/G/1$ systems when the system is initially not empty. These results may also be derived from the reversibility result of tandem queues and they also hold for the loss systems $M/M/1/N$, $GI/G/1/N$, and even for tandem queues.

Consider a $GI/G/1$ queueing system. The interarrival times $T_1, T_2, \cdots$ have distribution F and service times $S_0, S_1, \cdots$ have distribution G. Customers $C_0, C_1, \cdots$ arrive at $T_0, T_1, \cdots$. Assume that C_0 arrives at time $0\,(T_0=0)$. Letting R_n be the time instant that customer C_n enters service, it is shown in Ali (1970) that R_n has the same distribution after the interarrival time distribution F and service time distribution G are interchanged. This result easily follows from the reversibility result of Muth (1979) and may be extended to more general systems. For example, consider a $GI/G/1/N$ loss system, observe that a $GI/G/1/N$ queue that initially contains a single customer has the same departures as a $/G/1 \to /G/1/N$ tandem system in which a first customer has just completed service at the first station and an infinite number of customers remain queued in front of the first station. A small technicality arises because a customer is lost from the $GI/G/1/N$ system when it arrives and the buffer is full, whereas in the tandem system the blocking mechanism means that a customer must wait. So rather than defining R_n in terms of the nth event associated with the nth customer, R_n must be defined as the *nth instant at which a customer starts service in the $GI/G/1/N$ queue*. For this definition of R_n the following theorem is easily obtained from the reversibility result of Muth and an equivalence relationship with a two-station tandem queue.

Theorem 5. For a $GI/G/1/N$ queueing system with either type of blocking, R_n has the same distribution after F and G are interchanged.

For a two-station tandem queueing system with either finite or infinite buffers between stations $GI/G/1 \to /G/1$ and $GI/G/1 \to /G/1/N$, and either type of blocking, we define R_n as the instant of the nth customer starting service at the second station; if there is also a finite buffer in front of the first station, then similar to the previous theorem, we define R_n as the *nth instant that a customer starts service at the second station.*

Theorem 6. For the two-station tandem systems with either type of blocking mechanisms, either finite or infinite buffer at the first and/or second station, R_n has the same distribution after the interarrival and the service time distribution at the second station are interchanged.

Using these results it is readily seen the non-symmetric structure of the departure distribution in all the systems of Theorems 5 and 6. We summarize it in the following theorem.

Theorem 7. For all the queueing systems in Theorems 5 and 6 with a single initial customer, the expected number of departures in $(0, t]$ is not symmetric with respect to the interchange of the distributions of the interarrival and service times (service time of the second station in the case of tandem queues).

Remark 8. When there are infinite buffers it is clear that the result in Theorem 6 remains true when the first station is replaced by an arbitrary *reversible system,* in the sense that the departure process is stochastically the same when the service order is reversed. A tandem queue with infinite buffers and a single exponential server at each station is such an example.

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