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A Sato–Tate Law for Drinfeld Modules

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Abstract. We formulate and prove a Sato–Tate equidistribution law for Drinfeld modules.

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Introduction

The goal of this paper is to formulate and prove a Sato–Tate law for Drinfeld modules. The proof, based on an idea of Drinfeld, is not difficult. The formulation is not quite obvious. We will elaborate the motivation in this long introduction.

For this introduction, let $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$, and ∞ the 'infinite' place of K, defined as the pole of the function t.

Let E be a Drinfeld A-module over an A-field L. It is well known that the theory of Drinfeld modules is analogous to that of elliptic curves or abelian varieties in many ways. In particular, when L is a finite field, there is a Frobenius characteristic polynomial

$$P_{E/L}(X) = \det(1 - X\,\mathrm{Fr}_L | T_\ell(E))$$

associated to E/L, where ℓ is any prime of A not equal to the A-characteristic of L, Fr_L is the geometric Frobenius, and T_ℓ is the Tate module. The polynomial $P_{E/L}(X)$ has coefficients in A, independent of ℓ. Its roots satisfy the 'Riemann hypothesis' ([2, 5, 12]).

When L is a global A-field, the Drinfeld module E has good reduction at almost all places v of L, and therefore we get a Frobenius characteristic polynomial $P_v(X) = P_{\tilde{E}/\kappa(v)}$ for almost all places v of L.

In a recent paper of L.-C. Hsia and J. Yu [7], an equidistribution law is obtained for the leading terms of the coefficients of $P_v(X)$ as v varies. We notice that the 'leading term' of an element of $\mathbb{F}_q[t]$ is an ∞-adic concept. Therefore, their result should be regarded as a contribution toward understanding the distribution of the *polynomial* $P_v(X)$ *with* ∞*-adic coefficients* as v varies. The analogous question for an elliptic curve E/L (without complex multiplications) is to understand how

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$\det(1 - X\,\mathrm{Fr}_v | T_\ell(E))$ varies as a *polynomial with real coefficients*. The answer is the so-called Sato–Tate law, which is proved by Deligne in the function field case, but unknown for any single elliptic curve over a number field.

We refer to [1] or [8] for the precise formulation of the Sato–Tate law for an elliptic curve E over a global field L without complex multiplications, following Deligne. For the purpose of motivation, we state it slightly vaguely as follows: the Sato–Tate law for E/L is *as if the following were true*:

- There is a continuous surjective group homomorphism $\mathrm{Gal}(\bar{L}/L) \to \mathrm{SU}(2)$, unramified at all but finite many places, such that the induced map $\mathrm{Gal}(\bar{L}/L)^\natural \to \mathrm{SU}(2)^\natural$ is simply
$$\mathrm{Fr}_v \mapsto x_v = \imath(\mathrm{Fr}_v(\mathbf{N}v^{-1/2}) | T_\ell)^{\text{s.s.}}.$$
- Chebotarev density theorem holds as if SU(2) were a (pro)finite quotient of $\mathrm{Gal}(\bar{L}/L)$.

Here, ℓ is a prime invertible in L, and $\imath$ is a fixed embedding from $\mathbb{Q}_\ell$ to $\mathbb{C}$, $G^\natural$ denotes the set of conjugacy classes in G, and $g^{\text{s.s.}}$ denotes the semisimple part of an element g in a complex algebraic group.

Hsia and Yu considered
$$\bar{P}_v(X) = P_v(t^{-\deg v/r}X) \quad (\text{modulo } \pi_\infty = t^{-1}).$$

They describe the distribution of $\bar{P}_v \in \mathbb{F}_q[X]^{\deg = r}$ in terms of a homomorphism
$$\bar{\rho}\colon \mathrm{Gal}(\bar{L}/L) \to (\mathbb{Z}/r\mathbb{Z}) \ltimes \mathbb{F}_{q^r}^\times,$$

which comes from certain extension $L(\mathbb{F}_{q^r}, \delta^{1/(q^r-1)})/L$ via Kummer theory. In fact, their main result can be reformulated as follows: the polynomial $\bar{P}_v$ can be read-off from $\bar{\rho}(\mathrm{Fr}_v)$. Therefore, the distribution of $\bar{P}_v$ can be read-off via Chebotarev density theorem. See (4.1) for more details.

Observe that
$$(\mathbb{Z}/r\mathbb{Z}) \ltimes \mathbb{F}_{q^r}^\times \simeq D^\times / K_\infty^\times(1 + \mathfrak{m}_D),$$

where D is the division algebra over K_∞ with invariant $-1/r$ via local class field theory, and $\mathfrak{m}_D$ is the maximal ideal in the ring of integers in D. Therefore, the result of Hsia and Yu suggested that the Sato–Tate law should be formulated using D. Notice that $D^\times$ is almost compact: $D^\times/\pi_\infty^{\mathbb{Z}}$ is compact.

Recall that there is a bijection (Skolem–Noether)
$$\{\text{weakly elliptic elements in } \mathrm{GL}_r(K_\infty)\}^\natural \leftrightarrow (D^\times)^\natural,$$

where we say that g is *weakly elliptic* if $K_\infty(g)$ is a field. Recall also that P_v does define an elliptic element ([2, 12]).

The above discussion suggests: the Sato–Tate law for the Drinfeld module E/L is *as if the following were true*:

- There is a group homomorphism $W_L \to D^\times$, unramified at all but finitely many places, such that the induced map $W_L^\natural \to (D^\times)^\natural$ is simply $\mathrm{Fr}_v \mapsto x_v =$ the class corresponding to P_v by the Skolem–Noether theorem.
- Chebotarev density theorem applies to $\mathrm{Gal}(\bar{L}/L) = \hat{W}_L \twoheadrightarrow H$, where H is the profinite completion of the image of $W_L \to D^\times$.

Here, $W_L \subset \mathrm{Gal}(\bar{L}/L)$ is the Weil group of L (see 1.3).

Now we are ready to state the main result of this paper.

THEOREM. *The conjectural Sato–Tate law is true. In fact, the 'as if' statement is really true.*

We conclude the introduction with a few remarks.

Remark. In the classical case, the 'as if' statement can not be true: (i) the group $\mathrm{Gal}(\bar{L}/L)$, being profinite, can never map surjectively to SU(2); (ii) SU(2) is certainly not profinite.

In the Drinfeld case, both obstructions disappear. So the above result, which is a bit surprising at first, is indeed possible.

Remark. Let us compare the ways of attaching to P_v a conjugacy class in an (almost) compact, $\mathbb{Q}_\infty$- or K_∞-Lie group. In the classical case, the main facts used are: (i) Hasse's theorem ('Riemann hypothesis' for elliptic curves over a finite field), (ii) Cartan's conjugacy theorems about compact elements and maximal compact subgroups.

In the Drinfeld case, the main facts used are: (i) P_v defines an elliptic conjugacy class—a fact apparently weaker than the Riemann hypothesis, (ii) the Skolem–Noether theorem.

Remark. In the classical case, the Sato–Tate law is about the equidistribution of a sequence $\{x_v\}$ of points in the space $\mathrm{SU}(2)^\natural$, relative to the Haar measure of $\mathrm{SU}(2)^\natural$. The space does not depend on the elliptic curve. In the Drinfeld case, the analogous space is $H^\natural$, again equipped with the Haar measure. But it does depend on the Drinfeld module.

1. Notations

1.1. Let K be a global function field of characteristic p (i.e. a finitely generated field of transcendence degree 1 over $\mathbb{F}_p$). Single out a place ∞ of K and let $A \subset K$ be the subring consisting of functions regular away from ∞. The completion of K at ∞ is denoted by K_∞.

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Let π be a fixed prime element of K_∞ and κ the constant field of K_∞. Thus $K_\infty = \kappa((\pi))$. Let q be the cardinality of κ, $\bar\kappa$ the algebraic closure of κ, and n the integer such that $q = p^n$.

1.2. For a ring L in characteristic p, we let $L\{\tau\}$ (resp. $L\{\{\tau\}\}$) be the ring of twisted polynomials (resp. twisted formal power series) in τ, with the rule $\tau x = x^p\tau$. If L is perfect in the sense that $x \mapsto x^p$ is an isomorphism from L to L, we let $L(\{\{\tau^{-1}\}\})$ be the ring of twisted formal Laurent series in τ^{-1}. We define a function ord: $L(\{\{\tau^{-1}\}\}) \to \mathbb{Z} \cup \{+\infty\}$ by setting $\mathrm{ord}(\sum a_k\tau^{-k}) = \min\{k : a_k \neq 0\}$.

1.3. We define the Weil group W_L of a field L of characteristic p as a subgroup of $\mathrm{Gal}(\bar L/L)$ by $W_L = \{s \in \mathrm{Gal}(\bar L/L) \mid s|_{\bar{\mathbb{F}}_p} \in \sigma^{\mathbb{Z}}\}$, where $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is the arithmetic Frobenius $x \mapsto x^p$. We define $\mathrm{ord}\colon W_L \to \mathbb{Z}$ be such that $s|_{\bar{\mathbb{F}}_p} = \sigma^{-\mathrm{ord}(s)}$ for all $s \in W_L$.

1.4. The central division algebra over K_∞ of Brauer invariant $-1/r$ is denoted by D or D_r. For any central simple algebra R of degree r over a field F (i.e. $[R : F] = r^2$), and any $x \in R$, we define the characteristic polynomial of x to be $\mathrm{char}(x) = \det(1 - X(x \otimes_F 1))$, computed in $R \otimes_F \bar F \simeq M_r(\bar F)$.

2. A Construction of Drinfeld

2.1. This section is a detailed exposition of the construction in [2, Section 1]. Only K_∞ intervenes in the discussion. We assume that L is a *perfect* κ-algebra. We fix an integer $r \geqslant 1$ and give K_∞ the valuation ord such that $\mathrm{ord}(K_\infty^\times) = nr\mathbb{Z}$.

DEFINITION 2.2. An embedding of rings $\iota\colon \kappa \to L(\{\{\tau^{-1}\}\})$ is called *admissible* if $\iota(a) \equiv a \pmod{\tau^{-1}}$ for all $a \in \kappa$. An embedding of rings $\iota\colon K_\infty \to L(\{\{\tau^{-1}\}\})$ is called *admissible* if it is valuation preserving and $\iota|\kappa$ is admissible.

Admissible embeddings $K_\infty \to L(\{\{\tau^{-1}\}\})$ exist. In fact, choose any π' in $\bar L(\{\{\tau^{-n}\}\})$ of order rn, there is a unique admissible ι such that $\iota|\kappa = \mathrm{id}$ and $\iota(\pi) = \pi'$. The following lemma is due to Drinfeld, cf. [3, Section 2, Lemma] for an analogous result.

LEMMA 2.3. (i) *Any two admissible embeddings* $\iota\colon \kappa \to L(\{\{\tau^{-1}\}\})$ *are conjugate by an element of* $L\{\{\tau^{-1}\}\}^\times$.

(ii) *Any two admissible embeddings* $\iota\colon K_\infty \to L(\{\{\tau^{-1}\}\})$ *are conjugate by an element of* $\bar L\{\{\tau^{-1}\}\}^\times$.

Proof. (i) Let x be an element of $L(\{\{\tau^{-1}\}\})$ such that $\mathbb{F}_p(x) \simeq \kappa$. Then clearly $\mathrm{ord}(x) = 0$ and hence we can write $x = \sum_{k=0}^{\infty} x_k\tau^{-k}$. We have $x_0^{p^n} = x_0$. Let m be the smallest positive integer such that $x_0^{p^m} = x_0$. We claim that x is conjugate to x_0 by an element of $L\{\{\tau^{-1}\}\}^\times$. In particular, $m = n$ and $\mathbb{F}_p(x)$ is conjugate to $\mathbb{F}_p(x_0) = \kappa$.

We use induction to prove that there exists $u \in 1 + \tau^{-1}L\{\{\tau^{-1}\}\}$ such that $uxu^{-1} \equiv x_0 \pmod{\tau^{-(k+1)}}$. There is nothing to prove when $k = 0$. In general, we

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may assume that $x \equiv x_0 + b\tau^{-k} \pmod{\tau^{-(k+1)}}$ by induction hypothesis. Then $x = x^{p^n}$ is congruent to

$$x_0 + b\left(\sum_{j=0}^{p^n-1} x_0^{p^n-1-j} x_0^{j/p^k}\right)\tau^{-k} \equiv x_0 + b\delta_k \tau^{-k} \pmod{\tau^{-(k+1)}},$$

where $\delta_k = 0$ if m divides k, $\delta_k = 1$ otherwise. Thus we are already done when m divides k. If m doesn't divide k, the congruence formula

$$(1 + v\tau^{-k})(x_0 + b\tau^{-k})(1 + v\tau^{-k})^{-1} \equiv x_0 + (b + v(x_0^{1/p^k} - x_0))\tau^{-k}$$

modulo $\tau^{-(k+1)}$ shows that we can choose a suitable $u = 1 + v\tau^{-k}$ such that $uxu^{-1} \equiv x_0 \pmod{\tau^{-(k+1)}}$. This completes the proof the claim. Clearly, (i) follows from the claim.

(ii) Let ι_1 and ι_2 be admissible embeddings from K_∞ to $L(\{\{\tau^{-1}\}\})$. By (i), we are reduced to the case that $\iota_1|\kappa$ and $\iota_2|\kappa$ are both identities. Thus ι_1 is determined by $\iota_1(\pi)$, which lies in $L(\{\{\tau^{-n}\}\})$. The same goes for ι_2.

It is enough to choose w such that $\iota_2(\pi) = w\iota_1(\pi)w^{-1}$. Let us write out the conditions. Let

$$\iota_j(\pi) = \sum_{k=r}^{\infty} a_k^{(j)} \tau^{-nk}, \qquad w = \sum_{k=0}^{\infty} u_k \tau^{-nk}.$$

Then the condition $\iota_2(\pi)w = w\iota_1(\pi)$ is:

$$\begin{aligned}
&a_r^{(2)} u_0^{q^{-r}} = u_0 a_r^{(1)},\\
&a_{r+1}^{(2)} u_0^{q^{-(r+1)}} + a_r^{(2)} u_1^{q^{-r}} = u_0 a_{r+1}^{(1)} + u_1 (a_r^{(1)})^{q^{-1}},\\
&a_{r+2}^{(2)} u_0^{q^{-(r+2)}} + a_{r+1}^{(2)} u_1^{q^{-(r+1)}} + a_r^{(2)} u_2^{q^{-r}} = u_0 a_{r+2}^{(1)} + u_1 (a_{r+1}^{(1)})^{q^{-1}} + u_2 (a_r^{(1)})^{q^{-2}}, \ldots .
\end{aligned}$$

It is clear that w can be found by solving a Kummer equation (for u_0) and a series of Artin–Schreier equations (for $u_1, u_2, \ldots$). □

LEMMA 2.4. (i) *For any admissible ι, the centralizer D_ι of $\iota(K_\infty)$ in $\bar{L}(\{\{\tau^{-1}\}\})$ is isomorphic to $D = D_r$ as K_∞-algebras.*

(ii) *Any isomorphism $f\colon D_\iota \to D$ of K_∞-algebras satisfies* $\operatorname{ord} f(x) = \operatorname{ord} x$ *for all $x \in D_\iota$, where the valuation* ord *on D is normalized so that* $\operatorname{ord}(D^\times) = n\mathbb{Z}$.

(iii) *If $\iota(K_\infty) \subset \bar{\kappa}(\{\{\tau^{-1}\}\})$, D_ι lies in $\bar{\kappa}(\{\{\tau^{-1}\}\})$.*

Proof. By Lemma 2.3, it is enough to prove the result in the case $\iota|\kappa = \mathrm{id}$, $\iota(\pi) = \tau^{-rn}$. In this case, the centralizer is clearly $\kappa_r\{\tau^{-n}\}$ (where κ_r is the degree r extension of κ), and is isomorphic to D. Statements (ii) and (iii) are also clear. □

CONSTRUCTION 2.5. Now fix an admissible $\phi\colon K_\infty \to L(\{\{\tau^{-1}\}\})$, and choose an admissible $\iota\colon K_\infty \to \bar{\kappa}(\{\{\tau^{-1}\}\})$. Define

$$Y_\iota = \left\{u \in \bar{L}(\{\{\tau^{-1}\}\}) \;\middle|\; \iota(x) = u\phi(x)u^{-1} \quad \text{for all } x \in K_\infty\right\}.$$

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It is clear that $D_\iota^\times$ acts on the left of Y_ι by multiplying from the left, and this makes Y_ι a principal homogeneous space of $D_\iota^\times$.

There is a natural way to make $\mathrm{Gal}(\bar{L}/L)$ act on $\bar{L}(\{\{\tau^{-1}\}\})$, namely by the formula $s.(\sum a_k\tau^{-k}) = \sum s(a_k)\tau^{-k}$. The set Y_ι is not invariant under this action. However, we can define an action of the Weil group W_L on Y_ι as follows:

$$W_L \times Y_\iota \to Y_\iota, \qquad (s, u) \mapsto s_*(u) = \tau^{\mathrm{ord}(s)}(s.u).$$

It is easy to check that this is indeed a group action, and $s_*(du) = ds_*(u)$ for all $s \in W_L$, $d \in D_\iota$, $u \in Y_\iota$.

This implies immediately that the action of W_L on Y_ι factors through a suitable group homomorphism $\rho\colon W_L \to D_\iota$, unique up to conjugacy by an element of $D_\iota^\times$. More precisely, we pick a base point $u_0 \in Y_\iota$ and write $s_*(u_0) = d_s^{-1}.u_0$ for each $s \in W_L$. Then $\rho\colon s \mapsto d_s$ is a group homomorphism. Changing the base point u_0 amounts to changing this homomorphism by conjugation.

LEMMA 2.6. (i) *The homomorphism* $\rho\colon W_L \to D_\iota^\times$ *satisfies* $\mathrm{ord}\,\rho(s) = \mathrm{ord}\,(s)$.

(ii) *The homomorphism* ρ *induces a continuous homomorphism from the profinite completion of* W_L *to the profinite completion of* $\widehat{D}_\iota$.

(iii) *Let* $\iota_1, \iota_2\colon K_\infty \to \bar{\kappa}\{\tau^{-1}\}$ *be two admissible embeddings,* $w \in \bar{\kappa}\{\tau^{-1}\}$ *be of order* 0 *such that* $\iota_2(x) = w\iota_1(x)w^{-1}$ *for all* $x \in K_\infty$. *Then the maps* $D_{\iota_1} \to D_{\iota_2}$, $d \mapsto wdw^{-1}$ *and* $Y_{\iota_1} \to Y_{\iota_2}$, $u \mapsto wu$ *are bijections.*

The diagram

$$\begin{array}{ccc} W_L & \longrightarrow & D_{\iota_1} \\ \| & & \downarrow \\ W_L & \longrightarrow & D_{\iota_2} \end{array}$$

is commutative if the horizontal arrows are constructed using compatible base points u_0 *and* wu_0.

Proof. Statements (i) and (iii) are easy. Let's prove the continuity statement in (ii). For this we may assume that $L \supset \bar{\kappa}$ and hence $W_L = \mathrm{Gal}(\bar{L}/L)$, $\rho(W_L) \subset \mathcal{O}_{D_\iota}^\times$. We may also choose u_0 to be such that $\mathrm{ord}(u_0) = 0$.

Now the continuity statement amounts to the following: for any $N > 0$, the subgroup $U = \{s \in W_L : s(u_0) \equiv u_0 \pmod{\tau^N}\}$ is of finite index in W_L. This follows from the proof of Lemma 2.3: the first N coefficients of u_0 generate a finite extension of L. □

Thus given an admissible $\phi\colon K_\infty \to L\{\tau^{-1}\}$, we obtain a homomorphism $\rho\colon W_L \to D^\times$ by choosing an ι, an isomorphism $D_\iota \to D$, and a base point u_0. The homomorphism $\rho\colon W_L \to D^\times$ is canonically defined up to conjugacy by an element of $D^\times$.

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3. The Main Theorem

3.1. We refer to [6] for basic definitions and facts about Drinfeld modules. From now on, we assume that L is an A-field and E is a Drinfeld A-module of rank r. Thus L is a field with the structure of an A-algebra and we are given an injection $\phi = \phi_E \colon A \to L\{\tau\}$. It is easy to show that ϕ extends by continuity to an injection $\phi \colon K_\infty \to L^{1/p^\infty}(\{\{\tau^{-1}\}\})$. This implies that L^{1/p^∞} (hence L) has a unique structure of κ-algebra for which ϕ is admissible (Cf. [Goss, Remark 7.2.13]). Therefore, we can apply the construction of Section 2 to obtain a homomorphism $\rho \colon W_{L^{1/p^\infty}} = W_L \to D^\times$, canonically defined up to conjugacy.

LEMMA 3.2. *If L is finite, and $F_L : X \mapsto X^{1/\#L}$ is the geometric Frobenius element in* $\mathrm{Gal}(\bar{L}/L)$*, then the minimal polynomial of $\rho(F_L)$ in D is the minimal polynomial of τ^{mn} in* $\mathrm{End}_L(E) \otimes_A K_\infty$*, where $m = [L : \kappa]$.*

Here, we recall that $\mathrm{End}_L(\phi) \otimes_A K_\infty$ can be embedded in D, unique up to conjugacy by $D^\times$. This allows us to compare the minimal polynomials.

Proof of the Lemma. We may choose $\iota = \phi$, $u_0 = 1$. Then it is clear that $\rho(F_L) = \tau^{mn}$. Notice that $\mathrm{End}_L(\phi)$ is simply the centralizer of $\phi(A)$ in $L\{\tau\}$, and is contained in the centralizer D_ι of $\phi(K_\infty)$ in $\bar{L}\{\tau^{-1}\}$. This gives an embedding of $\mathrm{End}_L(\phi) \otimes_A K_\infty$ into $D \simeq D_\iota$. Finally, from [2] or [12], the minimal polynomials of τ^{mn} in $\mathrm{End}_L(E) \otimes_A K_\infty$ and in $\mathrm{End}_{\bar{L}}(E) \otimes_A K_\infty$ are the same. The lemma is now clear. □

3.3. Now assume that $L = \mathrm{Frac}\, R$, where R is a discrete valuation ring with the structure of an A-algebra. Moreover, we assume that $E_{/L}$ has good reduction mod $\mathfrak{m}_R$. In other words, E comes from a Drinfeld A-module over R: $\phi \colon A \to R\{\tau\}$. This implies that the coefficient of the lowest degree term of $\phi(x)$ has valuation 0 for all $x \in K_\infty$.

LEMMA. *Under the above hypothesis, the homomorphism $\rho \colon W_L \to D^\times$ is unramified. That is, it is trivial when restricted to the inertia group.*

Proof. This is easy to see from the construction of Section 2. The point is that the Kummer extensions and Artin–Schreier extensions in the proof of Lemma 2.3 are unramified. □

3.4. Now let L be a global field. We denote by ρ_∞ the homomorphism called ρ in (3.1). For each prime ideal ℓ of A such that $\ell \neq A\text{-char}\, L$, the Tate module $T_\ell(E)$ gives rise to a homomorphism

$$\rho_\ell \colon \mathrm{Gal}(\bar{L}/L) \to \mathrm{GL}_r(A_\ell).$$

THEOREM. *The continuous homomorphisms ρ_∞, ρ_ℓ are unramified at all places of good reduction. For such a place v, the characteristic polynomials*

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$\mathrm{char}(\rho_\infty(F_v)) \in K_\infty[X]$ *and* $\mathrm{char}(\rho_\ell(F_v)) \in A_\ell[X]$ *both have coefficients in* A*, and they are identical.*

Proof. This is immediate from Lemmas 3.2, 3.3, and [12]. □

We remark that the Skolem–Noether bijection mentioned in the introduction is characterized by the fact that corresponding elements have identical characteristic polynomial. Thus the above theorem is indeed the first part of the 'as if' statement in the introduction.

COROLLARY 3.5. *Let* H *be the image of* $\rho' : \mathrm{Gal}(\bar{L}/L) \to \widehat{D}^\times$. *Let* μ *be the Haar measure on* H *with total measure* 1. *Then for any open set* $U \subset H$ *stable under conjugacy, the set of places* $\{v : \rho_\infty(F_v) \in U\}$ *has Dirichlet density* $\mu(U)$.

3.6. We have tried to keep the exposition elementary. Drinfeld's construction works over a rather general base ring, and the discussion is this section can be rephrased as follows: for L global and ϕ defined over $\mathcal{O}_{L,S}$, the Galois representation $\rho\colon \mathrm{Gal}(\bar{L}/L) \to \widehat{D}^\times$ factors through $\mathrm{Gal}(\bar{L}/L) \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{L,S})$, and for any place v, the composition $\mathrm{Gal}(\overline{\kappa(v)}/\kappa(v)) \simeq \pi_1(\mathrm{Spec}\,\kappa(v)) \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{L,S}) \to \widehat{D}^\times$ is the same as the representation $\mathrm{Gal}(\overline{\kappa(v)}/\kappa(v)) \to \widehat{D}^\times$ associated to the reduction of ϕ at v.

4. Comments

4.1. As an illustration, we sketch a derivation of the result of Hsia and Yu from the current work. For simplicity, we assume that $A = \mathbb{F}_q[t]$, though everything works in a completely general setting.

Assume that the multiplication-by-t map on E is

$$\phi = \phi_E(t) = t\tau^0 + a_1\tau^n + \cdots + a_r\tau^{nr}.$$

Put $\delta = a_r, \pi = t^{-1}$. Then $\phi(\pi) \equiv (1/\delta)^{1/q^r}\tau^{-nr} \pmod{\tau^{-n(r+1)}}$ and the admissible embedding $\phi\colon \mathbb{F}_q\{\{\pi\}\} \to L(\{\{\tau^{-n}\}\})$ is the identity on $\kappa = \mathbb{F}_q$. Therefore, we can choose $\iota = \mathrm{id}$ on κ, $\iota(\pi) = \tau^{-rn}$, $u_0 \in \bar{L}\{\{\tau^{-n}\}\}^\times$ such that $\tau^{-rn} = u_0\phi(\pi)u_0^{-1}$. Then the coefficient c_0 of τ^0 in u_0 satisfies the equation $c_0^{q^r-1} = \delta$.

We would like to compute the composition

$$\bar{\rho}\colon W_L \to D^\times/(K_\infty(1+\mathfrak{m}_D)) \simeq (\mathbb{Z}/r\mathbb{Z}) \ltimes \mathbb{F}_{q^r}^\times.$$

This amounts to compute $s_*(u_0)$ modulo $\tau^{rn\mathbb{Z}}(1+\tau^{-n}\bar{L}\{\{\tau^{-n}\}\})$. From the preceding discussion, it is easy to see that $\bar{\rho}\colon \mathrm{Gal}(\bar{L}/L) \to (\mathbb{Z}/r\mathbb{Z}) \ltimes \mathbb{F}_{q^r}^\times$ is nothing but the homomorphism associated to the Kummer extension $L(\mathbb{F}_{q^r}, \delta^{1/(q^r-1)})/L$.

Let $g \in \mathrm{GL}_r(K_\infty)^\natural$ and $g' \in (D^\times)^\natural$ be classes related by the Skolem–Noether correspondence, and let $d = -\mathrm{ord}(\det(g))/n$. It is easy to show that g' modulo $K_\infty(1+\mathfrak{m}_D)$ determines $\det(1 - t^{-d/r}Xg)$ modulo $\pi = t^{-1}$. Thus we conclude that (with the notation of the introduction), $\bar{P}_v$ is determined by $\bar{\rho}(F_v)$ and $\bar{\rho}$ is described by a Kummer extension. This is the result of Hsia and Yu.

4.2. Let H_∞ (resp. H_ℓ) be the image of the homomorphism ρ_∞ (resp. ρ_ℓ). This is a closed subgroup of $D^\times$ (resp. $\mathrm{GL}_r(A_\ell)$). We raise a few questions about these groups.

4.2.1. How can we determine these groups?

4.2.2. Let G_∞ (resp. G_ℓ) be the Zariski closure of H_∞ (resp. H_ℓ). Can we determine G_∞ and G_ℓ? Can we determine their Lie algebras?

4.2.3. Is there any relation among G_ℓ for varying ℓ, and G_∞? Cf. the work of Pink and Larsen [11].

After the completion of the paper, the author found two articles of Pink [9, 10], which determines H_ℓ°. The referee also pointed out the the relevance of [4]. Hopefully, soon there will be more progresses toward these questions.

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