A COMPLETENESS THEOREM FOR A NONLINEAR PROBLEM

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1. Introduction

As is well known, the linear Sturm-Liouville eigenvalue problem on a bounded real interval $[a, b]$ possesses a family of eigenfunctions which is a complete orthonormal system for the real Hilbert space $L_2[a, b]$, i.e. there exists a sequence of eigenfunctions $\{u_n\}$ such that $(u_i, u_j) = \delta_{ij}$ (Kronecker delta) for $i, j \in \mathbf{N}$ (the set of positive integers) and, if $u \in L_2[a, b]$, $u = \sum_{j=1}^{\infty} c_j u_j$ where $c_j = (u, u_j)$. Pimbley (**4**, p. 113), raises the question as to whether similar completeness results hold for nonlinear problems. In this note we show that certain nonlinear Sturm-Liouville eigenvalue problems possess eigenfunctions which form a basis for $L_2[a, b]$, i.e. there exists a sequence of eigenfunctions $\{v_n\}$ for the nonlinear problem such that every $u \in L_2[a, b]$ can be expressed in the form $u = \sum_{j=1}^{\infty} c_j v_j$ by means of a unique sequence $\{c_n\}$ of real numbers.

Before stating our main result, we must first recall some properties of linear Sturm-Liouville problems. If $\mathbf{R}$ denotes the set of real numbers, let $p : [a, b] \to \mathbf{R}$ be continuously differentiable with $p(x) > 0$ for $x \in [a, b]$ and let $q : [a, b] \to \mathbf{R}$ be continuous. Consider the equations

$$-(pu')'(x) + q(x)u(x) = \lambda u(x), \tag{1}$$

$$a_1 u(a) + a_2 u'(a) = 0 = b_1 u(b) + b_2 u'(b); \; a_1^2 + a_2^2 \neq 0, \; b_1^2 + b_2^2 \neq 0. \tag{2}$$

Let $L : D(L) \to L_2[a, b]$ be such that $Lu = -(pu')' + qu$ where $u \in D(L)$ if and only if u satisfies (2), u is absolutely continuous on $[a, b]$ and

$$-(pu')' + qu \in L_2[a, b].$$

Then L is a self-adjoint operator on $L_2[a, b]$. Since L is closed, $D(L)$ is a Banach space with respect to the norm $|||\, u \,||| = \| u \| + \| Lu \|$ where $\| \; \|$ denotes the norm in $L_2[a, b]$.

With the above notation equations (1) and (2) may be expressed as

$$Lu = \lambda u \tag{3}$$

We study nonlinear perturbations of equation (3). We shall prove the following:

Theorem 1. *Let $N_i: D(L)\to L_2[a, b]$ be continuously Fréchet differentiable with $N_i(0) = 0$ and $N_i'(0) = 0$ for $i = 1, 2$. Then there exists a sequence of eigenfunctions $\{v_n\}$ for the problem*

$$Lu+N_1u = \lambda(u+N_2u) \tag{4}$$

such that $\{v_n\}$ is a basis for $L_2[a, b]$.

2. Proof of Theorem 1

Before proving Theorem 1 we state as propositions the two main facts on which the proof depends.

Proposition 1. *Let $\{u_n\}$ be a complete orthonormal system for a Hilbert space H. If $\{v_n\}$ is a sequence of vectors in H such that $\sum_{j=1}^{\infty} \| u_j-v_j \|^2<1$, then $\{v_n\}$ is a basis for H.*

Proof. See Kato (**3**), V 2.20 and the subsequent remarks.

Secondly we require a result from bifurcation theory due to Crandall and Rabinowitz (**1**). Let $A: D(A)\to L_2[a, b]$ be a densely defined closed linear operator on $L_2[a, b]$. Then $X = D(A)$ is a Banach space with respect to the norm $\| u \|_X = \| u \| + \| Au \|$. Proposition 2 is a special case of Theorem 2.4 in (**1**).

Proposition 2. *Let $N_i: X\to L_2[a, b]$ be continuously differentiable and $N_i(0) = 0$ and $N_i'(0) = 0$ for $i = 1, 2$. Regarding $A-\lambda_0 I$ as a map from X to $L_2[a, b]$, suppose that $N(A-\lambda_0 I)$, the null space of $A-\lambda_0 I$, is one dimensional and $R(A-\lambda_0 I)$ has codimension one i.e. there exist $u_0\in X$ and $y_0 \in L_2[a, b]$ such that $N(A-\lambda_0 I) = \operatorname{span}\{u_0\}$ and $R(A-\lambda_0 I) = \{y\in L_2[a, b]: (y_0, y) = 0\}$. If $(y_0, u_0) \neq 0$, and Z is any complement of* span $\{u_0\}$ *in X, then there exists a neighbourhood U of $(\lambda_0, 0)$ in $\mathbf{R}\times X$, a real interval $(-a, a)$ and continuous functions $m: (-a, a)\to \mathbf{R}$ and $l: (-a, a)\to Z$ such that $m(0) = \lambda_0$, $l(0) = 0$ and the set of all solutions of $Au+N_1u = \lambda(u+N_2u)$ contained in U is*

$$\{(m(s), su_0+sl(s))\in \mathbf{R}\times X: | s |<a\}\cup\{(t, 0): (t, 0)\in U\}.$$

We can now give the

Proof of Theorem 1. By the linear Sturm-Liouville theory there exists an increasing sequence of eigenvalues $\lambda_1<\lambda_2<\lambda_3<\ldots$ and a corresponding sequence of eigenfunctions $u_1, u_2, u_3, \ldots$ for equation (3). Let $k\in\mathbf{N}$. Then

$$N(L-\lambda_k I) = \operatorname{span}\{u_k\},\ R(L-\lambda_k I) = \{u\in L_2[a, b]: (u_k, u) = 0\}$$

and so we can apply Proposition 2 with $\lambda_0 = \lambda_k$ since $(u_k, u_k) \neq 0$. In the notation of Proposition 2 but replacing l, m and $\| \ \|_X$ by l_k, m_k and $|||\ |||$ respectively, there exists $\delta_k>0$ such that $||| l_k(s) |||<1/2^{k+1}$ if $| s |<\delta_k$. Choose and fix α_k such that $0<| \alpha_k |<\delta_k$ and let $v_k = u_k+l_k(\alpha_k)$. Since

$$\sum_{j=1}^{\infty} \| u_j - v_j \|^2 = \sum_{j=1}^{\infty} \| l_j(\alpha_j) \|^2 \leqq \sum_{j=1}^{\infty} ||| l_j(\alpha_j) |||^2 \leqq \sum_{j=1}^{\infty} 1/2^{j+1} < 1,$$

by Proposition 1, $\{v_n\}$ is a basis for $L_2[a, b]$ and so $\{\alpha_n v_n\}$ is also a basis for $L_2[a, b]$. Since, by Proposition 2, $\alpha_k v_k$ is an eigenfunction for (4) corresponding to the eigenvalue $m_k(\alpha_k)$, we have proved that $L_2[a, b]$ has a basis consisting of eigenfunctions of (4).

3. Applications

(*a*) The hypothesis that N maps $D(L)$ into $L_2[a, b]$ is more easily satisfied than the hypothesis that N maps $L_2[a, b]$ into itself. For example, if Nx is a polynomial in x the former hypothesis is satisfied but the latter is not. Hence, if $c_i : [a, b] \to \mathbf{R}$ is continuous and $k_i \in \mathbf{N}$, $k_i > 1$ for $i = 1, 2, \ldots, n$, then Theorem 1, with $N_1 = 0$ and $N_2 = 0$ respectively, shows that there exist bases $\{v_n\}$ and $\{w_n\}$ for $L_2[a, b]$ such that $\{v_n\}$ and $\{w_n\}$ consist of eigenfunctions of

$$-(pu')'(x) + q(x)u(x) = \lambda \left(u(x) + \sum_{i=1}^{n} c_i(x)[u(x)]^{k_i} \right) \tag{5}$$

and

$$-(pu')'(x) + q(x)u(x) + \sum_{i=1}^{n} c_i(x)[u(x)]^{k_i} = \lambda u(x) \tag{6}$$

respectively, satisfying boundary condition (2).

(*b*) It is clear from the proof that Theorem 1 will hold for appropriate nonlinear perturbations of any unbounded self-adjoint operator L on a Hilbert space H if L possesses a complete orthonormal system of eigenfunctions corresponding to simple eigenvalues. In particular the theorem is applicable in the case of perturbations of a linear Sturm-Liouville problem with discrete spectrum on the interval $[0, \infty)$.

Consider the differential expression $-u'' + qu$ on $[0, \infty)$ where $q : [0, \infty) \to \mathbf{R}$ is continuous and $\lim_{x \to \infty} q(x) = \infty$. Let $L : D(L) \to L_2[0, \infty]$ be such that $Lu = -u'' + qu$ where $u \in D(L)$ if and only if $u \in L_2[0, \infty]$, u' is absolutely continuous on $[0, T]$ for all $T > 0$, $-u'' + qu \in L_2[0, \infty]$ and $u(0) = 0$. Then

(i) since q is bounded below, $-u'' + qu$ is limit point, i.e. L is selfadjoint (Everitt (2));

(ii) since $\lim_{x \to \infty} q(x) = \infty$, L has discrete spectrum (Titchmarsh (6));

(iii) if $u \in D(L)$, then $u' \in L_2[0, \infty]$. (Everitt (2), Section 5.)

If $u \in D(L)$, then $u' \in L_2[0, \infty]$ and $u(0) = 0$ and so it can be shown that (see, for example, Stuart (5), Proposition 2.3) $u \in L_p[0, \infty]$ for $p > 2$. Hence, if $c_i : [0, \infty) \to \mathbf{R}$ is bounded, $k_i \in \mathbf{N}$ and $k_i > 1$ for $i = 1, 2, \ldots, n$,

$$N : u \to \sum_{i=1}^{n} c_i u^{k_i}$$

satisfies the hypotheses of Theorem 1 and so there exists a basis for $L_2[0, \infty]$ consisting of eigenfunctions of

$$-u''(x)+q(x)u(x) = \lambda\left(u(x)+\sum_{i=1}^{n} c_i(x)[u(x)]^{k_i}\right);\ u(0)=0,$$

and a basis for $L_2[0, \infty]$ consisting of eigenfunctions of

$$-u''(x)+q(x)u(x)+\sum_{i=1}^{n} c_i(x)[u(x)]^{k_i} = \lambda u(x);\ u(0)=0.$$

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