New Blackfriars (2024), **105**, 506–529
doi:10.1017/nbf.2024.55

ARTICLE

Are Mathematical Objects '*sui generis* Fictions'? Some Remarks on Aquinas's Philosophy of Mathematics

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(Received 20 April 2024; revised 18 July 2024; accepted 24 July 2024)

Abstract

This contribution proposes an interpretation of Thomas Aquinas's philosophy of mathematics. It is argued that Aquinas's philosophy of mathematics is a coherent view whose main features enable us to understand it as a moderate realism according to which mathematical objects have an *esse intentionale*. This *esse intentionale* involves both mathematicians' intellectual activity and natural things being knowable mathematically. It is shown that, in Aquinas's view, mathematics' constructive part does not conflict with mathematical realism. It is also held that mathematics' imaginative reasoning is coherent with Aquinas's doctrine of formal abstraction and his realistism. It focuses on some of Aquinas's texts, which it places within their textual and doctrinal context and interprets them in the light of some historical elements.

Keywords: fictionalism; formal abstraction; mathematical realism; philosophy of mathematics; Thomas Aquinas

1. Introduction

In their *Thomas Aquinas and Some Thomists on the Nature of Mathematics*, David Svoboda and Prokop Sousedik hold that Aquinas's philosophy of mathematics comprises two irreconcilable and contradictory parts.[1] Though Aquinas holds, sometimes, that mathematics is concerned with real abstracted quantity, that is, the real quantitative features of natural things (such as the 'curvature' of a snub nose for a snub nose or the 'five' fingers of a hand) considered only *qua* quantitative, he also seems to suggest that some mathematical objects lack reality. Such objects are mathematical constructions (i.e., diagrams) conceived, for the most part, with the aid of the imagination.

This 'constructivist' feature would conflict with Aquinas's realism. If mathematics deals with real abstracted quantity, why do some objects lack real counterparts? If, as

[1]David Svoboda and Prokop Sousedik, 'Thomas Aquinas and Some Thomists on the Nature of Mathematics', *Review of Metaphysics*, 73 (2020), 715–40 (p. 717).

a speculative or theoretical science, mathematics is limited to contemplation, then why has Aquinas also acknowledged that some mathematical disciplines deal with constructions and that their judgements must terminate in the imagination? If mathematics is a science of the real, why does it deal with objects or notions structurally similar to logical entities, commonly known as 'beings of reason'?

On this basis, Svoboda and Sousedik raise significant doubts regarding the coherence of Aquinas's doctrine of mathematical abstraction and his philosophy of mathematics, which deserve careful consideration.[2] In what follows, I propose to examine these doubts through historical and doctrinal lenses to appreciate their significance. I will proceed in three stages. First, I will focus on Aquinas's view on the status of mathematics concerning both the speculative sciences and the seven liberal arts to see whether, instead of a contradiction, such a view reflects Aquinas's account of mathematical knowledge as the *genus* encompassing mathematical disciplines and subdisciplines as specific instances. Second, I shall examine Aquinas's doctrine of formal abstraction in connection with the universality and the individuality of mathematical objects. I will argue that the role of formal abstraction, in Aquinas's view, has less to do with the logical 'quantity' of mathematical objects and more with their very formal structure, namely, 'to be intellectually independent of sensible matter', regardless of their universality or individuality. My point is that, in Aquinas's view, these characteristics derive from a mathematician's further consideration of both mathematical essences and constructed mathematical objects according to a 'part-whole' model, similar to total abstraction. Third, by examining Aquinas's comparison of mathematical objects with those of logic, I shall point out that despite their having only a remote foundation in reality and a direct foundation in the intellect's activity, aided by the imagination, mathematical objects need not be thought of as *sui generis* fictions or mere beings of reason, but rather as having an intentional mode of existence. Accordingly, mathematical and physical quantities would differ in their intelligible structure. They are essentially the same, although the attributes deriving from their respective mode of being (*intentionale–naturale*) are different.

2. Mathematical knowledge and mathematical disciplines

2.1 Mathematics: '*scientia speculativa*' or 'ars'?

Aquinas's view about speculative sciences is that their subject-matter (*speculabilia*) is such that it cannot be made or done by us, for the end of the speculative intellect is the truth it *simply* contemplates. Unlike the practical intellect, whose end is the truth ordered to action, the speculative intellect is limited to the apprehension of things.[3] This would exclude, in principle, any constructive reasoning. Speculative sciences are less about creating their objects than about contemplating real features of natural things abstractly.[4] Through this 'abstractive' way of consideration, we obtain

[2]Cf. David Svoboda and Prokop Sousedik (2020); David Svoboda, 'Formal Abstraction and its Problems in Aquinas', *American Catholic Philosophical Quarterly*, 96 (2022), 1–20.

[3]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 1, co.

[4]Ibid., q. 5, a. 1–3.

https://doi.org/10.1017/nbf.2024.55 Published online by Cambridge University Press

formal reasons of things, i.e., the principles of intelligibility through which we know an individual thing 'x' *qua* 'F'.[5]

According to Svoboda and Sousedik, the first doubt to be raised regarding Aquinas's philosophy of mathematics concerns its status as 'speculative science'. At first glance, the above account seems to be at odds with further claims made by Aquinas himself. Though he places mathematics within the three speculative sciences, in other passages, he also points out that mathematics is concerned with the *quadrivium*,

> [whose disciplines] among the other sciences, [..] are called arts because they consist not only of knowledge but also of a work which is directly a product of the reason itself; like the composition of syllogisms or discourses, counting, measuring, composing melodies, and reckoning the course of the stars.[6]

This wording suggests that mathematics would also involve *ratio factiva* and not only *ratio speculativa*. It would then be an *ars* rather than a *scientia*. Considering this, Svoboda and Sousedik point out that placing mathematics alongside physics and metaphysics would be problematic if one would keep thinking of mathematics as a speculative science. This would even be suggested by Andronicus of Rhodes's classification of Aristotle's writings, in which the higher science is not named meta-mathematics, as would be the case if mathematics had its supposed intermediate status, but metaphysics.[7]

Though Aquinas's language is indeed ambiguous to some extent, such a simple-minded contradiction on his part may seem strange. Why would have he declared that mathematics is not a *scientia speculativa* but an *ars* where he is supposed to argue that mathematics is one of the three speculative sciences? Light must be caston this apparently problematic feature of Aquinas's philosophy of mathematics.

2.2 'Pure' and 'applied' mathematics

In twelfth and thirteenth centuries, the coexistence of several models of division and classification of sciences was not uncommon. The range of available models reflects intellectual efforts to integrate newly discovered sciences and disciplines. Issues such as 'demarcation'[8] and approaches to ancient sources were prominent concerns.[9]

[5]In this context, a 'formal reason' corresponds to what Aquinas designates in his *Super Librum Boethii de Trinitate* as 'objects of speculation', i.e., the proper objects of speculative sciences in virtue of which they are distinguished. This is how he refers to them in his *De Anima*, II, cap. 3, lect. 6, No. 307, and in his *Summa Theologiae*, Ia, q. 1, a. 3, co., where he speaks of objects of speculation or 'subject-matters' as 'formal reasons' (*ratio formalis*) through which an individual thing is known (audible, visible, intelligible, etc.).

[6]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 1, ad. 3.

[7]Cf. David Svoboda and Prokop Sousedik (2020), p. 724.

[8]By 'demarcation', here, I am not referring to the *demarcation problem* in contemporary philosophy of science, but simply to medieval discussions on which of the newly discovered disciplines was to be legitimately considered as *scientia*.

[9]Cf. Joan Cadden, 'The Organization of Knowledge, Disciplines and Practices', in *The Cambridge History of Science, Vol. II, Medieval Science*, ed. by David C. Lindberg and Michael H. Shank (New York: Cambridge University Press, 2013), pp. 240–67.

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Significant effort was devoted to organizing all the available knowledge into a coherent framework, which was both an institutional and philosophical challenge. Medieval masters were familiar with Aristotelian tripartite divisions, Platonic-like bipartite divisions into divine and human sciences, and even purely pedagogical divisions inherited from Arabic sources. Prologues and introductions to their commentaries and treatises became ideal 'forums' for medieval thinkers to elaborate on these topics.[10]

At least two general kinds of division can be observed: *formal* divisions and *institutional* or 'didactical' divisions. The most popular were (1) liberal-mechanic arts, (2) theoretical-practical sciences, and (3) 'physics–logic–ethics' threefold distinction.[11] Both (1) and (2) are of particular interest, as it was common for both distinctions to coexist. Both (1) and (2) take *abstracted quantity* as the object of mathematics. They differ, however, in their treatment of mathematics. In the former, mathematics (i.e., the mathematical disciplines of the quadrivium) played an instrumental role in physics; in the latter, it was considered one of the three theoretical sciences.[12] The coexistence of those models strongly suggests that no division was universally accepted.

In light of this, Aquinas's mention of both the *quadrivium* and the speculative sciences appears to reflect such coexistence rather than an error. Aquinas's assertion that '[…] the seven liberal arts do not suitably divide theoretical philosophy',[13] in response to the third objection,[14] indicates that these divisions are not comparable due to their adherence to different classification criteria. Thus, although Aquinas acknowledges both the 'seven-liberal-arts' division, where mathematics corresponds to the quadrivium,[15] and the threefold division within the broader context of his response,[16] this is far from being controversial when we consider that these divisions differ in their principles and ends. Similarly, the distinction between understanding (*intellectus*) and science (*scientia*), on the one hand, and art (*ars*), on the other, stands for specific forms of knowledge *simpliciter* (*cognitio*). The former primarily involves acts of the speculative intellect, whereas the latter is guided by practical reason.[17] However, both may be seen as particular dispositions through which our intellect is perfected.[18]

In terms of their principles and ends, the threefold division arises from differences between the modes of definition of objects of speculation (i.e., *speculabilia*). On the other hand, the 'seven-liberal-arts' division seems to be grounded in an ideal order of progression toward speculative knowledge and wisdom.[19] From an epistemological

[10]Cf. Olga Weijers, 'The Organization and Content of Learning', in *A Scholar's Paradise: Teaching and Debating in Medieval Paris*, ed. by Olga Weijers (Turnhout: Brepols, 2015), pp. 45–58.

[11]Cf. Olga Weijers (2015) and Joan Cadden (2013).

[12]Cf. Joan Cadden (2013).

[13]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 1, ad. 3.

[14]Ibid., q. 5, a. 1, obj. 3.: '[…] Philosophy is commonly divided in seven liberal arts, which do not contain physics nor divine science, but only logic and mathematics. Therefore, physics and divine science should not be counted among speculative sciences.

[15]Ibid., q. 5, a. 3, ad. 3.

[16]Ibid., q. 5, a. 1, co.

[17]Thomas Aquinas, *In Aristotelis Libros Posteriorum Anayticorum*, I, lect. 44.

[18]Cf. Eleonore Stump, 'Aquinas on the Foundations of Knowledge', *Canadian Journal of Philosophy, Supplementary Volume*, 17 (1991), 125–58.

[19]This is why he refers to Hugh of Saint Victor's *Didascalicon*: '[…] seven arts are grouped [leaving out certain other ones] because *those who wanted to learn philosophy were first instructed on them*'. See Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 1, ad. 3, (italics mine).

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perspective, each division can be seen as corresponding to distinct acts performed by the speculative and the practical part of the intellect, respectively. When Aquinas indicates that the former is not adequately reflected by the latter, he suggests that what applies to the seven liberal arts (which also involves practical intellect) does not necessarily apply to theoretical philosophy (which is acted by the speculative intellect).[20]

To illustrate this, we may look at Aquinas's treatment of the so-called 'intermediate' or 'middle sciences' (*scientiae mediae*) as mathematical disciplines subordinated to higher pure mathematical sciences. In his commentary on Aristotle's *Posterior Analytics*, Aquinas outlines two ways in which middle sciences are subordinated to higher-order sciences: (1) as *species* to *genus* and (2) as *matter* to *form.*[21] To exemplify both modes, Aquinas cites optics and harmonics as respectively related to geometry and arithmetic in the second manner, by virtue of some material determination[22] – such as when we refer to some 'visual line' without affirming that it is a species of the mathematical line. Additionally, Aquinas posits that lower mathematical sciences agree with the higher in *genus*, though not in *species*. Then while the lower sciences are mathematical like the higher ones, they are also less so in that they only apply mathematical principles to sensible things.[23]

Aquinas appears to see the threefold and the 'seven-liberal-arts' divisions as respective instances of formal and institutional divisions of knowledge. While the former kind encompasses three types of knowledge (physical, mathematical, and metaphysical), the latter encompasses specific instances of such types, including the 'pure' mathematical sciences and the mathematical disciplines of the *quadrivium*. In the case of mathematics, this kind of *genus–species* distinction may help understand not only the difference between 'pure' and 'applied' mathematics but also the distinction

[20]One might think of Al-Fârâbî's scheme of division of sciences, in which any discipline of the quadrivium was conceived to have an 'active' or practical part and a 'speculative' or theoretical part. Cf. e.g. Jens Hoyrup, 'Jordanus de Nemore, 13th Century Mathematical Innovator: An Essay on Intellectual Context, Achievement, and Failure', *Archive for History of Exact Sciences*, 38 (1998), 307–63. About Al-Fârâbî's influence on Aquinas, see Claude Lafleur and Joanne Carrier, 'Abstraction, séparation et tripartition de la philosophie théorétique: Quelques éléments de l'arrière fond farabien et artien de Thomas d'Aquin, Super Boethium « De Trinitate », question 5, article 3', *Recherches de Theologie et Philosophie Medievales*, 67 (2000), 248–71.

[21]Thomas Aquinas, *In Aristotelis Libros Posteriorum Analyticorum*, I, lect. 25, cap. 13.

[22]We may here think of what Robert Grosseteste has referred to as an 'added condition' in his *Commentarius in Posteriorum Analyticorum Libros*, 1.18, (ed. P. Rossi), Florence, 1981: 'Scientia autem est subalternate alii cuius subjectum *addit conditionem* super subjectum subalternantis (…)' ('A science is subalternate to another when its object *adds some condition* to the subalternating object'), (italics mine). On Grosseteste's treatment of subalternated sciences, see W. R. Laird's, 'Robert Grosseteste on the Subalternate Sciences', *Traditio*, 43 (1987), 147–79, who gives a good overview on the topic and refer to relevant works as regards Grosseteste. As regards Thomas Aquinas, see C. A. Ribeiro do Nascimento, 'Le statut épistémologique des "sciences intermédiaires" selon S. Thomas d'Aquin', in *La Science de la Nature: Théories et pratiques* (Cahiers d'Études Médiévales 2, Montréal: Bellarmin, 1974), pp. 33–95; *De Tomás de Aquino a Galileu* (Campinas: IFCH, Unicamp, 1998–2ª ed), and W. A. Wallace, *Causality and Scientific Explanation*, 2 vols (Ann Arbor, 1972-74).

[23]Thomas Aquinas, *In Aristotelis Libros Posteriorum Analyticorum*, I, lect. 25, cap. 13.

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between the speculative and the demonstrative parts of mathematics.[24,25] Moving forward, I will now address further doubts raised by Svoboda and Sousedik.

3. Formal abstraction and universals of mathematics

3.1 Does formal abstraction lead to mathematical universals?

A second doubt raised by Svoboda and Sousedik concerning Aquinas's philosophy of mathematics has to do with formal abstraction. They argue that unlike total abstraction (which abstracts the universal from the singular), formal abstraction does not seem to result in knowledge of universals but rather of individual abstracted figures such as lines, triangles, and so on.[26] Consequently, this would lead to additional difficulties regarding the status of mathematical knowledge as 'speculative science'. If speculative sciences deal with the universal and necessary, then mathematics cannot be a speculative science, as it deals with individual objects.

Curiously, Aquinas addresses this question in his commentary on Boethius's *De Trinitate*, where he engages with a comparable objection:

> Again, all straight lines are specifically the same. But the mathematician treats of straight lines by numbering them; otherwise he would not treat of the triangle

[24]Such a '*genus-species*' distinction has some explanatory value for, at least, some medieval mathematicians. An example is the preface to the so-called 'Adelard III' version of Euclid's *Elements*, in which the author operates some divisions within mathematics according to such a *genus-species* model. Regarding the object of mathematics, for instance, he refers to 'quantity' as the *genus*, whose first *species* are 'discrete quantity' and 'continuous quantity' and on which the division between arithmetic and geometry is grounded. Speaking of geometry, he also says that regarding *supposition* (that is, the mathematical referents), the subject-matter of geometry is continuous quantity, which is a *subspecies* of the *genus* 'quantity'. Similarly, regarding the content, he also says that the *genus* of geometry is 'mathematics' in so far as geometry is contained within mathematics *simpliciter*, which is about quantity *simpliciter*. He even suggests that Euclid himself operates a distinction within geometry itself 'according to the parts of its matter', that is, the *species* or specific instances of continuous quantity such as 'line, surface, solid, and number, or according to the fifteenth distinctions made by [Euclid] himself (…) and called according to the distinction of principles and called by the mode of doing'. This appears to be a reference to the thirteen books of Euclid's *Elements* and other available treatises of Euclid. See *Johannes de Tinemue's redaction of Euclid's Elements, the So-called Adelard III version*, vol. 45, ed. by H. L. L. Busard (Stuttgart: Franz Steiner Verlag, 2001), and *The Commentary of Albertus Magnus on Book I of Euclid's Elements of Geometry*, vol. 3, ed. by A. Lo Bello (Boston: Brill Academic Publishers 2021) for the English translation.

[25]Pascale Bermon's approach gives some elements of Aquinas's epistemology which may prove helpful to understand this division. Bermon refers to Aquinas's theory of habits to treat the model of science he endorses as a large format (contrasting with those of other philosophers). To put it simply, Aquinas sees sciences as habits. Accordingly, the subordination of some sciences to another might be seen as a subordination of habits to other habits. Higher habits would be those whose perfecting objects are more general than those of lower habits. Since Aquinas characterises the objects of sciences as 'formal reasons' (see footnote No. 5), it could be said that objects of the quadrivial disciplines have a mathematical formal reason as their *genus*. Cf. Pascale Bermon, 'Tot Scibilia quot Scientiae? Are There as Many Sciences as Objects of Science? The Format of Scientific Habits from Thomas Aquinas to Gregory of Rimini', in *The Ontology, Psychology, and Axiology of Habits (Habitus) in Medieval Philosophy, Historical-Analytical Studies on Nature, Mind, and Action*, vol. 7, ed. by N. Faucher and M. Roques (Cham: Springer, 2018), pp. 301–19.

[26]Cf. David Svoboda, Prokop Sousedik, (2020), p. 726: '[…] [Contrasted with formal abstraction], total abstraction is often characterized as an activity of the intellect, in which the universal is separated from the singular, but a similar characteristic of formal abstraction is not mentioned by Aquinas'.

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> and the square. It follows that he considers lines as specifically the same and numerically different. But it is clear from the above that matter is the principle differentiating things specifically the same. So the mathematician treats of matter.[27]

The whole objection hinges on the idea that since matter is a principle of individuation, anything individual must be material. Accordingly,

i. Mathematicians regard their objects as numerically distinct yet specifically identical.
ii. Two numerically distinct yet specifically identical objects are different instances of the same species.
iii. Mathematicians consider individual instances of certain species.
iv. Something is individual by virtue of matter (since matter serves as a principle of individuation).
v. Therefore, mathematicians deal with material things and, consequently, with matter and motion.

While Aquinas's objector does not raise doubts about the 'speculative' nature of mathematics as Svoboda and Sousedik do, such a conclusion appears legitimate when one considers Aquinas's assertion that objects of speculative sciences derive one of its characteristics from the side of the intellect, namely, the immateriality. This is one reason why 'separation from matter and motion, or connection with them, essentially belongs to an object of speculation, which is the object of speculative science'.[28]

In responding to this objection, one might expect Aquinas to deny that mathematics deals with individuals, as he is supposed to demonstrate that mathematical objects do not involve matter and motion. However, that is different from what he does.[29] He rather seems to suggest that mathematicians may have some acquaintance with matter by referring to a sensible faculty such as 'imagination', which would allow conceiving of *individual* mathematical objects:

> […] even when the intellect has abstracted quantity from sensible matter, one still may imagine numerically different things of the same species, for instance, several equilateral triangles and several equal straight lines.[30]

[27]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, obj. 3.

[28]Ibid., q. 5, a. 1, co. By 'connection with them', Aquinas is referring to the degree of separation from matter and motion when not totally separated, which he adds at the end of the passage.

[29]As indicated by Svoboda and Sousedik, (2020), p. 727: '[…] [That Aquinas speaks of formal abstraction as concerned by the universal is difficult to defend]. This is attested to by Aquinas's reaction to the objection according to which a mathematician works with geometrical objects as if they were singulars […]. Aquinas does not say that mathematics does not deal with singular objects because it is a theoretical science. He chooses a different strategy, according to which, even when quantity is abstracted "from sensibly available matter, it is still possible to imagine numerically different beings of the same kind" […]. Thus, geometry deals with singular, not universal, mathematical objects because objects obtained by formal abstraction are numerically different and can even be imagined'.

[30]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, ad. 3.

Considering this, the doubt raised by Svoboda and Sousedik can be formulated as follows: if mathematics is a speculative science, then the objects it obtains through 'formal abstraction' must be universal rather than individual. However, in the above-mentioned passages, Aquinas asserts that mathematicians deal with individual objects. Therefore, if mathematics deals with individual objects, and then somehow with matter, mathematics cannot be counted among the speculative sciences.

This places 'formal abstraction' at the core of the discussion, given Svoboda's and Sousedik's assumption that objects obtained through formal abstraction '*ought* to be universal'.[31] If mathematics deals with individual objects, it would follow, according to them, that formal abstraction fails in explaining the universality of mathematical objects on which mathematics' character of 'speculative science' would rest.

Addressing these issues will involve questioning Svoboda's and Sousedik's interpretation of Aquinas's doctrine of formal abstraction. Not only because there is nothing to prevent Aquinas from thinking that mathematics deals with universal as well as individual objects. But also, and above all, because it is not necessary, philosophically speaking, for the abstracted object to be universal through formal abstraction *alone*. Assuming that the universality of mathematical objects depends only on formal abstraction, the specific difference between 'total abstraction' as the 'abstraction of the universal from the particular' and 'formal abstraction' becomes obscure. Furthermore, if 'total abstraction' as the 'abstraction of the universals' pertains exclusively to physics, since one necessary condition to be a 'speculative science' is 'universality', then it would follow that *only* physics would be a speculative science, because even the separate beings studied in metaphysics (or divine science) may be individual despite their immateriality (take, for instance, an angel). Would not it have been easier to say that any abstraction is an abstraction of the universal from the particular?

3.2 Some words on Aquinas's doctrine of 'formal abstraction'

To shed light on these points, one should recall first what Aquinas says about 'formal abstraction' in his *Super Librum Boethii*, q. 5, a. 3. From the beginning of the article, Aquinas distinguishes two operations of the intellect: what he calls '*intelligentia indivisibilium*' and '*compositio et divisio*'. By the first operation, the intellect 'knows what a thing is', i.e., 'the nature itself of a thing, in virtue of which the object known holds a certain rank among beings, whether it be a complete thing, *like some whole*, or an incomplete thing, *like a part or an accident*'.[32] The second one is concerned with the '*esse*' of things, forming affirmative or negative statements. In the rest of the article, he calls the first '*abstractio*' and the second '*separatio*', ascribing the latter to metaphysics.[33]

As regards abstraction, Aquinas notes that it cannot be made except upon things which are united in reality, either as '*part to whole*' or as '*form to matter*'. Accordingly, depending on the kind of union it is acted upon, 'abstraction' will be

[31]Cf. Svoboda and Sousedik, (2020), p. 726–27: '[Regarding formal abstraction, Aquinas] mostly speaks about "abstraction of form from sensibly available matter", but he does not add, as we would expect, that an object grasped in this way is universal. But the form of a circle of a certain radius *ought to be universal* because it is really present in many singular ten-crown coins and can be predicated of them' (italics mine).

[32]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, co, (italics mine).

[33]Ibid., q. 5, a. 3, co.

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'total' or 'formal'. At first glance, the former corresponds to 'physics' while the latter to 'mathematics'.[34]

At this point, when looking carefully at Aquinas's wording about both the abstracted 'whole' and 'form', it may be noted that while he refers to the first as a nature *absolutely considered,*[35] he refers to the second as 'the accidental forms of quantity and figure'.[36] Similar wording is found in Aquinas's *De Ente et Essentia.*[37] In the third chapter, Aquinas claims that it is from the intellectual existence of an 'essence absolutely considered' that a universal is obtained when it is compared to the individuals of which such an essence is predicated. As regards the 'accidental forms', he only deals with them in the last chapter to say that they are 'incomplete essences' since their notion (as accidents) always depend on the notion of something other (substance).[38] Accordingly, the quiddities considered through formal abstraction have nothing to do, from the outset, with any logical quantity. They are, *en l'état*, just accidental forms (quantitative essences), not yet considered as 'wholes' with respect to any part.

Does this entail that these objects ought to be individual and that mathematics does not deal with universal objects? In my opinion, Aquinas's doctrine of formal abstraction in his *Super Librum Boethii de Trinitate* does not support such a conclusion. At this point, the only thing one can say is that 'formal abstractions' are forms *considered* without sensible matter. Even if one could expect them to be universal, this aspect does not derive from formal abstraction alone, for an individual quantity can also be *considered* without sensible matter. This is why outlining Aquinas's distinction, in his corpus, between 'objects of speculation' (*speculabilia*) and 'objects scientifically knowable' (*scibilia*) may be helpful in addressing this topic.

3.3 'Objects of speculation' and 'objects scientifically knowable'

According to Aquinas, the difference between the three speculative sciences (*physics, mathematics*, and *metaphysics*) rests on the difference between their objects of speculation (*speculabilia*).[39] *Speculabilia* are defined as 'objects *qua* objects', that is, as things considered by a cognitive power according to a certain formal reason. In Aquinas's words, just as

> (…) it is incidental to a sense object as such whether it be an animal or a plant, (…) [and that] the distinction between the senses is not based upon this difference but rather upon the difference between colour and sound, [so] the speculative sciences must be divided according to the differences between objects of speculation, considered precisely as such.[40]

[34]Ibid., q. 5, a. 3, co.

[35]Ibid., q. 5, a. 3, co: 'This is the abstraction of a whole, in which we consider a nature absolutely, according to its essential character'.

[36]Ibid., q. 5, a. 3, co.

[37]Thomas Aquinas, *De Ente et Essentia*, cap. 3.

[38]Thomas Aquinas, *De Ente et Essentia*, cap. 6.

[39]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 1.

[40]Ibid., q. 5, a. 1.

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A characteristic feature of *speculabilia* is their degree of immateriality, which is known by looking at what the definitions (especially the *mode of definition*) reveal.[41] Regarding mathematics, it is by virtue of formal abstraction that we are told that mathematical definitions do not involve sensible matter. If we take any definition of a mathematical or geometrical treatise of Aquinas's time, like Euclid's *Elements*, one may note that, just as any definition *tout court*, it only states *what* a mathematical object is, i.e., a mathematical essence such as 'line', 'surface', 'circle', etc. It does that without reference to sensible matter. By *speculabilia*, in mathematics, Aquinas probably designates the essences expressed in the definitions at the beginning of a geometrical treatise such as the *Elements*. It is these *speculabilia* that one would obtain via formal abstraction.

Speculabilia must be distinguished from what Aquinas calls *scibilia* or 'objects scientifically knowable'. He defines these as 'the conclusions of a demonstration wherein proper attributes are predicated of their appropriate subjects'.[42] By looking again at the *Elements*, to extend the parallel, we may take the individual mathematical objects in the conclusions of the proofs to correspond to the *scibilia*. So, objects like 'this triangle *ABC*' (I, prop. 1), 'this rectilinear angle *BAC*' (I, prop. 9), or 'this line *AB* touching the circle *BCD*' (III, prop. 17) are probably what Aquinas has in mind when he speaks of *scibilia*. These are individual instances of the quantitative essences 'triangle', 'rectilinear angle', 'line', and 'circle'. They are related to them as conclusions derived from first principles, which Aquinas will further treat as 'definitions laid down *as* [universal] principles'.[43]

Yet, though *scibilia* differ from *speculabilia* as individual instances from their universals, they seem to share an intelligible structure deriving from the way in which *speculabilia* are grasped (formal abstraction): *being considered without sensible matter*. It would be worth noting that formal abstraction is only about considering forms without *sensible* matter at all, so

> [...] mathematics does not abstract from all matter, but *only from sensible matter*. However, the parts of the quantity from which the [mathematical] demonstration proceeds as from a material cause are not sensible matter, but they pertain to intelligible matter, which is indeed found in mathematics, as is clear in the Book VII of the Metaphysics.[44]

To be considered without sensible matter is common to both individual and universal objects of mathematics. Their 'material' difference lies on the side of the individuality or commonness of their *intelligible* matter,[45] which one must judge from the logical quantity of the object (as it is a 'universal' or an 'individual').

[41]Ibid., q. 5, a. 1.

[42]Thomas Aquinas, *In Aristotelis Libros Posteriorum Anayticorum*, I, Lect. 10, cap. 4.

[43]Ibid., I, Lect. 5: 'Another type of position is the one which does not signify existence or non-existence: in this way a definition is a position. For the definition of "one" is laid down in arithmetic as a principle, namely, that "one is the quantitatively indivisible"'.

[44]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, ad. 4, (italics mine).

[45]On Aquinas's fourfold distinction of matter as regards 'mathematical abstractions', see *Summa Theologicae*, q. 85, a. 1, ad. 2, where he distinguishes 'individual sensible matter', 'common sensible matter', 'individual intelligible matter', and 'common intelligible matter'.

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It seems, then, that Aquinas's view on formal abstraction is that it gives any mathematical object, whether individual or universal, its structure. The aim of Aquinas's doctrine of formal abstraction is less to account for the universality or individuality of mathematical objects than for their intellectual separation from sensible matter. This is what the mathematical mode of definition tells us about mathematical objects: they are objects existing in matter but considered without [sensible] matter at all.[46] Accordingly, there is nothing wrong in conceiving of mathematical objects as formal abstractions even when they are singular.

Passages of the sixth question of his *Super Librum Boethii de Trinitate* and further texts of his corpus are along these lines. They ascribe a central role to the imagination regarding the method of mathematics and the grasping of individual mathematical objects, which presents some structural similarities with 'formal abstraction':

> But singulars are known only as long as they come under the senses or imagination, which is called an intellect here because it considers things without the senses just as the intellect does. But when singular circles of this kind are removed from a state of actuality, i.e., when they are no longer considered by the senses (as sensible circles) or by imagination (as mathematical circles), it is not evident whether they exist as singulars; yet they are always referred to and known by their universal formula.[47]
>
> [...] Accordingly, the knowledge we have through judgment in mathematics must terminate in the imagination and not in the senses because mathematical judgment goes beyond sensory perception. Thus, the judgment about a mathematical line is not always the same as that about a sensible line. For example, that a straight line touches a sphere at only one point is true of an abstract straight line but not of a straight line in matter, as is said in the De Anima.[48]

Strictly speaking, to *formally abstract* is different from *to imagine.*[49] However, they need not be entirely at odds with each other, for, as Aquinas puts it, the imaginative apprehension of individual mathematical objects somehow parallels the intellectual apprehension of intelligible objects:

> Now that some singulars are considered among the objects of mathematics is clear from the fact that in this order many things of the same species are observed as many equal lines and many similar figures. And such singulars are said to be intelligible insofar as they are grasped without the senses using imagination alone, which is sometimes referred to as an intellect, according to the statement in Book III of The Soul. [...] singulars are known only as long as they

[46]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, co.

[47]Thomas Aquinas, *In Duodecim Libros Metaphysicorum Aristotelis Expositio*, VII, lect. 10, No. 1495.

[48]Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 6, a. 2, co.

[49]According to Aquinas, formal abstraction is a purely intellectual operation (*Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, co.) whereas imagination is, at best, an internal power which stores traces of sensory acts.

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come under the senses or imagination, which is called an intellect here because it considers things without the senses just as the intellect does.[50]

This is because, just as the latter, the former is autonomous from what the sense-organs reveal. If my suggestion about formal abstraction as giving mathematical objects their structure (i.e., objects considered without *sensible* matter at all) is correct, then one can say that Aquinas's point is that the imaginative apprehension of individual mathematical objects may be seen as an instance of formal abstraction. For, as far as 'it considers things without the senses just as the intellect does', i.e., in autonomy from the sense-organs, the imagination would not deal with *individual* nor *common* sensible matter, but only with *individual intelligible* matter, which is also the reason why such objects, though individual, would be intelligible.[51] To some extent, it is as if imagination could somehow formally abstract, which, though it appears to conflict with Aquinas's doctrine of formal abstraction at first glance, is not irreconcilable with.[52]

3.4 Aristotelian subtleties of the doctrine of abstraction

Part of the difficulty in conciliating Aquinas's doctrine of formal abstraction with the imagination stems from limiting the explanatory scope of his general doctrine of abstraction. This is partly what Alexander of Aphrodisias made in answering to Porphyry's questionnaire about universals. Some scholars have suggested that he reduced Aristotle's inductive model (somehow abstractive) to a purely geometrical model of dematerialisation. Accordingly, grasping universals was no longer the term of inductive reasoning (going from particular to universal propositions) but that of a process of setting aside sensible matter, which in Aristotle's view was strictly the preserve of geometry.[53]

Along these lines is the interpretation of Aristotle's doctrine of abstraction as a *sui generis* dialectical process consisting of a selective consideration of some features of

[50]Thomas Aquinas, *In Duodecim Libros Metaphysicorum Aristotelis Expositio*, VII, lect 10, No. 1494-1495.

[51]In commenting on Aristotle's *Metaphysics VII*, Aquinas suggests that being individual, and then, material, does not entail being sensible. See Thomas Aquinas, *In Duodecim Libros Metaphysicorum Aristotelis Exposition*, VII, lect. 11, No. 1521: '[Aristotle] answers that it makes no difference to his thesis whether the material parts [of mathematical objects] are sensible or not, because there is intelligible matter even in things which are not sensible'. The parts at stake are not parts of the species. These are called 'matter' as they are the principle of individuation of an individual (this line) whose being is not identical to its species (line). In other words, Aquinas's point is that 'anything sensible must be material whereas anything material need not be sensible'.

[52]More should be said about Aquinas's view on the status of imagination in mathematics. However, I have limited myself, here, to an explanation of the passage in question which serves my purpose without extending the content of the paper, since I deal with this question in an article based on a talk given at the 'Nancy-Liège Workshop on Mathematical Intuition', hosted by Andrew Arana, Yacin Hamami, Gerhard Heinzmann and Bruno Leclercq in May and November 2023. Cf. Daniel E. Usma Gomez, 'Mathematical Intuition as Imagination: The Case of Aquinas Philosophy of Mathematics'. The volume is currently being edited by the journal *Logique et Analyse* and scheduled for publication in 2025.

[53]Cf. Clelia Crialesi, 'The Status of Mathematics in Boethius: Remarks in the Light of his Commentaries on the Isagoge', in *The Sustainability of Thought: An Itinerary Through the History of Philosophy*, ed. by Lorenzo Giovannetti (Napoli: Bibliopolis, 2020), pp. 95–124, who provides an overview of this phenomenon as regards Boethius's philosophy of mathematics. Cf. also Alain de Libéra, *L'art des généralités: théories de l'abstraction* (Paris: Aubier, 1999).

any substance independently of the others.[54] Thereupon, any selective consideration would be an instance of abstraction *simpliciter*. If this was indeed Aristotle's view, one might ask whether the same held for Aquinas. For, though he defines formal abstraction as a purely intellectual operation, he never says that it must be independent of other powers.[55] In that way, just as abstraction *simpliciter* is a *sui generis* process of selective consideration of a substance's features independent of the others, so may be formal abstraction. It would be a *sui generis* process of considering features such as the 'quantitative essences' and their derived attributes setting aside any sensible matter to keep only intelligible matter. Thus, grasping a mathematical object with the aid of the imagination would be an instance of formal abstraction provided that sensible matter is not involved.

Accordingly, imagination would function in mathematics *like* the intellect when it selectively considers things without sensible matter at all, despite these things being singular. These are the individual mathematical objects resulting from mathematical demonstrations or constructions (*scibilia*). If they are also 'formal abstractions', this is because the imagination grasps them without the sense-organs, that is, without sensible matter, just as the intellect grasps the quantitative essences. They differ, however, on the 'individuality' and 'commonness' of the remaining intelligible matter, whose consideration is beyond the field of formal abstraction.

3.5 Mathematical universals

Whereas Aquinas's 'intellect-imagination' parallel based on the structure of formal abstraction accounts for the knowledge of individual mathematical objects, it appears to say nothing about the universal mathematical objects or the former *qua* individual. Svoboda and Sousedik conclude that Aquinas's doctrine of formal abstraction falls short of accounting for mathematics's status as a speculative science, whose object must be universal.[56] However, as I have suggested, such an account is not to be expected from formal abstraction but from *total abstraction*.

Many have been written about Aquinas's view on universals and common natures.[57] Common natures and universals differ insofar as, while the former are ontological principles of natural composites, the latter are principles of intelligibility of such composites as they are predicable of many. Universality is generally understood as an underlying nature our intellect discovers when, considering the notions of common natures in comparing them with individual extramental things, it understands

[54]Cf. Allan Bäck, *Aristotle's Theory of Abstraction* (Cham: Springer, 2014). Cf. also John J. Cleary, 'On the Terminology of "Abstraction" in Aristotle', *Phronesis*, 30 (1985), 13–45.

[55]Admitting that would have led him to a sort of dualism. Indeed, Aquinas's epistemology would have consisted of an account of knowledge through intellectual powers independent of an account of knowledge through an inner-sense power, as imagination.

[56]Cf. David Svoboda and Prokop Sousedik (2020).

[57]For a general overview, cf. Ralph W. Clark, 'Saint Thomas Aquinas's Theory of Universals', *The Monist*, 58 (1974), 163–72; Gabriele Galluzzo, 'Aquinas on Common Nature and Universals', *Recherches de Theologie Et Philosophie Medievales*, 71 (2004), 131–71; Jeffrey E. Brower, 'Aquinas on the Problem of Universals', *Philosophy and Phenomenological Research*, 92 (2016), pp. 715–35; Luiz Marcos da Silva Filho, 'Fundamento do universal no singular em Tomás de Aquino: Natureza Comum, Similitude e/ou Ideia?', *Dois Pontos*, 18 (2021), 86–112.

that they can be said of many.[58] And to be strictly 'scientific', in an Aristotelian sense, sciences must deal with such natures 'predicable' of many, i.e., universals.

Aquinas appears to restrict the knowledge of universals to natural sciences:

> [...] whereas individuals contain determinate matter in their nature [*ratione*], whereas universals contain common matter [..], the [abstraction of a common nature from determinate matter] is not said to be the abstraction of form from matter absolutely, but the abstraction of the universal from the particular.[59]

In other words, it is not 'formal abstraction' but 'total abstraction' which leads to universals.

As far as Aquinas is concerned, there is no oversight or omission regarding the universality of formal abstractions. He does not think that objects grasped only through formal abstraction ought to be universal. However, this does not necessarily mean that mathematics is not about the universal and cannot be a speculative science. Mathematicians can further consider what they obtain through formal abstraction (the quantitative essences) as universals when, according to something like a 'part-whole' model, they think of them as natures predicable of many individual mathematical objects. In this direction,

> We conclude that there are three kinds of distinction in the operation of the intellect. There is one through the operation by which the intellect joins and divides, which is properly called separation, and this belongs to divine science or metaphysics. There is another through the operation by which the quiddities of things are conceived, namely the abstraction of form from sensible matter, which belongs to mathematics. And there is a third through the same operation which is the abstraction of a universal from a particular, which belongs *to physics and to all the sciences in general*, because science disregards accidental features and treats necessary matters.[60]

Leaving aside the first operation, which is not 'abstraction' but 'separation' and is proper of metaphysics, note that after having said that 'the operation by which the quiddities of things are conceived', i.e., formal abstraction, 'belongs to mathematics', Aquinas notes that it is 'through the same operation' that the universal is obtained from the particular. When formal abstraction and total abstraction would be specific instances of abstraction *simpliciter*, what deserves special attention is the statement that total abstraction pertains to '*all* the sciences in general', mathematics included.

This rests on Aquinas's distinction, at the very outside of his reply, regarding how natural things are united in reality: as form to matter (*hylomorphic* union) and as parts to their whole (*mereological* union).[61,62] Formal and total abstraction correspond to

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[58]Thomas Aquinas, *Scriptum Super Libros Sententiarum Magistri Petri Lombardi Episcopi Parisiensis*, I, d. 2, q. 1, a. 3; *In Doudecem Libros Metaphysicorum Aristotelis expositio*, VII, 13, No. 1570.

[59]Thomas Aquinas, *Espositio Super Librum Boethii de Trinitate*, q. 5, a. 2, co.

[60]Ibid., q. 5, a. 3, co, (italics mine).

[61]Ibid., q. 5, a. 3, co.

[62]I take the terms '*hylomorphic union*' and '*mereological union*' from Claude Lafleur and Joanne Carrier, 'Abstraction et séparation: de Thomas d'Aquin aux néoscolastiques, avec retour à Aristote et aux artiens',

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these kinds of union as the principle of their specific difference: abstraction is formal when the intellect operates on a hylomorphic union; it is 'total' when it acts upon a mereological union to abstract the whole (*totius*), the universal, from its parts, the singulars.

In this sense, whereas the proper of mathematics is to abstract quantitative forms or quiddities from sensible matter, when 'absolutely considered', these forms are not *per se* 'universals' but mere essences expressed in definitions. Seeing them as universals requires understanding them as 'predicable of many', which is only possible by comparing them with the individuals of which they will be predicated in the conclusions of demonstrations. In other words, the quantitative or mathematical essences are conceived as universals when thought of as wholes of which individual mathematical objects are parts. Taken *simpliciter*, a definition like 'an equilateral triangle is that which has its three sides equal' signifies nothing but a quantitative essence. It is a 'formal abstraction' because its definition involves no reference to sensible matter, though it cannot be seen as universal yet. Doing this requires comparing it with individual constructed triangles like, for instance, *ABC, DEF*, and so on, whose sides *AC, AB, BC*, and *DF, DE, EF* equal one another. Only when 'to have three sides equal' is predicated of individual triangles *ABC, DEF*, and so on, can the quantitative essence be thought of as 'predicable of many', which is what being a universal is about.

Aquinas appears to grant for the scientific character of mathematics (in an Aristotelian sense) by somehow committing it also with total abstraction. This is in line with previous passages in which Aquinas states that:

> Science treats of something in two ways: in one way, primarily and principally; and in this sense science is concerned with universal natures, which are its very foundation. In another way it treats of something secondarily, as by reflection; and in this sense it is concerned with the things whose natures they are. And, using the lower powers, it relates those natures to the particular things possessing them. For a universal nature is used by a knower both as a thing known and as a means of knowing. Thus, through the universal of man we can judge of this or that particular man.[63]

If mathematics is a speculative science, then mathematicians must further consider the mathematical essences they obtain through formal abstraction as the universal natures they will truly predicate of individual mathematical objects constructed in demonstrations. This occurs with the aid of the lower powers, which in the case of mathematics corresponds to the role of the imagination. In other words, they should conceive the quantitative forms as universal natures of individual objects of mathematics (i.e., according to a part/whole model).[64] Accordingly, though a formal-abstractive mode of consideration is the proper of mathematics, it is from

Laval Théologique et Philosophique, 66 (2010), 105–26. The term 'hylomorphic' refers to 'form-matter' compounds, where 'mereological' refers to 'whole-part' relations. In this case, since Aquinas sees the union of singulars to universals as a 'part-whole' relation, it could be named a 'mereological union'.

[63] Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 2, ad. 4.

[64] Cf. Gabriele Galluzzo (2004).

a total-abstractive mode of consideration that mathematics derives its 'scientific' character.

If I am correct, one might adopt another attitude regarding Aquinas's account of formal abstraction. Formal abstraction is not at odds with mathematics' status of speculative science because Aquinas never intended such status to stem from formal abstraction. Formal abstraction only gives a mathematical object its formal structure and characterises the mathematical habit regardless of whether the objects are universal or individual. Now, since a formally abstracted mathematical essence can prove to be predicable of many, it is only secondarily that the mathematician thinks of them as universals. At that point, it would be worth distinguishing the apprehension of those mathematical essences as what they indeed are, i.e., *quiddities*, from their further apprehension as *universals, i.e.*, predicable of many. Though both cases may be instances of *speculabilia*, they differ inasmuch as in the second case the *speculabilia* correspond to the starting point of mathematical demonstrations through which mathematical *scibilia* are known.

4. Aquinas's '*realist constructivism*': a media via between fictionalism and Platonism?

4.1 Does mathematics deal with 'real' or 'fictional' objects?

The last doubt regards Aquinas's texts in which he appears to acknowledge that mathematical objects lack real counterparts.[65] This doubt is raised by Svoboda and Sousedik relying on Aquinas's ascription of properties to mathematical objects that their real counterparts do not have. Since the mathematical properties of a physical and those of a mathematical tangent are different, it would follow that 'the geometrical tangent could not have been obtained from the real one by formal abstraction because the one who abstracts does not lie'.[66] Accordingly, mathematics would investigate 'not a really existing order but an order created by the human intellect when it cognises reality'.[67] Mathematics would deal with fictions or, at best, with *beings of reason*, considering the relevant role Aquinas ascribes to imagination in mathematical activity:

> Thus, the judgment about a mathematical line is not always the same as that about a sensible line. For example, that a straight line touches a sphere at only one point is true of an abstract straight line but not of a straight line in matter, as is said in the De Anima.[68]

For this reason, at least, part of the mathematical discourse would not be about a real order of things. Following this interpretation, such a fact would contravene the correspondence criterion for truth. Then, mathematics cannot be true because truth consists in '*adequatio rei et intellectu*'.

[65] Cf. Thomas Aquinas, *In Duodecem Libros Metaphysicorum Aristotelis Expositio*, XI, lect. 1, No. 2161.

[66] Cf. David Svoboda and Prokop Sousedik, (2020), p. 729. The authors do refer to the passage of *In Duodecem Libros Metaphysicorum Aristotelis Expositio*, XI, lect. 1, No. 2161, in which Aquinas states that some mathematical objects do not exist in the reality as the mathematician investigates them and, at best, that mathematical objects do not consist of *physical* properties. I deal below with this passage.

[67] Ibid., (2020), p. 733.

[68] Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 6, a. 2, co.

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To some extent, this reading disregards relevant elements of Aquinas's approach. One might first note that Aquinas deals with a similar interpretation in many passages. This is even the reason why he states that '*abstrahentium non est mendacium*'. Since formal abstraction does not result in a real but only in an intellectual separation from sensible matter, there is no lie in formally abstracting. Mathematicians can legitimately do that in virtue of two principles: (i) *ontological* or *essential dependence*[69] and (ii) epistemic or *intelligible dependence.*[70] To make it clear:

(i) *Essential dependence*: 'p' depends ontologically on 'q' if the existence of 'p' requires that of 'q' as its substrate.
(ii) *Intelligible* or *epistemic dependence*: 'p' depends intellectually on 'q' if the understanding 'p' depends on the understanding of the notion of 'q'.

In his account of abstraction in his *Super Librum Boethii*, Aquinas goes from the first to the second. Following Aristotle, he first states that according to the order of generation, quantity befalls substance before the other qualities.[71] This means that quantity is ontologically prior to sensible qualities. This ontological priority serves as a ground for what Aquinas treats as an *intellectual priority of quantity* concerning sensible qualities: quantity does not depend on sensible qualities to be understood.[72] As regards substance, however, quantity is posterior, which means that it depends on it to exist and to be understood.[73] Thus, the conditions making formal abstraction a legitimate and truthful operation are:

(a) *Cognitive, intellectual*, or *epistemic* independence of sensible qualities, given the genetic priority of quantity.
(b) *Cognitive, intellectual*, or *epistemic* dependence on substance, given substance's ontological (and logical) priority.
(c) *Ontological* dependence on substance (which holds for any accident or attribute).

Contrary to what Svoboda and Sousedik suggest, the structural difference between mathematical objects and their real counterparts is not a problem for Aquinas. Even though 'touching a sphere at one point' is *only* true of a mathematical line, not of a sensible line,[74] this does not entail that mathematical objects are fictions or beings of reason. This is what Armand Maurer points to.[75] Relying on some works of his contemporaries and on one singular passage of Aquinas's commentary on Peter Lombard's

[69] I take the expression '*essential dependence*' from Ross Inman, 'Essential Dependence, Truthmaking, and Mereology: Then and Now', in *Metaphysics: Aristotelian, Scholastic, Analytic*, ed. by Lukás Novák, Daniel D. Novotný, Prokop Sousedik, David Svoboda (Heusenstamm: Ontos Verlag, 2012), pp. 73–90.
[70] I take the expression '*intelligible dependence*' from Thomas C. Anderson, 'Intelligible Matter and the Objects of Mathematics in Aquinas', *New Scholasticism*, 43 (1969), pp. 555–76.
[71] Thomas Aquinas, *Expositio Super Librum Boethii de Trinitate*, q. 5, a. 3, co.
[72] Ibid., q. 5, a. 3.
[73] Ibid., q. 5, a. 3.
[74] Ibid., q. 6, a. 2, co.
[75] Armand Maurer, 'A Neglected Thomistic Text on the Foundation of Mathematics', *Pontifical Institute of Mediaeval Studies* 21, (1959), 185–92; 'Thomists and Thomas Aquinas on the Foundation of Mathematics', *Review of Metaphysics*, 47 (1993), 43–61.

Sentences in which Aquinas speaks of '*abstractio mathematicorum*' as having a remote foundation in reality though an immediate foundation in the intellect's activity,[76] Maurer concludes that mathematical objects are not merely real beings nor merely mind inventions, but intellectual constructions with a real basis.[77] They result from the (mathematical) way of knowing reality. In light of this, Aquinas's statement in his commentary on Aristotle's Metaphysics that '(..) sensible things do not consist of such lines and circles that the mathematical sciences investigate'[78] is in no way a denial of the reality of mathematical objects. Situating it within its textual context, this passage summarises Aristotle's opposition to the Platonist claim that mathematical objects have a separate existence and does not yet express Aquinas's view on the matter. At that very moment, Aquinas is simply discussing Aristotle's account of Plato's theory of intermediates to point out a difficulty arising from Aristotle's opposition to this theory:

> Then [Aristotle] argues on the other side of the question [that is, if separate Forms do not exist, then what about mathematical objects?]; for, if the objects of mathematics are not separate, it is difficult to say what the mathematical sciences deal with. For they do not seem to deal with sensible things as such, because no lines and circles such as the mathematical sciences investigate are found in sensible things. It seems necessary to hold, then, that there are certain separate lines and circles.[79]

Denying the separateness of mathematical objects, as Aristotle does, leads to a difficulty which, if not addressed, would necessitate the belief in separate mathematical objects: explaining why mathematics *seems to deal* with objects one cannot find in sensible things. Without such an explanation, Aquinas notes, 'it *seems* [*videtur*] necessary to hold, then, that there are certain separate lines and circles'.[80]

To avoid the 'necessity' of believing in separate mathematical objects, Aquinas holds that,

> [...] the truth of the matter is that mathematical objects are not separate from sensible things as regards their being but only as regards *their intelligible structure*, as has been shown above (…) and will be considered below.[81]

Which, in other words, is to say that mathematical and sensible quantities differ in so far as the latter depend *intellectually* on sensible matter, whereas the former do not, without this implying that they are mere beings of reason. '*Abstrahentium non est mendacium*' because the separateness of abstractions (here, formal abstractions) is no more than *intellectual*. On the one hand, the truth of the apprehension of quantitative essences is grounded on the substantial priority of quantity regarding the

[76] Thomas Aquinas, *Scriptum Super Libros Sententiarum Magistri Petri Lombardi Episcopi Parisiensis*, ed. by R. P. Mandonnet (Paris: Lethielleux, 1929), I, d. 2, q. 1, a. 3, co.

[77] Cf. Armand Maurer (1993).

[78] Thomas Aquinas, *In Duodecem Libros Metaphysicorum Aristotelis Expositio*, XI, lect. 16, No. 2161.

[79] Ibid., XI, lect. 16, No. 2161.

[80] Ibid., XI, lect. 16, No. 2161 (italics mine).

[81] Ibid., XI, lect. 16, No. 2162 (italics mine).

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sensible qualities. On the other hand, the truth of the judgments about individual mathematical objects rests on the correspondence, not with physical quantities, but with the intellectual or 'intentional' existence of the quantitative essences from which they derive. This is why the notion of *esse intentionale* deserves special attention.

4.2 Mathematical objects' '*esse intentionale*'

As informative and enlightening as Maurer's account is, one might still ask what kind of objects mathematical objects are. Maurer seems to have shifted from thinking of them as *beings of reason* of another kind than those of logic[82] to conceiving them as objects between real beings and beings of reason.[83] But what did Aquinas himself say?

As previously indicated, Maurer relies on a passage of Aquinas's commentary on Peter Lombard's *Sentences*[84] (which I will refer to as 'the inserted *quaestio*') to support his interpretation. This 'inserted *quaestio*' has a special status, as it is an addition of Aquinas's own to his commentary around 1265 and 1267 during his Roman stay.[85] The passage is from its mature years. Part of it states that:

> […] Sometimes, however, what the name signifies is not a similitude of the thing existing out of the soul, but something resulting from the way of understanding it [*ex modo intelligendi*]. These are the *intentions* which our intellect discovers [*adinvenit*]. For instance, the signification of the name "genus" is not a similitude of something existing out of the soul; but the intellect, understanding "animal" as being in many species, attributes to it the intention of "genus". And although those intentions have a direct foundation in the intellect, they have a remote foundation in the thing itself. It follows that the intellect is not false when it discovers these intentions. The same holds for all those [intentions] resulting from the way of understanding, as is the case for *mathematical abstractions* (*abstractio mathematicorum*) and the like.[86]

Aquinas distinguishes three kinds of notions:

(1) Notions having an immediate foundation in reality. They are *likenesses* [*similitudines*] of real things.
(2) Notions having an immediate foundation in *the way of considering reality* and only a *remote* foundation in reality.
(3) Notions having neither *immediate* nor *remote* foundation in reality.

His insistence on the remote foundation that (2) have in real things themselves, despite their having an immediate foundation in the intellect's activity, rules out any

[82]Cf. Armand Maurer (1959).

[83]Cf. Armand Maurer (1993).

[84]Thomas Aquinas, *Scriptum Super Libros Sententiarum Magistri Petri Lombardi Episcopi Parisiensis*, I, d. 2, q. 1, a. 3, co.

[85]Cf. Antoine Dondaine, 'Saint Thomas et la dispute des attributs divins (I Sent., d. 2, a. 3): authenticité et origine', Archivum Fratrum Praedicatorum, (1938), pp. 253–62. Maurer refers to this paper in his two articles.

[86]Thomas Aquinas, *Scriptum Super Libros Sententiarum Magistri Petri Lombardi Episcopi Parisiensis*, I, d. 2, q. 1, a. 3, co, (italics mine).

fictionalist or non-realist interpretation such as Svoboda's and Sousedik's.[87] If they were fictions or mere 'beings of reason', they should have been counted among (3). What deserves special attention is that Aquinas classes the '*abstractio mathematicorum*' alongside the notions of logic, which are notions or 'intentions' (*intentiones*) resulting from the way of knowing reality.

Maurer notes twice that Aquinas's view on that resemblance has nothing to do with the mathematician's act of abstracting but with the common structure between the notions in question: having an immediate foundation in the intellect and a remote foundation in reality.[88] Their resemblance does not stem from a same intellectual act, for there is nothing like a 'logical' abstraction. However, this does not mean that the mathematician's act of abstracting is irrelevant. Maurer does not seem to address the question. Still, mathematical objects's intentionality should derive from some intellectual act, as from its direct foundation.

Looking again at the *De Ente et Essentia* may help to better understand this aspect of Aquinas's thought. In the third chapter of this opuscule, he notes that an abstracted essence or quiddity may be considered 'according to the existence it has in one thing or another' and that anything we predicate of it in virtue of this or that existence will be accidentally predicated of it.[89] To this statement, he adds that a 'nature has a twofold existence: one in singular things, another in the soul, and *accidents follow upon the nature according to either existence*'.[90] This passage is of significant relevance, for it is from this claim that Aquinas will explain precisely the logical notions of *genus* and *species* as well as that of *universal*. The commonness of an abstracted nature, in virtue of which it is called a *genus*, a *species*, or a *universal*, results from considering it as *predicable of many*. Considering the case of 'human nature', Aquinas concludes that this is why

> […] the [logical] notion of the species attaches to human nature according to the existence it has in the intellect. For human nature exists in the intellect in abstraction from all that individuates; and this is why it has a content which is the same in relation to all individual men outside the soul […].[91]

In other words, logical notions would be like *accidents* following from an essence's *intellectual* existence or *esse intentionale*, i.e., from reflecting upon a previously abstracted notion. This is what 'to have a direct foundation in the mind's activity though a remote foundation in reality' means for logical notions. But what about mathematical abstractions?

What the mathematician obtains by reflecting upon previously abstracted mathematical essences are individual mathematical objects which, according to their formal reason,[92] can also be seen as mathematical abstractions. In other words, they are the

[87]David Svoboda and Prokop Sousedik (2020), p. 737: 'Both absolute numbers and geometrical objects are thus understood as *sui generis* fictions, as constructs of our minds (…). It seems that according to Aquinas, even whole numbers are invented by human beings'.

[88]Cf. Armand Maurer (1959), p. 189; (1993), p. 53.

[89]Thomas Aquinas, *De Ente et Essentia*, cap. 3.

[90]Ibid., cap. 3, (italics mine).

[91]Ibid., cap. 3.

[92]The 'mathematical' formal reason 'being intellectually separated from sensible matter'.

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mathematical constructions resulting from mathematical demonstrations (i.e., the *scibilia*). Aquinas explains this in his commentary on Aristotle's *Posterior Analytics*, where he gives an outline of the structure of a geometrical demonstration

> (…) in [mathematical] sciences those things which are first in the genus of quantity are postulated, as the unity, the line, the surface, and the like. After these have been postulated, some others are sought through demonstration, as, for instance, equilateral triangles and squares and the like in geometry. In these cases, demonstrations are said to be, as it were, operational, as when it is required to construct a triangle from a given straight line. Once constructed, some proper attributes are proven of it as, for instance, that its angles are equal, and other features of this kind. In the first case, the triangle is treated as a proper attribute, in the second case, it is taken as a subject. Then, when the Philosopher says of the triangle that we must first know what 'triangle' means, he is taking triangle as a proper attribute, not as a subject.[93]

The first stage consists in postulating those objects '*which are first in the genus of quantity*' that one can assimilate to the abstracted mathematical essences (i.e., *speculabili*a) expressed in the definitions.[94] Once laid down, the geometer reflects upon those essences to search for further attributes through an operative demonstration, i.e., construction, which, in turn, would be the starting point for investigating further mathematical attributes and so on.

If I am correct, the 'mathematical abstractions' Aquinas refers to in the 'inserted *quaestio*' are the objects or notions given in the conclusions of mathematical demonstrations (i.e., the *scibilia*). The expression 'mathematical abstractions' should be taken here to function as a generic term for mathematical objects (without sensible matter at all). They satisfy all the criteria for 'second intentions'. On the one hand, they have a remote foundation in reality through the abstracted mathematical essences given in the definitions. On the other hand, they have an immediate foundation in the *operative* way of considering mathematical essences to raise individual constructed mathematical objects of which one can truly predicate mathematical attributes. That is how we can take notions like 'equilateral' to be proved of a triangle constructed on a given straight line (Euclid, *Elements,* I, 1).

As regards the operative demonstration, Aquinas describes it as an actualisation of truths or properties which are 'potentially' or 'virtually'[95] in the first principles of the demonstration, with a special mention of the '*drawing of geometrical figures*' or, as it were, the diagrammatic activity, which makes room for imagination in mathematics as to conceive of the *individual* figures (this circle, this line, etc.) serving to construct the diagrams:[96]

[93]Thomas Aquinas. *In Aristotelis Libros Posteriorum Analyticorum*, I, lect. 2.

[94]Cf. Ibid., I, lect. 5, where Aquinas asserts that, sometimes, specifically in mathematics, a definition can be laid down as a (first) principle of a syllogism (*positio*). As such, definitions can be, at least virtually, truly predicated on a subject in the conclusions.

[95]Cf. Jean W. Rioux, *Thomas Aquinas' Mathematical Realism* (Cham: Springer Verlag, 2023), chapters 6–7.

[96]I develop further elsewhere the interaction between intellectual powers and imagination in Aquinas's view on mathematical activity. See footnote 52.

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> Geometers discover the truth which they seek by dividing lines and surfaces. And division brings into actual existence the things which exist *potentially* before division takes place. However, if all had been divided to the extent necessary for discovering the truth, the conclusions which are being sought would then be evident. But since divisions of this kind exist *potentially* in the *first drawing of geometrical figures*, the truth which is being sought does not therefore become evident immediately […].[97]

As such, the 'mathematical abstractions' in the 'inserted *quaestio*' are not free creations but actualisations of what is virtually contained in the abstracted mathematical essences existing in the intellect and laid down as the first principles of mathematical demonstrations. Rather than flirting with some form of 'fictionalism',[98] Aquinas's philosophy of mathematics can be labelled as a '*realist constructivism*' according to which objects of mathematical activity cannot be known except through construction upon quantitative essences previously abstracted from reality through 'formal abstraction'.[99] As regard the truth of these constructions, it does not result from some correspondence to physical quantities. Rather, it would involve a kind of consistency with the quantitative essences laid down as first principles, which are true by virtue of their *essential* and *intellectual dependence* on substance, on the one hand, and their *intellectual independence* of sensible qualities, on the other hand.

Their mode of existence may be called 'intentional', which holds for both the quantitative essences and the constructions, although to different degrees.[100] Focusing on the term '*intentio*' may prove helpful in reconstructing Aquinas's philosophy of mathematics. Aquinas's use of '*esse intentionale*' to designate the intellectual mode of existence of a form having an '*esse naturale*' in the physical world suggests that mathematical knowledge involves both the cognisor's act of receiving a form through abstraction and the real object from which this form is abstracted. This seems to endow the notion of intentionality with a twofold status or function insofar as it may serve to

[97]Thomas Aquinas. *In IX Metaphys*., lect. 10, No. 1888 and ff, (italics mine).

[98]Broadly speaking, 'fictionalism' is the nominalist idea that mathematical concepts are like fictional terms since they refer to nothing in the real world. Mathematical objects would be fictional or imaginary entities like fictional characters. The best-known contemporary source on this subject is Hartry Field (1946–present). To avoid any anachronism, I would like to note that in the context of a discussion of Aquinas's philosophy of mathematics, the term 'fictionalism' has a purely philosophical meaning (just as 'mathematical Platonism' does not necessarily refer to Plato's own position).

[99]Cf. Jean W. Rioux, *Thomas Aquinas' Mathematical Realism* (Cham: Springer Verlag, 2023), chapters 6–7. I recommend considering Rioux's use of this line of interpretation to shed some new light on the well-known debate 'classical-intuitionist' mathematicians about the law of the excluded-middle.

[100]The existence of the quantitative essences may be seen as 'intentional' in so far as the intellectual grasping of them through (formal) abstraction may be described in terms of reception of a form by its appropriate powers (in this case, quantitative forms), which, according to some prominent commentators, is what *intentionality* is about. See John Haldane, 'Brentano's Problem', *Grazer Philosophische Studien*, 35 (1989), 1–32; Anthony J. Lisska, 'Axioms of Intentionality in Aquinas's Theory of Knowledge', *International Philosophical Quarterly*, 16 (1976), 305–22; *Aquinas's Theory of Perception: An Analytic Reconstruction* (New York: Oxford University Press, 2016); Anthony Kenny, *Aquinas on Mind* (New York: Routledge, 1993); Roger Pouivet, *Après Wittgenstein, Saint Thomas* (Paris: Librairie Philosophique Vrin, 2014), to find out more about debates on Aquinas's notion of 'intentionality'. The constructions can be seen, in turn, as 'second intentions' since they result from the intellectual reflection upon a previously abstracted notion.

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designate the intellectual mode of existence of 'mathematical abstractions' (whether mathematical essences or constructions) as well as to characterise the way of knowing them ('formal abstraction' or reflection upon 'formal abstractions').

5. Conclusion

Does Aquinas conceive of mathematical objects as *sui generis* fictions? Although some of his views point in that direction, they do not seem sufficient to conclude that Aquinas might have endorsed a fictionalist view. Aquinas's historical context and corpus give us some clues to deal with doubts regarding his doctrine of formal abstraction and his philosophy of mathematics to clarify some ambiguities.

I tried to show that, in Aquinas' view, there is no contradiction in speaking of mathematics as a speculative science and as being concerned with the *quadrivium* if we consider the '*genus-species*' scheme according to which Aquinas distinguishes the general kind of knowledge of which some (if not all) quadrivial disciplines are specific instances. As a speculative science, mathematics would be a kind of knowledge dealing with objects with a 'mathematical formal reason' given by formal abstraction. As part of the *quadrivium*, mathematical disciplines would be specific instances of mathematical knowledge differing according to their application domain.

In this vein, Aquinas's silence about the causal relation between formal abstraction and the universality of mathematical objects necessary for mathematics' scientific character is not an issue. I intended to show that the proper of formal abstraction, according to Aquinas, is providing any mathematical object with its very formal structure of 'being intellectually independent of sensible matter'. Such structure holds for the intellectually abstracted mathematical essences (*speculabilia*), which can be *posited as* universals but are not themselves universals, as well as for the individual mathematical objects constructed with the aid of the imagination (*scibilia*). As regards the logical quantity of such objects, i.e., their universality and individuality, Aquinas seems to suggest that it derives from a further consideration of the mathematical essences and their derived constructed individual objects according to a 'part-whole' model. This would be why he speaks of 'total abstraction' as pertaining to all the sciences in general, mathematics included, in considering their *speculabilia* as being 'predicable of many'.

This highlights the 'intentional' nature of mathematical objects. Conceiving of 'formal abstractions' as mathematical essences existing in the mind may enable us to understand why Aquinas speaks of mathematical objects as being like logical notions. For, just as the logical quantity (i.e., the universality or the individuality) of any concept derives from the intellect's reflection upon abstractions, so do individual mathematical objects from quantitative essences through operative demonstrations or constructions. In other words, constructed individual mathematical objects follow from the intellectual or intentional existence of quantitative essences. Accordingly, if mathematical constructions, postulations or '*descriptiones*' lack totally identical real counterparts, this is not due to their 'fictional' character but to the mathematical accidents deriving from the *esse intentionale* of an abstracted form, which is structurally different from its *esse naturale* in physical things.

We seem to obtain a philosophy of mathematics that can be called '*realist-constructivism*', accounting for (1) why the mathematical disciplines can be called

https://doi.org/10.1017/nbf.2024.55 Published online by Cambridge University Press

'mathematical', (2) the way of grasping the mathematical essences (definitions) as the starting point necessary for mathematical activity (purely demonstrative or applied), and (3) the mode of existence of mathematical objects in a moderately realist way.

I am not pretending there is no difficulty in Aquinas's philosophy of mathematics. I aimed no more than to suggest an examination of some recently raised doubts regarding Aquinas's doctrine of formal abstraction in the light of historical, doctrinal, and textual elements I take to help deal with some of its main ambiguities. This should not dispel all issues but, on the contrary, help to address them more effectively by bringing them closer to their context without reducing them to it. Thus, one could go beyond some commonplaces such as reading Aquinas in an exclusive dialogue with Plato, Aristotle, and Arab commentators, to start taking more seriously his more direct relations with the so-called 'thirteenth century Oxford Platonists', Albert the Great, and medieval discussions about the status of mathematics.

Acknowledgements. Thanks to the reviewers for their careful reading and thoughtful comments on the first draft of this article. They were a great help in correcting the manuscript. Thanks also to fr. Jean-Christophe de Nadaï O.P. and fr. Marc Millais O.P. for their welcome and help in consulting the printed editions of Aquinas' corpus when I was preparing the manuscript. Thanks also to those who re-read the first version and helped me to clarify certain points.

Cite this article: Daniel Eduardo Usma Gómez, 'Are Mathematical Objects "*sui generis* Fictions"? Some Remarks on Aquinas's Philosophy of Mathematics', *New Blackfriars*, 105 (2024), 506–529. https://doi.org/10.1017/nbf.2024.55