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GENERALIZED MIXED QUASI-COMPLEMENTARITY PROBLEMS IN TOPOLOGICAL VECTOR SPACES

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Abstract

In this paper, we introduce and consider a new class of complementarity problems, which are called the generalized mixed quasi-complementarity problems in a topological vector space. We show that the generalized mixed quasi-complementarity problems are equivalent to the generalized mixed quasi variational inequalities. Using a new type of KKM mapping theorem, we study the existence of a solution of the generalized mixed quasi-variational inequalities and generalized mixed quasi-complementarity problems. Several special cases are also discussed. The results obtained in this paper can be viewed as extension and generalization of the previously known results.

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1. Introduction

Complementarity problems theory, which was introduced and studied by Lemke [18] and Cottle and Dantzig [10] in the early 1960s, has enjoyed a vigorous and dynamic growth. Complementarity problems have been extended and generalized in various directions to study a large class of problems arising in industry, finance, optimization, regional, physical, mathematical and engineering sciences, see [1, 2, 5–19]. Equally important is the mathematical subject known as variational inequalities, which was introduced in the early 1960s. For applications, physical formulation, numerical methods, dynamical system and sensitivity analysis of the mixed quasi-variational inequalities, see [1, 2, 5, 12, 17–19] and the references therein. It has been shown that if the set involved in complementarity problems and variational inequalities is a convex cone, then both the complementarity problems and variational inequalities are

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equivalent, see Karamardian [17]. This equivalence has played a central and crucial role in suggesting new and unified algorithms for solving the complementarity problem and its various generalizations and extensions; see [1, 3, 4] and the references therein for more details.

Inspired and motivated by the research going on in these fascinating and interesting fields, we introduce and analyze a new class of complementarity problems in topological vector spaces, which are called generalized mixed quasi-complementarity problems. This class is quite general and unifies several classes of complementarity problems in a general framework. Under suitable conditions, we establish the equivalence between the generalized mixed quasi-complementarity problems and generalized mixed quasi-variational inequalities. This alternative equivalence is used to discuss several existence results for the solution of the generalized mixed quasi-variational inequalities in topological vector spaces in conjunction with generalized KKM theorem, which is due to Fakhar and Zafarani [11]. Since the generalized mixed quasi-variational inequalities include generalized mixed quasi-complementarity problems, f-complementarity problems, general complementarity problems, various classes of variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems.

Let X be a real Hausdorff topological vector space with topological dual X^*, with $\langle ., .\rangle$ the duality pairing between them and let K, 2^A and $\mathcal{F}(A)$ denote a closed convex cone subset of X, the family of all subsets of the set A, and the family of all nonempty finite subsets of A, respectively. Let $T : K \to 2^{X^*}$ be a set-valued mapping with nonempty values and let $F : K \times K \to \mathbb{R}$ be a bicontinuous function, unless otherwise specified.

We consider the problem of finding $\bar{x} \in K$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, \bar{x}\rangle + F(\bar{x}, \bar{x}) = 0, \quad \langle \bar{t}, y\rangle + F(\bar{x}, y) \geq 0, \quad \forall y \in K, \tag{1.1}$$

which is called the *generalized mixed quasi-complementarity problem (GMQCP).*

We note that if $F(x, y) = f(x)$, $\forall x \in K$, then problem (1.1) is equivalent to finding $\bar{x} \in K$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, \bar{x}\rangle + f(\bar{x}) = 0, \quad \langle \bar{t}, y\rangle + f(y) \geq 0, \quad \forall y \in K, \tag{1.2}$$

which is known as the f-complementarity problem and generalized f-complementarity problem, introduced and studied by Itoh *et al.* [16] for a single-valued mapping T and by Huang *et al.* [14] for a set-valued mapping T, respectively. For the applications and numerical methods of problem (1.2), see [1, 2, 9, 11, 13, 18, 19].

If $F(0, .) \equiv 0$, and $K^* \equiv \{u \in X^* : \langle u, v\rangle \geq 0, \forall v \in K\}$ is a polar (dual) cone of the convex cone K, then the GMQCP is equivalent to finding $\bar{x} \in K$ and $\bar{t} \in T(\bar{x})$ such that

$$\bar{t} \in K^* \quad \text{and} \quad \langle \bar{t}, \bar{x}\rangle = 0, \tag{1.3}$$

which is called the general complementarity problem for set-valued mappings. For the recent applications, numerical results and formulation of the complementarity problems, see [2, 6, 7, 9, 11, 13] and the references therein.

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Related to the generalized mixed quasi-complementarity problem (1.1), we consider the problem of finding $\bar{x} \in K$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, y - \bar{x} \rangle + F(y, \bar{x}) - F(\bar{x}, \bar{x}) \geq 0, \quad \forall y \in K, \tag{1.4}$$

which is called the generalized mixed quasi-variational inequality problem (GMQVIP). Note that if $F(x, y) = f(x), \forall x \in K$, then GMQVI reduces to the generalized variational inequality problem introduced in [14]. For the formulation, numerical results, existence results, sensitivity analysis and dynamical aspects of the generalized mixed quasi-variational inequalities for a single-valued mapping, see [1, 2, 5, 12, 14, 17–19] and the references therein.

It is obvious that any solution of the GMQCP is a solution of the GMQVIP. The following example shows that, even when T is a single-valued mapping, the converse does not hold in general.

EXAMPLE 1.1. Let $X = Y = \mathbb{R}$, $K = [0, +\infty)$, $F(x, y) = 1$, for all $x, y \in K$ and define $T : K \to \mathbb{R}^* = \mathbb{R}$ by

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{-1\} & \text{otherwise.} \end{cases}$$

One can easily see that $(\bar{x} = 0, \bar{t} = 0)$ is a solution of the GMQVIP but it is not a solution of the GMQCP.

We now show that the problems (1.1) and (1.4) are equivalent, that is their solution sets are equal, under some conditions and this is the main motivation of our next result.

THEOREM 1.2. *Let K be a nonempty subset in X with $2K \subseteq K$, and $0 \in K$. If $F(2x, y) = 2F(x, y), \forall x, y \in K$, then the GMQCP* (1.1) and the GMQVIP (1.4) are equivalent.

PROOF. Let $\bar{x} \in K$ and $\bar{t} \in T(\bar{x})$ be a solution of the GMQVI (1.4). Then by taking $y = 0$ and $y = 2\bar{x}$ in (1.4),

$$\langle \bar{t}, -\bar{x} \rangle + F(0, \bar{x}) - F(\bar{x}, \bar{x}) \geq 0,$$
$$\langle \bar{t}, \bar{x} \rangle + F(2\bar{x}, \bar{x}) - F(\bar{x}, \bar{x}) \geq 0,$$

which implies, using $F(0, \bar{x}) = 0$ and $F(2\bar{x}, \bar{x}) = 2F(\bar{x}, \bar{x})$, that

$$\langle \bar{t}, \bar{x} \rangle + F(\bar{x}, \bar{x}) = 0. \tag{1.5}$$

Also, from (1.5) and (1.4),

$$\begin{aligned} 0 &\leq \langle \bar{t}, y - \bar{x} \rangle + F(y, \bar{x}) - F(\bar{x}, \bar{x}) \\ &= \langle \bar{t}, y \rangle + F(y, \bar{x}) - \langle \bar{t}, \bar{x} \rangle + F(\bar{x}, \bar{x}) \\ &= \langle \bar{t}, y \rangle + F(y, \bar{x}), \end{aligned}$$

that is,

$$\langle \bar{t}, y\rangle + F(y, \bar{x}) \geq 0, \quad \forall y \in K. \tag{1.6}$$

This shows that $(\bar{x}, \bar{t}) \in K \times T(\bar{x})$ is a solution of the GMQCP (1.1).

Conversely, let $(\bar{x}, \bar{t}) \in K \times T(\bar{x})$ be a solution of the GMQCP (1.1). Then, from (1.5) and (1.6), we conclude that $(\bar{x}, \bar{t})$ is also a solution of the GMQVIP (1.4), and so the proof is complete. □

REMARK 1.3.

(a) If K is a closed convex cone, then $0 \in K$, and $2K \subset K$, however the converse does not hold in general, for instance, $K = N \cup \{0\}$, where the set N denotes natural numbers, is not a convex cone while is a nonempty set with $2K \subseteq K$, and $0 \in K$. Hence Theorem 1.2 improves Theorem 2.1 in [14].

(b) If F is positively homogeneous in the first variable then $F(2x, y) = 2F(x, y)$, $\forall x, y \in K$. However the converse is not true; for instance, $F(x, y) = 0$, for x rational and $F(x, y) = x$, for x irrational, which is not positively homogeneous. Thus Theorem 1.2 improves Theorem 2.1 in [14].

In the rest of this section, we recall some definitions and preliminary results that are used in next sections.

Let X be a nonempty set, Y a topological space, and $\Gamma : X \to 2^Y$ a multi-valued map. Then, Γ is said to be transfer closed-valued if, for every $y \notin \Gamma(x)$, there exists $x' \in X$ such that $y \notin \operatorname{cl}\Gamma(x')$, where cl denotes the closure of a set. It is clear that $\Gamma : X \to 2^Y$ is transfer closed-valued if and only if

$$\bigcap_{x\in X} \Gamma(x) = \bigcap_{x\in X} \operatorname{cl}\Gamma(x).$$

Let K be a nonempty convex subset of a topological vector space (t.v.s.) X and let K_0 be a subset of K. A multi-valued map $\Gamma : K_0 \to 2^K$ is said to be a KKM map if

$$\operatorname{co} A \subseteq \bigcup_{x\in A} \Gamma(x), \quad \forall A \in \mathcal{F}(K_0),$$

where co denotes the convex hull.

By checking the proof of Theorem 2.1 in [11, Page 113, lines 15–19], one can realize that one can change condition (ii) in Theorem 2.1 into the following condition:

(ii)′ for each $x, y \in K$, $\operatorname{cl}_K(\bigcap_{z\in[x,y]} \Gamma(z)) \bigcap [x, y] = (\bigcap_{z\in[x,y]} \Gamma(z)) \bigcap [x, y]$.

The mapping $\Gamma : K \to 2^{X^*}$ is said to be C-pseudomonotone (see [15]) if the following implication, for every $x, y \in X$, and every net $\{x_i\}$ in X with $x_i \to x$, holds:

$$\sup_{f\in\Gamma(x_i)} \langle (1-t)x + ty - x_i, f\rangle \geq 0, \quad \forall t \in [0, 1], \ \forall i \in I \Rightarrow \sup_{f\in\Gamma(x)} \langle y - x, f\rangle \geq 0.$$

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Note that C-pseudomonotonicity of Γ is equivalent to (ii)$'$ (see [15, Theorem 7]).

The set-valued mapping T is said to be upper semi-continuous (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$ there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

PROPOSITION 1.4 ([6]). *Let D be a convex, compact set and let K be a convex set. Let $f : D \times K \to \mathbb{R}$ be concave and u.s.c. in the first variable and convex in the second variable. Assume that*

$$\max_{x \in D} f(x, y) \geq 0, \quad \forall y \in K.$$

Then there exists $\bar{x} \in D$ such that $f(\bar{x}, y) \geq 0 \; \forall y \in K$.

2. Main results

Our aim in this section is to establish an existence theorem of a solution to the problems (GMQVIP) and (GMQCP) in the topological vector spaces setting. The following theorem provides sufficient conditions under which the solution set of the (GMQVIP) is nonempty and compact. Furthermore, it improves Theorem 15 of [15] whenever the mapping $F \equiv 0$.

THEOREM 2.1. *Suppose that:*

(a) *T is C-pseudomonotone with nonempty compact values;*
(b) *for each $A \in \mathcal{F}(K)$, T and F are u.s.c. and continuous on* co A, *respectively;*
(c) *for each $x \in K$ $F(x, \cdot)$ is convex;*
(d) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that, for each $x \in K \backslash B$, there exists $y \in D$ such that* $\sup_{f \in T(x)} \langle y - x, f\rangle + F(x, y) - F(x, x) < 0$.

Then the set

$$\{x \in K \mid \forall y \in K, \exists f \in T(x); \langle y - x, f\rangle + F(x, y) - F(x, x) \geq 0\}$$

is a nonempty compact subset of B.

PROOF. We define $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \left\{ x \in K \;\middle|\; \sup_{f \in T(x)} \langle f, y - x\rangle + F(x, y) - F(x, x) \geq 0 \right\}.$$

(I) Γ is a KKM mapping. Otherwise, there exists $A = \{x_1, x_2, \ldots, x_n\} \subseteq K$, and $z \in \operatorname{co} A$ such that $z \notin \bigcup_{i \in \{1,2,\ldots,n\}} \Gamma(x_i)$. Hence, by definition of Γ and by choosing a fixed element $f \in T(z)$ we deduce that $\langle x_i - z, f\rangle + F(z, x_i) - F(z, z) < 0$, for $i = 1, 2, \ldots, n$. Thus, from $z \in co(A)$ and (ii) we get $0 = \langle z - z, f\rangle + F(z, z) - F(z, z) < 0$, which is a contradiction. Hence Γ is a KKM mapping.

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It is clear that Γ satisfies the following relations.

(II) $\bigcap_{x\in \mathrm{co}\, A} \mathrm{cl}_{\mathrm{co}\, A}\Gamma(x) \cap \mathrm{co}\, A = \bigcap_{\mathrm{co}\, A} \Gamma(x) \cap \mathrm{co}\, A$, $\forall A \in \mathcal{F}(A)$.

(III) For each $x, y \in K$, $\mathrm{cl}_K(\bigcap_{z\in[x,y]} \Gamma(z)) \bigcap [x, y] = (\bigcap_{z\in[x,y]} \Gamma(z)) \bigcap [x, y]$. Note that (III) follows by condition (a) [15, Theorem 7].

(IV) By the condition (d) we have $\mathrm{cl}(\bigcap_{x\in D} \Gamma(x)) \subseteq B$.

Consequently, Γ fulfills all the assumptions in [11, Theorem 2.1] and hence $\bigcap_{x\in K} \Gamma(x)$ is nonempty. It is obvious that

$$\{x \in K \mid \forall y \in K, \exists f \in T(x); \langle y - x, f\rangle + F(x, y) - F(x, x) \geq 0\} = \bigcap_{x\in K} \Gamma(x) \neq \emptyset.$$

One can easily see (by (d)) that $\bigcap_{x\in K} \Gamma(x)$ is a closed subset of B and so is compact in B. This completes the proof. $\square$

As an application of Theorem 2.1 we state the next theorem, which presents an existence theorem of the solutions for the GMQVIP in the topological vector space setting. Moreover, it improves, in the case $F = 0$, on [15, Corollary 16].

THEOREM 2.2. *Suppose that all the assumptions of Theorem* 2.1 hold. If the values of T are convex then the solution set of the GMQVIP is nonempty.

PROOF. By Theorem 2.1, there exists $\bar{x} \in K$ such that

$$\sup_{f\in T(\bar{x})} \langle f, y - x\rangle + F(x, y) - F(x, x) \geq 0, \quad \forall y \in K.$$

Now we define $P : T(\bar{x}) \times K \to \mathbb{R}$ to be

$$P(f, y) = \langle f, y - \bar{x}\rangle + F(\bar{x}, y) - F(\bar{x}, \bar{x}).$$

The mapping P satisfies all of the assumptions of Proposition 1.4 and so there exists $\overline{f} \in T(\bar{x})$ such that

$$\langle \overline{f}, y - \bar{x}\rangle + F(\bar{x}, y) - F(\bar{x}, \bar{x}) \geq 0, \quad \forall y \in K.$$

This means that $(\bar{x}, \overline{f})$ is a solution of the GMQVIP and the proof is complete. $\square$

THEOREM 2.3. *Suppose that all the assumptions of Theorems* 2.2 are satisfied. If K is a closed convex cone and $F(2u, v) = 2F(u, v)$, $\forall u, v \in K$, then the GMQCP has a solution.

PROOF. The result follows by Theorems 1.2 and 2.2. $\square$

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