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ON CONDENSED NOETHERIAN DOMAINS WHOSE INTEGRAL CLOSURES ARE DISCRETE VALUATION RINGS

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ABSTRACT. A condensed domain is an integral domain such that $IJ = \{xy : x \in I, y \in J\}$ holds for each pair I, J of ideals. We prove that, under suitable conditions, a subring of a discrete valuation ring is condensed if and only if it contains an element of value 2. We also define the concept strongly condensed.

1. **Introduction.** The concept of a condensed domain was introduced by David F. Anderson and David E. Dobbs in [1] and further developed in [2]. A condensed domain is, by definition, an integral domain such that for any two ideals I and J we have $IJ = P(I,J)$ where $P(I,J) = \{xy : x \in I, y \in J\}$. Thus each element in IJ is a product of an element of I by an element of J and not only a sum of such products. A condensed domain can be Noetherian as well as non-Noetherian. Indeed to show that a domain is condensed it is enough to consider two-generated ideals. This is proved in [1] by induction but here we take the opportunity of giving a somewhat shorter argument. Thus, suppose that $IJ = P(I,J)$ holds whenever I and J are two-generated ideals. Given two arbitrary ideals I and J we must show that $P(I,J)$ is an *ideal* ie is closed under addition. Take x_1, x_2 in I and y_1, y_2 in J. Then $x_1y_1 + x_2y_2$ is in $(x_1, x_2)(y_1, y_2)$ and hence is of the form xy for some x in (x_1, x_2) and some y in (y_1, y_2). It follows from [1] Corollary 2.9 that the dimension of a Noetherian condensed domain is at most one. In this paper we consider the following local case. R is a Noetherian ring with integral closure R' and R' is a discrete valuation ring. First, in section 2, we prove that if R is condensed then R must contain an element of value 2 in the discrete valuation on R'. In section 3 we show that the existence of an element of value 2 together with a few extra conditions on R is sufficient to imply that R is condensed in the following stronger sense.

DEFINITION. *We shall say that a domain is strongly condensed if for every pair I, J of ideals we have $IJ = aJ$ for some a in I or $IJ = Ib$ for some b in J.*

2. Let (t) be the maximal ideal of R'. Then $R[t]$ is finitely generated as an R-module, whence it follows that the conductor $C = R : R[t]$ of $R[t]$ in R is different

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from zero. Thus $R[t]/C$ is Artinian and hence t^n is in C for all $n \geqq N$, some N. But C is an ideal of R so t^n is an element in R for all $n \geqq N$. After this preamble we are ready for the proof of our first proposition.

PROPOSITION 1. *Suppose that R is condensed. Then R must contain an element of value* 2.

PROOF. Following the notation above we have $t^N, t^{N+1}, t^{N+2}, \ldots \in R$. Now let $I = (t^N, t^{N+1})$ and $J = (t^N + t^{N+1}, t^{N+2})$ be two ideals of R. Then $t^{2N+1} \in IJ$, say $t^{2N+1} = yz$ where $y \in I$ and $z \in J$. Let $y = a_0 t^N + a_1 t^{N+1}$ and $z = b_0 t^N + b_0 t^{N+1} + b_2 t^{N+2}$ where $a_0, a_1, b_0, b_2 \in R$. First we assume that a_0 is a unit (of R and hence of R') ie $v(a_0) = 0$ where v is the valuation function on R'. It follows that $v(y) = N$, hence $v(z) = N + 1$, hence $v(b_0) = 1$ and finally $v(b_0^2) = 2$. Next we assume that a_0 is not a unit ie $v(a_0) > 0$. Thus $v(y) \geqq N + 1$ but $v(z) \geqq N$ and $v(y) + v(z) = 2N + 1$, whence it follows $v(y) = N + 1$, $v(z) = N$ and $v(b_0) = 0$. We have $t^{N+1} = y(b_0 + b_0 t + b_2 t^2)$. Letting $w = y(b_0 t + b_2 t^2)$ we get $v(w) = N + 2$ but also $w = t^{N+1} - y b_0 \in I$. Thus for some $\alpha, \beta \in R$ we have $w = \alpha t^N + \beta t^{N+1}$. Thus $v(\alpha + \beta t) = 2$ and hence $v(\alpha) = 1$, $v(\alpha) = 2$ or (if $v(\alpha) > 2$)$v(\beta) = 1$. In any case this provides us with an element in R of value 2.

3. We can not prove in general that the condition in Proposition 1 is sufficient for a ring to be condensed. We must restrict ourselves to the following situation. We assume henceforth that R', the integral closure of R, is a discrete valuation ring which is *finite over* R, ie R' is finitely generated as an R-module. Further we assume that R' and R have the *same residue class* field ie that R'/M' and R/M are isomorphic, where M and M' are the unique maximal ideals of R and R' respectively. Since R' is finite over R the conductor $R{:}R'$ is an R'-ideal which is different from zero. Thus $R{:}R'$ contains some power of M'. But $R{:}R' \subseteq M$ and hence every nonzero ideal I of R contains some power of M' and we may define $N(I)$ as the smallest integer n such that $(M')^n \subseteq I$. In other words I contains all elements in R' of value greater than or equal to $N(I)$. The next lemma and the two propositions which follow are well known and easy.

LEMMA 2. *Let r_1 and r_2 be elements of R' such that $v(r_1) = v(r_2)$. Then there is a unit element u of R such that $v(r_2 - ur_1) > v(r_1)$.*

PROOF. r_2/r_1 is a unit of R'. Since $R/M \simeq R'/M'$ there is a unit u of R and an element r of M' such that $r_2/r_1 = u + r$. Thus $v(r_2 - ur_1) = v(r_1 r) > v(r_1)$.

PROPOSITION 3. *Let I and J be two ideals of R and suppose I contains J and that $v(I) = v(J)$. Then $I = J$.*

PROOF. Let a be an element of I. Then by Lemma 2 there is an element b of J such that $v(a) = v(b) = v$, say, but $v(a - b) > v$. By repeated use of Lemma 2 we get an element c of J such that $v(a - c) > N(J)$. Thus $a - c$ belongs to J and hence a belongs to J.

PROPOSITION 4. *Let I be a nonzero ideal of R and suppose that $N(I) \leqq N$ and that I contains some element of value $N - 1$. Then $N(I) \leqq N - 1$.*

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PROOF. This follows immediately from Lemma 2. □

If there is an element in R of value 2 then all ideals of R are two-generated, as is shown next.

PROPOSITION 5. *Suppose that R contains an element of value 2. Then every nonzero ideal of R is of the form (a, b) where $v(a)$ is even and $v(b)$ is odd. Further* $N(I) = \max(v(a), v(b)) - 1$.

PROOF. Let I be a nonzero ideal of R and let a be an element in I of least possible even value and b an element in I of least possible odd value. Suppose first that $v(a) < v(b)$. Then $v(I) = \{v(a), v(a) + 2, \ldots, v(b) - 1, v(b), v(b) + 1, \ldots\} = v((a, b))$ and hence $I = (a, b)$ according to Proposition 3. In this case $N(I) = v(b) - 1$. The case $v(a) > v(b)$ is of course analogous.

PROPOSITION 6. *Suppose that R contains an element of value 2. Then R is strongly condensed.*

PROOF. Let I and J be nonzero ideals of R. From the preceding proposition it is clear that there are elements a, b, c and d of R such that $I = (a, b)$ and $J = (c, d)$ and such that $v(b) - v(a)$ and $v(d) - v(c)$ are odd positive integers. Suppose, for example, that $v(b) - v(a) \geqq v(d) - v(c)$. We propose to show that $IJ = aJ$ ie that $bJ \subseteq aJ$. Take an element x in J. We shall show that $bx \in aJ$ ie that $bx/a \in J$. We have $v(bx/a) = v(b) + v(x) - v(a) \geqq v(d) - v(c) + v(x) \geqq v(d)$ as $v(x) \geqq v(c)$. But $N(J) = v(d) - 1$. Thus $bx/a \in J$ and the proof is complete. □

Note that under the assumptions of this section we have three equivalent conditions on R. 1) R is condensed. 2) R is strongly condensed. 3) R contains an element of value 2. Also note that the implication 1) $\Rightarrow$ 3) was proved (Proposition 1) in a slightly more general setting. We do not know, but it is conceivable that conditions 1) and 2) are generally equivalent ie that all condensed domains are in fact strongly condensed.

EXAMPLES. Let $R' = k[[t]]$ be the ring of formal power series over a field k. Then the rings $k[[t^2, t^p]]$ are all strongly condensed, where p is any integer greater than 1, whereas for example the ring $k[[t^3, t^4, t^5]]$ is not condensed.

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