THE KERNEL OF m-QUOTA GAMES

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1. Introduction. In **(1)**, M. Davis and M. Maschler define the kernel K of a characteristic-function game; they also prove, among other theorems, that K is a subset of the bargaining set $M_1^{(i)}$ and that it is never void, i.e. that for each coalition structure b there exists a payoff vector x such that the payoff configuration (x, b) belongs to K. The main advantage of the kernel, as it seems to us, is that it is easier to compute in many cases than the bargaining set $M_1^{(i)}$. Also, in the case when interpersonal comparisons of utility are meaningful, it seems that the kernel describes an adequate way of bargaining among the players, (**1**, Section 6). In this paper we continue the study of the kernel by proving some theorems on the kernel of m-quota games.

2. Definitions. An *n-person game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is a set with n members, and v is a real function defined on the power set of N.

N is the *set of players* and v is the *characteristic function* of the game. We always assume that v is normalized such that $v(\{i\}) = 0$, $i = 1, \ldots, n$, and $v(B) \geqslant 0$ for all $B \subset N$.

Let (N, v) be an n-person game. A *coalition structure* (c.s.) is a partition of N. An *individually rational payoff configuration* (i.r.p.c.) is a pair (x, b), where b is a c.s. and x is an n-tuple of real numbers that satisfies: $x_i \geqslant 0$, $i = 1, \ldots, n$, and $\sum_B x_i = v(B)$, for all $B \in b$. An i.r.p.c. (x, b) represents a possible outcome of the game: b specifies the coalition structure and x determines the distribution of the payoff among the players. Let (x, b) be an i.r.p.c. If $B \subset N$, we denote

$$e(B, x) = v(B) - \textstyle\sum_B x_i.$$

Also, let $i, j \in B \in b$ and $i \neq j$; we denote

$$T_{ij} = \{D : D \subset N, i \in D, j \notin D\}$$

and

$$s_{ij}(x) = \max_{D \in T_{ij}} e(D, x).$$

We say that i *outweighs* j with respect to (x, b) if $s_{ij}(x) > s_{ji}(x)$ and $x_j > 0$. The i.r.p.c. (x, b) is *balanced* if there exists no pair of players h and k such that h outweighs k.

The *kernel* K of the game (N, v) is the set of all balanced i.r.p.c.'s.

The reader is referred now to **(1)** for a comprehensive introduction to the kernel.

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We now proceed to define m-quota games. Quota games were first discussed, in connection with solution theory, in (**6** and **2**); the theory of bargaining sets of quota games is developed mainly in (**3**, **4**, and **5**).

If $S \subset N$, then $|S|$ will denote the number of members of S.

An n-person game (N, v) is an *m-quota game*, $1 < m < n$, if there exist n real numbers $w_1, \ldots, w_n$ such that $v(S) = \sum_S w_i$ when $|S| = m$, and $v(S) = 0$ when $|S| \neq m$. w_i is called the *quota* of players i. If $w_i < 0$, then player i is called *weak*. The quota vector, if it exists, is unique (**4**, Lemma 4.2). We shall use the symbol (N, m, w) to denote the m-quota game with the set of players N and the quota vector w.

3. The case $n \geqslant 2m$. Let (N, m, w) be an n-person m-quota game. If (x, b) is an i.r.p.c. and $i, j \in B \in b$, $i \neq j$, then we denote

$$Q_{ij} = \{D: D \subset N, i, j \notin D, |D| = m - 1\}$$

and

$$A_{ij}(x) = \max_{D \in Q_{ij}} \sum_D (w_k - x_k).$$

Thus

$$A_{ij}(x) = A_{ji}(x).$$

We have that

$$s_{ij}(x) = \max\{-x_i, A_{ij}(x) + w_i - x_i\}.$$

So if $x_i < x_j$ and $w_i - x_i > w_j - x_j$, i outweighs j.

LEMMA 3.1. *Let (x, b) be an* i.r.p.c., *$i, j \in B \in b$, $i \neq j$, and let $w_i \geqslant w_j$; if $(x, b) \in K$, then $x_i \geqslant x_j$.*

Proof. Suppose that $x_j > x_i$. Our assumption also implies that $w_i - x_i > w_j - x_j$. Hence i outweighs j, which is impossible since $(x, b) \in K$.

LEMMA 3.2. *Let (x, b) be an* i.r.p.c., *$i, j \in B \in b$, $i \neq j$, and let $w_i \geqslant w_j$; if $(x, b) \in K$, then $w_i - x_i \geqslant w_j - x_j$.*

Proof. Suppose that $w_j - x_j > w_i - x_i$. In this case we have $x_i > x_j + w_i - w_j \geqslant x_j$, and therefore j outweighs i, which is impossible since $(x, b) \in K$.

When we investigate m-quota games we can, without loss of generality, consider only c.s.'s that consist only of 1-person and m-person coalitions. So, in what follows, b will designate a c.s. of the above type. The players that do not belong to the m-person coalitions of b will be called *isolated players*.

LEMMA 3.3. *Let $(x, b) \in K$. If for each $B \in b$ there is a player $i \notin B$ such that $w_i \geqslant 0$, then $x_j \leqslant \max(0, w_j)$ for $j = 1, \ldots, n$.*

Proof. Suppose that there exists a player h such that $x_h > \max(0, w_h)$. Then h must belong to an m-person coalition $B_0 \in b$. There is a $k \in B_0$ such that $x_k < w_k$. If B_0 is the only m-person coalition of b, then there is $i \notin B_0$ such

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that $0 \leqslant \omega_i = w_i - x_i$. If there are other m-person coalitions, then let $B_1 \in b$, $B_1 \neq B_0$, and $|B_1| = m$; there is $i \in B_1$ such that $w_i - x_i \geqslant 0$. So we can always find $i \notin B_0$ such that $w_i - x_i \geqslant 0$. Now

$$A_{kh}(x) + w_k - x_k \geqslant \textstyle\sum_{B_0 - \{h\}} (w_j - x_j) + w_i - x_i \geqslant 0.$$

So

$$s_{kh}(x) = A_{kh}(x) + w_k - x_k > \max\{-x_h, A_{kh}(x) + w_h - x_h\} = s_{hk}(x),$$

and therefore k outweighs h, which is impossible.

COROLLARY 3.4. *Suppose that there are no weak players and let b be a* c.s.; *then an* i.r.p.c. $(x, b) \in K$ *if and only if the isolated players receive zero and the players that belong to the m-person coalitions of b receive their quotas.*

Proof. By the existence theorem for the kernel (**1**, Theorem 5.4), there exists at least one payoff vector x such that $(x, b) \in K$; Lemma 3.3 completes the proof.

LEMMA 3.5. *Suppose that $n \geqslant 2m$ and let (x, b) be an* i.r.p.c., *$i, j \in B \in b$, $i \neq j$. If $(x, b) \in K$, $x_i > 0$ and $x_j > 0$, then $w_i - x_i = w_j - x_j$.*

Proof. Suppose that $w_i - x_i > w_j - x_j$. Since $n \geqslant 2m$, the conditions of Lemma 3.3 are satisfied and therefore $w_i \geqslant x_i$ and $w_j \geqslant x_j$; also there is a coalition S such that $S \cap B = \emptyset$, $|S| = m - 1$, and $\sum_S (w_k - x_k) \geqslant 0$. It follows that

$$s_{ij}(x) = A_{ij}(x) + w_i - x_i > A_{ij}(x) + w_j - x_j = s_{ji}(x).$$

So i outweighs j, which is impossible.

LEMMA 3.6. *Suppose that $n \geqslant 2m$ and let b be a* c.s. *There is a unique payoff vector x such that the* i r.p.c. $(x, b) \in K$.

Proof. Let y be a payoff vector such that the i.r.p.c. $(y, b) \in K$ (by the existence theorem, (**1** Theorem 5.4), there exists at least one such y). Let $B \in b$ be an m-person coalition (if there is no such B, then $y = 0$ and the lemma is proved). Without loss of generality, let $B = \{1, 2, \ldots, m\}$, $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_m$ and

$$\sum_{i=1}^{m} w_i > 0.$$

By Lemma 3.1, there is a $1 \leqslant p \leqslant m$ such that $y_i > 0$ for $1 \leqslant i \leqslant p$ and $y_i = 0$ for $p < i \leqslant m$. By Lemma 3.5 $w_i - y_i = w_j - y_j$ for $1 \leqslant i, j \leqslant p$. From these equations and from

$$\sum_{i=1}^{p} (w_i - y_i) = \sum_{i=1}^{p} w_i - v(B)$$

we conclude that

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$$y_i = w_i + \frac{1}{p}\left\{v(B) - \sum_{i=1}^{p} w_i\right\}, \qquad i = 1, \ldots, p.$$

We now denote

$$p_0 = \max\left\{q : q \leqslant m, w_q + \frac{1}{q}\left[v(B) - \sum_{i=1}^{q} w_i\right] > 0\right\}.$$

We shall prove that $p = p_0$. For $1 \leqslant q \leqslant p_0$ we define

$$f_q = w_q + \frac{1}{q}\left\{v(B) - \sum_{i=1}^{q} w_i\right\}.$$

We now compute

$$\begin{aligned} f_q - f_{q+1} = w_q - w_{q+1} + \frac{1}{q(q+1)}\Big\{(q+1)v(B) \\ - (q+1)\sum_{i=1}^{q} w_i - qv(B) + q\sum_{i=1}^{q+1} w_i\Big\} \\ = w_q - w_{q+1} + \frac{1}{q}\left\{w_{q+1} + \frac{1}{q+1}\left[v(B) - \sum_{i=1}^{q+1} w_i\right]\right\} \\ = \frac{1}{q} f_{q+1} + w_q - w_{q+1}, \end{aligned}$$

so $f_q - (1 + 1/q)f_{q+1} = w_q - w_{q+1}$. This equation implies that $f_q > 0$ for $1 \leqslant q \leqslant p_0$. Suppose now that $p < p_0$. Then

$$w_{p+1} - (w_p - y_p) = w_{p+1} - w_p + f_p = (1 + 1/p)f_{p+1} > 0.$$

So $p + 1$ outweighs p, which is impossible. So y is determined uniquely by the above equations. We have thus shown that there is at most one payoff vector x such that the i.r.p.c. $(x, b) \in K$, and the proof is completed.

4. The case $n < 2m$. Let (N, m, w) be an n-person m-quota game. In what follows we suppose that $n < 2m$.

LEMMA 4.1. *Let b be a* c.s. *Then there is a unique payoff vector x such that the* i.r.p.c. $(x, b) \in K$.

Proof. We know that there is at least one payoff vector x such that the i.r.p.c. $(x, b) \in K$ (**1**, Theorem 5.4). We shall now prove that there is at most one such x. We assume that b contains an m-person coalition B; if b does not contain an m-person coalition the proof is immediate. Suppose that there exist two distinct payoff vectors x and y such that the i.r.p.c.'s (x, b) and (y, b) belong to K. Denote

$$R = \{i : x_i > y_i\} \quad \text{and} \quad L = \{i : y_i > x_i\}.$$

(R and L are necessarily non-empty.) Also let Q be an m-person coalition that satisfies

$$\min\{w_i:i \in Q\} \geqslant \max\{w_i:i \in N - Q\}.$$

We shall now show that there exist players i and j such that one of the following conditions is satisfied:

$$H_1:i \in L, j \in R, w_i \geqslant w_j \quad \text{and} \quad A_{ij}(x) + w_i - x_i \geqslant A_{ij}(y) + w_i - y_i,$$

or

$$H_2:i \in R, j \in L, w_i \geqslant w_j \quad \text{and} \quad A_{ij}(y) + w_i - y_i \geqslant A_{ij}(x) + w_i - x_i.$$

We distinguish the following possibilities:

(a) $R \cup L \subset Q$. Let i be a player with the maximal quota in $R \cup L$. If $i \in L$, let $j \in R$. $w_i \geqslant w_j$. Since $R \cup L \subset Q$, using Lemma 3.2, we have

$$A_{ij}(x) + w_i - x_i = A_{ij}(y) + w_i - y_i + x_j - y_j > A_{ij}(y) + w_i - y_i.$$

So i and j satisfy H_1.

If $i \in R$, let $j \in L$. It can be shown similarly that i and j satisfy H_2 in this case.

(b) $(R \cup L) \cap Q = \emptyset$. In this case, if $h, k \notin Q$, then $A_{kh}(x) = A_{kh}(y)$. Let i be a player with the maximal quota in $R \cup L$. If $i \in L$, we can find a $j \in R$ such that i and j will satisfy H_1. If $i \in R$, we can find a $j \in L$ such that i and j will satisfy H_2.

(c) $L \cap Q = \emptyset$ and $R \cap Q \neq \emptyset$. Let $i \in R \cap Q$ and $j \in L$. We have $w_i \geqslant w_j$ and

$$A_{ij}(y) + w_i - y_i = \sum_Q(w_k - y_k) > \sum_Q(w_k - x_k) = A_{ij}(x) + w_i - x_i;$$

so i and j satisfy H_2.

(d) $L \cap Q \neq \emptyset$ and $R \cap Q = \emptyset$. A similar reasoning to that in (c) shows that we can find i and j that satisfy H_1.

(e) $L - Q = \emptyset$ and $R - Q \neq \emptyset$. Let $i \in L$ and $j \in R - Q$. $w_i \geqslant w_j$ and

$$\begin{aligned} A_{ij}(x) + w_i - x_i &= \sum_Q (w_k - x_k) = \sum_{Q-B} w_k + \sum_{Q\cap B} (w_k - x_k) \\ &= \sum_{Q-B} w_k - \sum_{B-Q} (w_k - x_k) > \sum_{Q-B} w_k - \sum_{B-Q} (w_k - y_k) \\ &= \sum_{Q-B} w_k + \sum_{B\cap Q} (w_k - y_k) = \sum_Q (w_k - y_k) \\ &= A_{ij}(y) + w_i - y_i. \end{aligned}$$

So i and j satisfy H_1.

(f) $R - Q = \emptyset$ and $L - Q \neq \emptyset$. A similar reasoning to that in Case (e) shows that we can find i and j that satisfy H_2.

(g) $R \cap Q \neq \emptyset$, $L \cap Q \neq \emptyset$, $R - Q \neq \emptyset$, and $L - Q \neq \emptyset$. If $\sum_Q (w_h - x_h) \geqslant \sum_Q (w_h - y_h)$, we can choose $i \in L \cap Q$ and $j \in R - Q$ that satisfy H_1. If $\sum_Q (w_h - x_h) < \sum_Q (w_h - y_h)$, we can choose $i \in R \cap Q$ and $j \in L - Q$ that satisfy H_2.

We shall now prove that each of the cases H_1 and H_2 leads to a contradiction. Suppose that there exist i and j that satisfy H_1. The inequality $w_i \geqslant w_j$ implies that $x_i \geqslant x_j$ and $w_i - y_i \geqslant w_j - y_j$. It follows that

$$x_i \geqslant x_j > y_j \geqslant 0 \quad \text{and} \quad w_i - x_i > w_i - y_i \geqslant w_j - y_j > w_j - x_j.$$

Therefore $s_{ij}(x) = s_{ji}(x)$ and, since $w_i - x_i > w_j - x_j$, $s_{ji}(x) = -x_j$. Since $y_i > x_i > 0$, we have that $s_{ij}(y) \geqslant s_{ji}(y)$. The inequalities $-y_i < -x_i$ and $A_{ij}(x) + w_i - x_i \geqslant A_{ij}(y) + w_i - y_i$ show that $s_{ij}(x) \geqslant s_{ij}(y)$. So we have

$$s_{ji}(x) = s_{ij}(x) \geqslant s_{ij}(y) \geqslant s_{ji}(y).$$

On the other hand we have

$$s_{ji}(x) = -x_j < -y_j \leqslant s_{ji}(y),$$

and the desired contradiction is reached. A similar reasoning shows that when a pair of players satisfy H_2, a contradiction is reached; so the proof of the lemma is completed.

We now summarize the results in Theorem 4.2.

THEOREM 4.2. *Let* (N, m, w) *be an n-person m-quota game and let* b *be a* c.s. *Then there exists a unique payoff vector* x *such that the* i.r.p.c. $(x, b) \in K$.

Proof. Lemmas 3.6 and 4.1.

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