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CONTINUITY OF THE HAUSDORFF DIMENSION FOR GRAPH-DIRECTED SYSTEMS

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Abstract

In this paper we discuss the continuity of the Hausdorff dimension of the invariant set of generalised graph-directed systems given by contractive infinitesimal similitudes on bounded complete metric spaces. We use the theory of positive linear operators to show that the Hausdorff dimension varies continuously with the functions defining the generalised graph-directed system under suitable assumptions.

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1. Introduction

Given a generalised graph-directed iterated function system where the maps are contractions and 'infinitesimal similitudes', there is an associated invariant set list for the system (see [14]). The definitions and the precise results are given in the next section. We would like to emphasise that we work in the general setting of complete metric spaces rather than only Euclidean spaces. Graph-directed systems are generalisations of the well-known iterated function systems (IFSs). Such constructions have been studied by several authors (see, for example, [2–4, 10–12] and [15]). An IFS can be thought of as a graph-directed IFS with a single vertex. On the other hand, it is known (see [2]) that there are graph-directed IFSs whose invariant sets (or attractors) cannot be attractors of standard IFSs. Thus the study of graph-directed IFSs indeed gives something more than the study of IFSs. Hutchinson [6] has obtained results for the Hausdorff dimension of the invariant set for a finite IFS consisting of contractive similitudes on Euclidean space. He has shown that the Hausdorff dimension matches the 'similarity dimension' under the assumption of the 'open set condition'. Schief [18] has worked on IFSs consisting of similitudes on complete metric spaces and has shown that in this generality, the open set condition is no longer sufficient and must be strengthened to the 'strong open set condition'. Nussbaum *et al.* [14] have generalised the concept of similitudes to that of infinitesimal similitudes on general metric spaces (a generalisation of conformal maps on Euclidean spaces), and have obtained a formula for the Hausdorff dimension of the invariant set for a generalised graph-directed IFS under appropriate disjointness assumptions.

In this paper we investigate how the Hausdorff dimensions of invariant sets vary if one varies the infinitesimal similitudes defining the generalised graph-directed IFS. If we consider IFSs defined by contractive similitudes, it is easy to see that the similarity dimensions (and hence the Hausdorff dimensions of the invariant sets, in the setting of Hutchinson [6]) vary continuously with the similarity ratios. We shall show that similar results are true in our general setting too. The precise statements and proofs are given in Section 3 of this paper.

The setting and conclusion of the main result (Theorem 3.3) are as follows. Let $(V, \mathcal{E}, \Gamma, \alpha)$ be a generalised directed graph with a sequence of graph-directed IFSs given by the maps $\{\theta_{(j,e),m} : (j, e) \in \Gamma\}$, $m \in \mathbb{N}$. Let σ_m be the Hausdorff dimension of each $C_{j,m}$, $1 \leq j \leq p$, where $\{C_{j,m} : 1 \leq j \leq p\}$ denotes the unique invariant set list for the system $\{\theta_{(j,e),m} : (j, e) \in \Gamma\}$ (see (2.1)). For $(j, e) \in \Gamma$ and $x \in S_j$, assume that $\lim_{m\to\infty} \theta_{(j,e),m}(x) = \theta_{(j,e)}(x)$ and $\lim_{m\to\infty} D\theta_{(j,e),m}(x) = D\theta_{(j,e)}(x)$, where these limits define $\theta_{(j,e)}(x)$ and we assume that the convergences are uniform in $x \in S_j$. Let $\{C_j : 1 \leq j \leq p\}$ be the unique invariant set list for the system $\{\theta_{(j,e)} : (j, e) \in \Gamma\}$. If σ_0 denotes the Hausdorff dimension of each C_j, $1 \leq j \leq p$, then $\sigma_0 = \lim_{m\to\infty} \sigma_m$. In other words, the Hausdorff dimension of the invariant set list of the limiting graph-directed system given by the maps $\{\theta_{(j,e)} : (j, e) \in \Gamma\}$ is the same as the limit of the Hausdorff dimension of the invariant set list of the systems $\{\theta_{(j,e),m} : (j, e) \in \Gamma\}$, $m \in \mathbb{N}$.

2. Preliminaries

We begin by recalling some definitions and notation from [14] for a 'generalised graph-directed IFS'. Throughout this paper, $V = \{j \in \mathbb{N} : 1 \leq j \leq p\}$, $\mathcal{E}$ is a finite set and S_j, $j \in V$, are bounded complete metric spaces. Let d_j denote the distance metric on S_j. Let Γ be a given subset of $V \times \mathcal{E}$ and $\alpha : \Gamma \to V$ a given map. For each $(j, e) \in \Gamma$, let $\theta_{(j,e)} : S_j \to S_{\alpha(j,e)}$ be a Lipschitz map with the Lipschitz constant $\mathrm{Lip}(\theta_{(j,e)}) \leq c$ for a fixed positive constant $c < 1$. We call $(V, \mathcal{E}, \Gamma, \alpha, \{\theta_{(j,e)} : (j, e) \in \Gamma\})$ a generalised graph-directed IFS. For $j \in V$, we shall consistently denote sets Γ_j and $\mathcal{E}_j$ by

$$\Gamma_j := \{(k, e) \in \Gamma : \alpha(k, e) = j\}$$

and

$$\mathcal{E}_j := \{e \in \mathcal{E} : (j, e) \in \Gamma\}.$$

In the original construction of Mauldin and Williams [12] of a 'graph-directed IFS', V is the set of vertices and $\mathcal{E}$ is the set of edges of a directed multigraph. Also, Γ is the collection of all $(t(e), e) \in V \times \mathcal{E}$, where $t(e)$ denotes the terminal vertex of the edge e and $\alpha(t(e), e)$ is the initial vertex of the edge e. Furthermore, $\mathcal{E}_j$ is the collection of all edges with terminal vertex j and Γ_j is the collection of all $(t(e), e)$ for which j is the initial vertex of e. We refer the reader to [5, Ch. 4.3] for a discussion of the Mauldin–Williams graph.

Definition 2.1 (Hypothesis H2.1). We say that the generalised graph $(V, \mathcal{E}, \Gamma, \alpha)$ is *strongly connected* if for each pair j and k in V, there exist $n \geq 1$ and $(J, E) := [(j_1, e_1), \ldots, (j_n, e_n)]$ such that $(j_i, e_i) \in \Gamma$ for $1 \leq i \leq n$, $j_1 = j$, $\alpha(j_i, e_i) = j_{i+1}$, $1 \leq i < n$, and $\alpha(j_n, e_n) = k$.

Note that for a directed multigraph, strong connectedness means that for any two vertices j and k there is a *path* (that is, a finite sequence of directed edges such that the terminal vertex of an edge is the initial vertex of the next edge) starting at j and terminating at k.

For a strongly connected generalised graph $(V, \mathcal{E}, \Gamma, \alpha)$, the sets Γ_j and $\mathcal{E}_j$ are nonempty sets for each $j \in V$. For the directed multigraph, this means that for every vertex j there is an edge starting at j and there is an edge terminating at j.

Under the assumption of H2.1 (in fact, one needs only to assume that Γ_j is nonempty for all $j \in V$), an application of the contraction mapping theorem gives the existence of a unique invariant set list for the system $\{\theta_{(j,e)} : (j, e) \in \Gamma\}$, that is, a unique list of nonempty compact sets $C_j \subseteq S_j$ for $j \in V$ such that

$$C_i = \bigcup_{(j,e)\in\Gamma_i} \theta_{(j,e)}(C_j) \qquad \forall i \in V. \tag{2.1}$$

See [14, Theorem 2.3] for a proof.

For $j \in V$, we denote by X_j the space of all continuous and bounded real-valued functions on S_j, that is,

$$X_j := \{f : S_j \to \mathbb{R} : f \text{ is continuous and bounded}\}. \tag{2.2}$$

This is a Banach space with $\|f\| = \sup_{s\in S_j} |f(s)|$. For $M > 0$, $\lambda \geq 0$ and $j \in V$, define

$$K_j(M, \lambda) := \{f \in X_j : 0 \leq f(s) \leq f(t) \exp(M(d_j(s, t))^\lambda) \ \forall s, t \in S_j\}. \tag{2.3}$$

If Y is a real Banach space, a closed set $K \subseteq Y$ is called a closed *cone* if $aK + bK \subseteq K$ for all $a \geq 0, b \geq 0$ and $K \cap (-K) = \{0\}$. It is not hard to prove the following lemma.

Lemma 2.2 ([14], Lemma 3.2). *Let $K_j := K_j(M, \lambda)$ be defined by (2.3) with $\lambda > 0$. Then K_j is a closed cone in $(X_j, \|.\|)$ and $\{f \in K_j : \|f\| \leq 1\}$ is equicontinuous.*

Next, we need to define the concept of an *infinitesimal similitude*.

Definition 2.3. Let (S_1, d_1) be a perfect metric space and let (S_2, d_2) be any metric space. A map $\theta : S_1 \to S_2$ is said to be an *infinitesimal similitude* at $s \in S_1$ if for any sequences $(s_k)_k$ and $(t_k)_k$ in S_1 with $s_k \neq t_k$ for $k \geq 1$ and $s_k \to s$, $t_k \to s$, the limit

$$\lim_{k\to\infty} \frac{d_2(\theta(s_k), \theta(t_k))}{d_1(s_k, t_k)} =: (D\theta)(s)$$

exists and is independent of the particular sequences $(s_k)_k$ and $(t_k)_k$. We shall say that θ is an infinitesimal similitude on S_1 if θ is an infinitesimal similitude at s for all $s \in S_1$.

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REMARK 2.4. The concept of an infinitesimal similitude generalises the concept of a similitude, which is a map $\theta : S_1 \to S_2$ satisfying $d_2(\theta(x), \theta(y)) = cd_1(x, y)$ for all $x, y \in S_1$ and a fixed constant c. Note that every similitude $\theta : S_1 \to S_2$ is an infinitesimal similitude with $(D\theta)(s) = c$ for all $s \in S_1$. Furthermore, if $\theta : S_1 \to S_2$ is a Lipschitz map and also an infinitesimal similitude, then $(D\theta)(s) \leq \mathrm{Lip}(\theta)$ for all $s \in S_1$. Thus for an infinitesimal similitude $\theta : S_1 \to S_2$ which is also a contraction map, $(D\theta)(s) < 1$ for all $s \in S_1$. The idea of an 'infinitesimal similitude' allows us to extend the concept of 'conformality' to the more general metric space setting.

DEFINITION 2.5 (Hypothesis H2.2). We say that a generalised strongly connected graph-directed IFS satisfies H2.2 if, for each $(j, e) \in \Gamma$, the map $\theta_{(j,e)} : S_j \to S_{\alpha(j,e)}$ is an infinitesimal similitude, $(D\theta_{(j,e)})(s) > m > 0$ for all $s \in S_j$ and the function $D\theta_{(j,e)} : S_j \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\lambda > 0$ for each $(j, e) \in \Gamma$.

Under Hypothesis H2.2, it can be shown that for each $(j, e) \in \Gamma$, the function $D\theta_{(j,e)}$ is in the cone $K_j(M_0, \lambda)$ defined by (2.3) for some $M_0 > 0$ (see [14, Lemma 4.5]).

For $\sigma \geq 0$, define the so-called Perron–Frobenius operator

$$L_\sigma : \prod_{j=1}^{p} X_j \to \prod_{j=1}^{p} X_j$$

by

$$(L_\sigma f)_j(s) := \sum_{e \in \mathcal{E}_j} ((D\theta_{(j,e)})(s))^\sigma f_{\alpha(j,e)}(\theta_{(j,e)}(s)) \quad \text{for } s \in S_j, 1 \leq j \leq p, \tag{2.4}$$

where X_j is the space in (2.2) and the norm on $\prod_{j=1}^{p} X_j$ is given by $\|f\| = \max_{1 \leq j \leq p} \|f_j\|$ if $f = (f_1, f_2, \ldots, f_p)$. Under the assumptions made in this section, it is easy to see that L_σ is a positive bounded linear operator (that is, $f_j \geq 0$ on S_j, $1 \leq j \leq p$, implies that $(L_\sigma f)_j \geq 0$ on S_j for $1 \leq j \leq p$) and it maps the cone $K := \prod_{j=1}^{p} K_j(M, \lambda)$ into itself for some $M > 0$ (see [14, Lemma 3.3]). Furthermore, the linear operator L_σ has an eigenvector $u_\sigma \in K \setminus \{0\}$ with eigenvalue $r(L_\sigma)$, the spectral radius of L_σ, and $r(L_\sigma) > 0$ (see [14, Theorem 3.6]). The proof involves generalisations of the Kreĭn–Rutman theorem to noncompact linear operators (see [1, 7–9, 13, 16] and [17]). Let us recall that the spectral radius of a bounded linear operator L is given by $r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}$.

The following two propositions can be found in [14].

PROPOSITION 2.6 ([14], Lemma 4.6). *The map $\sigma \mapsto r(L_\sigma), \sigma \geq 0$, is continuous and monotonically decreasing. Furthermore, there is a unique $\sigma_0 \geq 0$ such that $r(L_{\sigma_0}) = 1$.*

PROPOSITION 2.7 ([14], Lemma 4.8). *Assume that the hypotheses* H2.1 *and* H2.2 *are satisfied and let $u_\sigma \in K \setminus \{0\}$ be an eigenvector of L_σ with eigenvalue $r(L_\sigma)$. Then each component $(u_\sigma)_j$ is a strictly positive function on S_j for $1 \leq j \leq p$. Furthermore, there are constants l_1 and l_2 with $0 < l_1 \leq l_2 < \infty$ such that for every j, $1 \leq j \leq p$,*

$$l_1 \leq (u_\sigma)_j(t) \leq l_2 \quad \text{for all } t \in S_j. \tag{2.5}$$

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Lemma 2.8. *Assume hypotheses* H2.1 *and* H2.2. *Let* $u = (u_1, u_2, \ldots, u_p) \in \prod_{j=1}^{p} X_j$ *be such that* $l_1 \le u_j \le l_2$ *on* S_j, $1 \le j \le p$, *for some* $0 < l_1 \le l_2 < \infty$. *Then* $\lim_{n\to\infty} \|L_\sigma^n u\|^{1/n} = r(L_\sigma)$.

Proof. First, we prove that $\|L_\sigma^n\| = \|L_\sigma^n e\|$ for all $n \in \mathbb{N}$, where e denotes the function in $\prod_{j=1}^{p} X_j$ with each component identically equal to one. For $f = (f_1, f_2, \ldots, f_p)$ and $g = (g_1, g_2, \ldots, g_p)$ we write $f \le g$ to mean $f_j \le g_j$ on S_j for $1 \le j \le p$. Since L_σ is a positive bounded linear operator, it follows that $L_\sigma f \le L_\sigma g$ whenever $f \le g$. If $f \in \prod_{j=1}^{p} X_j$ with $\|f\| \le 1$, then $-e \le f \le e$. Thus $-L_\sigma^n e \le L_\sigma^n f \le L_\sigma^n e$, which implies that $\|L_\sigma^n f\| \le \|L_\sigma^n e\|$. Taking the supremum over the set $\{f \in \prod_{j=1}^{p} X_j : \|f\| \le 1\}$, we get $\|L_\sigma^n\| \le \|L_\sigma^n e\|$. The reverse inequality obviously holds as $\|e\| = 1$. Thus $\|L_\sigma^n\| = \|L_\sigma^n e\|$ for all $n \in \mathbb{N}$.

Now the assumption on the given function u can be rewritten as $l_1 e \le u \le l_2 e$ for some $0 < l_1 \le l_2 < \infty$. This implies $l_1 L_\sigma^n e \le L_\sigma^n u \le l_2 L_\sigma^n e$, for any $n \in \mathbb{N}$, which gives $l_1\|L_\sigma^n e\| \le \|L_\sigma^n u\| \le l_2\|L_\sigma^n e\|$. Using $\|L_\sigma^n e\| = \|L_\sigma^n\|$, we have $l_1\|L_\sigma^n\| \le \|L_\sigma^n u\| \le l_2\|L_\sigma^n\|$. Taking the nth root and passing to the limit, we find $\lim_{n\to\infty} \|L_\sigma^n u\|^{1/n} = \lim_{n\to\infty} \|L_\sigma^n\|^{1/n} = r(L_\sigma)$. This completes the proof. □

Finally, we shall need the following important theorem about the Hausdorff dimension of the invariant set.

Theorem 2.9 ([14], Theorem 4.17). *Assume that the hypotheses* H2.1 *and* H2.2 *are satisfied and let* $C_j \subseteq S_j$, $1 \le j \le p$, *be the unique invariant set list such that*

$$C_i = \bigcup_{(j,e)\in\Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \le i \le p.$$

Also assume that $\theta_{(j,e)}|_{C_j}$ *is injective for all* $(j,e) \in \Gamma$ *and that* $\theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C_{j'})$ *is empty whenever* $\alpha(j,e) = \alpha(j',e')$ *and* $(j,e) \ne (j',e')$. *Let* σ_0 *be the unique nonnegative real number such that* $r(L_{\sigma_0}) = 1$. *Then the Hausdorff dimension of each* C_i, $1 \le i \le p$, *is the same and equals* σ_0.

3. Continuity of the Hausdorff dimension

We shall show that the Hausdorff dimension of the invariant set of a generalised graph-directed IFS varies continuously with the functions $\theta_{(j,e)}$, $(j,e) \in \Gamma$ under the assumptions of Theorem 2.9. Let $(V, \mathcal{E}, \Gamma, \alpha)$ be a generalised directed graph and let $S_1, S_2, \ldots, S_p$ be bounded complete metric spaces. For each integer $m \ge 1$ and $(j,e) \in \Gamma$, suppose that $\theta_{(j,e),m} : S_j \to S_{\alpha(j,e)}$ is a Lipschitz map with $\mathrm{Lip}(\theta_{(j,e),m}) \le c < 1$. Assume that, for each $m \ge 1$, the graph-directed IFS $(V, \mathcal{E}, \Gamma, \alpha, \{\theta_{(j,e),m} : (j,e) \in \Gamma\})$ satisfies the assumptions of Theorem 2.9 and let σ_m be the Hausdorff dimension of each $C_{j,m}$, $1 \le j \le p$, where $\{C_{j,m} : 1 \le j \le p\}$ denotes the unique invariant set list for the system $\{\theta_{(j,e),m} : (j,e) \in \Gamma\}$ (see (2.1)). For $(j,e) \in \Gamma$ and $x \in S_j$, assume that $\lim_{m\to\infty} \theta_{(j,e),m}(x) = \theta_{(j,e)}(x)$ and $\lim_{m\to\infty} D\theta_{(j,e),m}(x) = D\theta_{(j,e)}(x)$, where these limits define $\theta_{(j,e)}(x)$ and we assume that the convergences are uniform in $x \in S_j$. Assume that the system $\{\theta_{(j,e)} : (j,e) \in \Gamma\}$ satisfies H2.1 and H2.2 and that $\theta_{(j,e)}|_{C_j}$ is injective, where

$\{C_j : 1 \le j \le p\}$ denotes the unique invariant set list for the system $\{\theta_{(j,e)} : (j,e) \in \Gamma\}$. For $\sigma \ge 0$, we have, in the obvious notation, linear operators $L_{\sigma,m}$ corresponding to $\{\theta_{(j,e),m} : (j,e) \in \Gamma\}$ and L_σ corresponding to $\{\theta_{(j,e)} : (j,e) \in \Gamma\}$ (see (2.4)). By Theorem 2.9, we know that $r(L_{\sigma_m,m}) = 1$. Let σ_0 denote the unique value of σ for which $r(L_\sigma) = 1$. We shall show that $\lim_{m\to\infty} \sigma_m = \sigma_0$.

If K is a closed cone in a Banach space Y and $L : Y \to Y$ is a bounded linear operator with $L(K) \subseteq K$, we define

$$\|L\|_K = \sup\{\|L(y)\| : y \in K, \|y\| \le 1\}. \tag{3.1}$$

Lemma 3.1. *The notations and assumptions are as given in this section. Let $\sigma \ge 0$ be fixed. Assume that there exist $M_0 > 0$ and $\lambda > 0$ such that $D\theta_{(j,e),m} \in K_j(M_0, \lambda)$ and $D\theta_{(j,e)} \in K_j(M_0, \lambda)$ for all $(j,e) \in \Gamma$ and for all $m \in \mathbb{N}$. Choose $M > 0$ such that the cone $K(M, \lambda) = \prod_{j=1}^p K_j(M, \lambda)$ in $\prod_{j=1}^p X_j$ is mapped into itself by the operators $L_{\sigma,m}$, $m \in \mathbb{N}$ and L_σ. Then $\|L_{\sigma,m} - L_\sigma\|_{K(M,\lambda)} \to 0$ as $m \to \infty$.*

Proof. Let $\epsilon > 0$ be given and $1 \le j \le p$. By Lemma 2.2, $\{f_j \in K_j(M, \lambda) : \|f_j\| \le 1\}$ is equicontinuous. Therefore, we can find a $\delta > 0$, independent of j, such that $d_j(s,t) < \delta$ implies $|f_j(s) - f_j(t)| < \epsilon$ for all $f_j \in K_j(M, \lambda)$ with $\|f_j\| \le 1$. Suppose $f = (f_1, f_2, \ldots, f_p) \in K(M, \lambda)$, $\|f\| \le 1$ and let $(j,e) \in \Gamma$. Since $\theta_{(j,e),m} \to \theta_{(j,e)}$ and $D\theta_{(j,e),m} \to D\theta_{(j,e)}$ as $m \to \infty$, uniformly on S_j, there exists a positive integer m_0 such that $d_{\alpha(j,e)}(\theta_{(j,e),m}(t), \theta_{(j,e)}(t)) < \delta$ and $|(D\theta_{(j,e),m}(t))^\sigma - (D\theta_{(j,e)}(t))^\sigma| < \epsilon$ for all $t \in S_j$ and $m \ge m_0$. This implies that $|f_{\alpha(j,e)}(\theta_{(j,e),m}(t)) - f_{\alpha(j,e)}(\theta_{(j,e)}(t))| < \epsilon$ for every $t \in S_j$ and $m \ge m_0$. So, for $1 \le j \le p$ and $t \in S_j$,

$$\begin{aligned}
&|(L_{\sigma,m}f)_j(t) - (L_\sigma f)_j(t)| \\
&\quad \le \sum_{e\in\mathcal{E}_j} |(D\theta_{(j,e),m}(t))^\sigma f_{\alpha(j,e)}(\theta_{(j,e),m}(t)) - (D\theta_{(j,e)}(t))^\sigma f_{\alpha(j,e)}(\theta_{(j,e)}(t))| \\
&\quad \le \sum_{e\in\mathcal{E}_j} |(D\theta_{(j,e),m}(t))^\sigma - (D\theta_{(j,e)}(t))^\sigma|\,|f_{\alpha(j,e)}(\theta_{(j,e),m}(t))| \\
&\qquad\quad + \sum_{e\in\mathcal{E}_j} (D\theta_{(j,e)}(t))^\sigma |f_{\alpha(j,e)}(\theta_{(j,e),m}(t)) - f_{\alpha(j,e)}(\theta_{(j,e)}(t))| \\
&\quad \le 2|\mathcal{E}_j|\epsilon \le 2|\mathcal{E}|\epsilon
\end{aligned}$$

for all $m \ge m_0$, where $|\mathcal{E}_j|$ is the cardinality of the set $\mathcal{E}_j$ and $|\mathcal{E}|$ is the cardinality of the set $\mathcal{E}$. Using the definition in (3.1), we have $\|L_{\sigma,m} - L_\sigma\|_{K(M,\lambda)} \le 2|\mathcal{E}|\epsilon$ for all $m \ge m_0$. Since $\mathcal{E}$ is a finite set and $\epsilon > 0$ was arbitrary, we have proved that $\|L_{\sigma,m} - L_\sigma\|_{K(M,\lambda)} \to 0$ as $m \to \infty$. □

Lemma 3.2. *The assumptions are as in the previous lemma. Then, for $\sigma \ge 0$, we have $r(L_{\sigma,m}) \to r(L_\sigma)$ as $m \to \infty$.*

Proof. Let $u_\sigma \in K(M, \lambda) \setminus \{0\}$ be an eigenvector of L_σ with eigenvalue $r_\sigma := r(L_\sigma)$. By Lemma 3.1, it follows that $\|L_{\sigma,m}u_\sigma - L_\sigma u_\sigma\| \to 0$ as $m \to \infty$. Since $L_\sigma u_\sigma = r_\sigma u_\sigma$, $\|L_{\sigma,m}u_\sigma - r_\sigma u_\sigma\| \to 0$ as $m \to \infty$. By Proposition 2.7, there exists $l_1 > 0$ such that

$(u_\sigma)_j > l_1$ on S_j for $1 \le j \le p$. So, given $0 < \delta < 1$, there exists $m_0(\delta) \in \mathbb{N}$ such that $(1-\delta) r_\sigma (u_\sigma)_j \le (L_{\sigma,m} u_\sigma)_j \le (1+\delta) r_\sigma (u_\sigma)_j$ on S_j for all $m \ge m_0(\delta)$ and $1 \le j \le p$. That is, $(1-\delta) r_\sigma u_\sigma \le L_{\sigma,m} u_\sigma \le (1+\delta) r_\sigma u_\sigma$ for $m \ge m_0(\delta)$. Iterating this step, for $m \ge m_0(\delta)$, $(1-\delta)^n r_\sigma^n u_\sigma \le L_{\sigma,m}^n u_\sigma \le (1+\delta)^n r_\sigma^n u_\sigma$ for all $n \in \mathbb{N}$. This implies that for every $n \in \mathbb{N}$, $(1-\delta)^n r_\sigma^n \|u_\sigma\| \le \|L_{\sigma,m}^n u_\sigma\| \le (1+\delta)^n r_\sigma^n \|u_\sigma\|$ for all $m \ge m_0(\delta)$. Taking the nth root and taking the limit as $n \to \infty$, $(1-\delta) r_\sigma \le \lim_{n\to\infty} \|L_{\sigma,m}^n u_\sigma\|^{1/n} \le (1+\delta) r_\sigma$ for $m \ge m_0(\delta)$. From Lemma 2.8 and Proposition 2.7, $(1-\delta) r_\sigma \le r(L_{\sigma,m}) \le (1+\delta) r_\sigma$ for $m \ge m_0(\delta)$. Since $0 < \delta < 1$ was arbitrary,

$$r_\sigma \le \liminf_{m\to\infty} r(L_{\sigma,m}) \le \limsup_{m\to\infty} r(L_{\sigma,m}) \le r_\sigma \quad \text{and} \quad \lim_{m\to\infty} r(L_{\sigma,m}) = r(L_\sigma). \qquad \square$$

Theorem 3.3. *The assumptions are as given in this section. Suppose that $r(L_{\sigma_m,m}) = 1$ for $m \ge 1$ and $r(L_{\sigma_0}) = 1$. Then $\lim_{m\to\infty} \sigma_m = \sigma_0$. Furthermore, if we also assume that $\theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C_{j'}) = \emptyset$ whenever $(j,e) \ne (j',e')$ and $\alpha(j,e) = \alpha(j',e')$, then, for $1 \le j \le p$, the Hausdorff dimension of C_j is the limit of the Hausdorff dimension of $C_{j,m}$ as $m \to \infty$.*

Proof. We argue by contradiction. Suppose that $\lim_{m\to\infty} \sigma_m \ne \sigma_0$. Then there exist $\delta > 0$ and a subsequence $\{m_i\}_{i\ge1}$ such that either $\sigma_{m_i} > \sigma_0 + \delta$ for all $i \ge 1$ or $\sigma_{m_i} < \sigma_0 - \delta$ for all $i \ge 1$. Assume $\sigma_{m_i} > \sigma_0 + \delta$ for all $i \ge 1$. Then by the strictly decreasing property (Proposition 2.6), $r(L_{\sigma_{m_i},m_i}) < r(L_{\sigma_0+\delta,m_i})$ for all $i \ge 1$. By Lemma 3.2, $\lim_{i\to\infty} r(L_{\sigma_0+\delta,m_i}) = r(L_{\sigma_0+\delta})$, which is strictly less than $r(L_{\sigma_0}) = 1$. On the other hand, $r(L_{\sigma_{m_i},m_i}) = 1$ for all $i \ge 1$, which gives $r(L_{\sigma_0+\delta}) \ge 1$. Thus we arrive at a contradiction. Similarly, $\sigma_{m_i} < \sigma_0 - \delta$ for all $i \ge 1$ leads to a contradiction. Hence we must have $\lim_{m\to\infty} \sigma_m = \sigma_0$.

If we further assume that $\theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C_{j'}) = \emptyset$ whenever $(j,e) \ne (j',e')$ and $\alpha(j,e) = \alpha(j',e')$, then Theorem 2.9 implies that σ_0 is the Hausdorff dimension of C_j for $1 \le j \le p$. We already know that σ_m is the Hausdorff dimension of $C_{j,m}$. Thus $\lim_{m\to\infty} \sigma_m = \sigma_0$ implies that the Hausdorff dimension of C_j is the limit of the Hausdorff dimension of $C_{j,m}$ as $m \to \infty$. $\square$

Remark 3.4. If we could allow some overlap in Theorem 2.9, for instance, if Theorem 2.9 is true under the open set condition or the strong open set condition (in the context of graph-directed systems on Euclidean spaces or graph-directed systems given by similitudes), then the first half of Theorem 3.3 still gives the continuity of the Hausdorff dimension.

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