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EXTENSIONS OF A PROBLEM OF PAUL ERDÖS ON GROUPS

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Abstract

The main results are as follows. A finitely generated soluble group G is polycyclic if and only if every infinite set of elements of G contains a pair generating a polycyclic subgroup; and the same result with "polycyclic" replaced by "coherent".

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1. Introduction and main results

The origin of the results in this note is a question of P. Erdös. Suppose that G is a group such that the subsets consisting of mutually noncommuting elements are all finite. Are they boundedly finite? B. H. Neumann (1976) answered this question affirmatively by proving that G has the property stated if and only if it is centre-by-finite. From the very wording of the problem, it is not surprising that Ramsey's infinite theorem was used in the proof: the graph considered by Neumann has the elements of G as vertices, two vertices being connected by an edge if and only if they do not commute.

We were inspired to ask questions of a similar nature. Given a group property $\mathfrak{X}$ and a group G, we can consider the $\mathfrak{X}$-graph $G_{\mathfrak{X}}$ on G whose vertices are the elements, as before, and two vertices x, y are connected by an edge if and only if $\langle x, y\rangle \notin \mathfrak{X}$. Thus Neumann's graph is $G_{\mathfrak{A}}$, $\mathfrak{A}$ being the class of abelian groups. By Ramsey's theorem, every infinite subgraph of $G_{\mathfrak{X}}$ contains either an infinite complete subgraph or an infinite null set. In keeping with the spirit of Erdös's

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question we define the class $\mathfrak{X}^{\#}$ to consist of those groups G such that $G_{\mathfrak{X}}$ contains no infinite complete subgraph. Thus G is in $\mathfrak{X}^{\#}$ if and only if every infinite subset of G contains a pair x, y of different elements such that $\langle x, y\rangle \in \mathfrak{X}$.

How, then, to recognise $\mathfrak{X}^{\#}$ when $\mathfrak{X}$ is one or other of the well-known group-theoretical properties? Neumann's theorem says that $\mathfrak{A}^{\#}$-groups are just the centre-by-finite groups. A next step might be to consider $\mathfrak{N}^{\#}$, where $\mathfrak{N}$ is the class of all nilpotent groups. Unfortunately, $\mathfrak{N}^{\#}$ contains some monsters. For instance, M. F. Newman's lovely example (unpublished) of an infinite 3-generator p-group in which *every* two-generator subgroup is (finite and) nilpotent; or that of Vaughan-Lee and Wiegold (1980), which is infinite, perfect, of prime exponent $p \geqslant 5$ and locally finite, so that every two-generator subgroup is even nilpotent of bounded class (see also L′vov and Khukhro (1978)). However, if one restricts attention to finitely generated soluble groups, one gets sensible answers. Unfortunately, Ramsey's theorem has been no help to us here.

Before starting our results, we introduce some notation for group properties:
$\mathfrak{F}$ finite,
$\mathfrak{S}$ soluble,
$\mathfrak{G}$ finitely generated,
$\mathfrak{P}$ polycyclic,
$\mathfrak{U}$ supersoluble,
$\mathfrak{C}$ coherent.

THEOREM A. $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{N}^{\#} = \mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{F}\mathfrak{N}$.

Of course, the difficult piece here is to prove that $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{N}^{\#} \subseteq \mathfrak{F}\mathfrak{N}$.

Next a somewhat surprising result, generalizing the first authors's result (Lennox (1973)) that a finitely generated soluble group is polycyclic if and only if every two-generator subgroup is polycyclic:

THEOREM B. $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{P}^{\#} = \mathfrak{P}$.

Thus a finitely generated soluble group is polycyclic if and only if every infinite subset contains a pair of elements generating a polycyclic subgroup.

We have tried our best to establish

CONJECTURE C. $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{U}^{\#} = \mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{F}\mathfrak{U}$,

without success. Even the inclusion $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{F}\mathfrak{U} \subseteq \mathfrak{U}^{\#}$, which we have proved, was not altogether obvious. Of course, Theorem B tells us that $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{U}^{\#} \subseteq \mathfrak{P}$, and this is surely significant; but we have to admit defeat here.

Finally, recall that a group is coherent if and only if every finitely generated subgroup is finitely presented. Groves (1978) showed that a finitely-generated soluble group G is coherent if and only if it is semi-polycyclic, that is, if and only if for every pair x, y of elements of G, either the subgroup $\langle x^{y^i}: i > 0\rangle$ or the subgroup $\langle x^{y^i}: i < 0\rangle$ is finitely generated. We use this and other results to prove

THEOREM D. $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{C}^{\#} = \mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{C}$.

The interested reader might like to do the finger-exercises necessary to replace "soluble" by "hyper-abelian" throughout.

2. Proofs

Firstly we deduce Theorem A from Theorem B.

Theorem B $\Rightarrow$ *Theorem* A. Let G be a group in $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{N}^{\#}$. Then every infinite set of elements of G contains a pair generating a nilpotent and thus a polycyclic group. So by Theorem B, G is polycyclic. Since G satisfies the maximum condition, we may assume that every factor-group of G by a non-trivial normal subgroup is in $\mathfrak{F}\mathfrak{N}$ and that G has a non-trivial abelian normal torsion-free subgroup A such that $G/A \in \mathfrak{F}\mathfrak{N}$. Let y, x be any elements of A, G respectively, and consider the infinite sequence

$$yx, y^2x, \ldots, y^rx, \ldots.$$

By hypothesis $\langle y^m x, y^n x\rangle$ is nilpotent for two different positive integers m, n and thus, with $r = n - m$ we have that $\langle y^r, y^m x\rangle$ is nilpotent. It follows that the repeated commutator

$$[y^r, {}_t x] = 1$$

for some t. So $[y, {}_t x]^r = 1$ as A is abelian and normal, so that $[y, {}_t x] = 1$ since A is torsion-free. Thus by a result of Baer (see Vol. II, 7.21 of Robinson (1972)), $y \in \zeta_\omega(G)$ so that in particular the centre $\zeta_1(G)$ is non-trivial. Thus G is finite-by-nilpotent since $G/\zeta_1(G)$ is $\mathfrak{F}\mathfrak{N}$ (see Hall (1956)).

To prove Theorem B, we need the following result, which is embedded in the proof of Theorem B of Lennox (1973).

LEMMA. *Let G be a finitely generated soluble group and A an abelian normal subgroup such that G/A is polycylic and $\langle a, x\rangle$ is polycyclic whenever $a \in A$, $x \in G$. Then G is polycyclic.*

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Thus, let G be a finitely generated soluble $\mathfrak{P}^{\#}$-group. We prove by induction on the solubility length that G is polycyclic. This is true in the abelian case, so we assume that G is non-abelian, that the theorem is true for groups of lower solubility length and that A is the last non-trivial term of the derived series. By induction G/A is polycyclic, and to apply the lemma we have to prove that $\langle a, x\rangle$ is polycyclic for all $a \in A$, $x \in G$. Clearly, we may assume that $G = \langle a, x\rangle$. Suppose that G is not polycyclic; then x is obviously of infinite order (otherwise $\langle a^{x^i}: i \in \mathbf{Z}\rangle$ is finitely generated) and the sequence

$$xa, x^2a, \ldots, x^na, \ldots$$

is infinite. As before, there exists $\alpha \neq 0$ and r such that $\langle x^\alpha, x^r a\rangle$ is polycyclic. Then $\langle [a, x^\alpha], x^\alpha\rangle$ is polycyclic and if $[a, x^\alpha] = 1$, G is obviously polycyclic. So we may assume that $u = [a, x^\alpha]$ is non-trivial, and that $\langle u^{\langle x^\alpha\rangle}\rangle$ is finitely generated. But then $\langle u^{\langle x\rangle}\rangle$ is finitely generated; since $G/\langle u^{\langle x\rangle}\rangle$ is polycyclic, we are through.

The proof of Theorem D is harder, and the reader is advised to have Groves's paper (1978) to hand. Suppose that G is not coherent but is in $\mathfrak{G} \cap \mathfrak{S} \cap \mathfrak{C}^{\#}$. All relevant hypotheses go over to homomorphic images, so we may assume that G has an abelian normal subgroup A such that G/A is coherent, that is, semi-polycyclic. By Proposition 8 of Groves (1978), G must fail to act semi-polycyclically on A; what this means here is that there are elements $a \in A$, $x \in G$ such that $\langle a, x\rangle$ is not semi-polycyclic. To achieve our contradiction we may assume that $G = \langle a, x\rangle$ and that G/N is semi-polycyclic for all $N \lhd G$, $N \neq 1$.

Evidently x is of infinite order and so we have an infinite sequence

$$xa, x^2a, \ldots, x^ra, \ldots.$$

By hypothesis $\langle x^r a, x^s a\rangle$ is semi-polycyclic for some different positive integers r, s, so that $K = \langle x^\alpha, x^r a\rangle$ is semi-polycyclic, with $\alpha = s - r$. Now set $u = [x^\alpha, a]$, so that as $u \in K$, $V = \langle u, x^\alpha\rangle$ is semi-polycyclic. If $u = 1$ then $\langle a, x\rangle$ is even polycyclic; so certainly $u \neq 1$.

Our next task is to prove that a has infinite order. Suppose to the contrary that $a^\gamma = 1$ for some $\gamma \neq 0$. By what we just saw, we have that $|W : V| < \infty$, where $W = \langle a, x^\alpha\rangle$. Thus by Lemma 1 of Groves (1978), W is semi-polycyclic. Now we need to consider the action of $y = x^\alpha$ on $a^{\langle x\rangle} = a^{\mathbf{Z}\langle x\rangle}$. Now suppose that $B = \langle a^{y^i}: i \in \mathbf{Z}^+\rangle$ is finitely generated, so that $B^\omega = \langle a^{\omega y^i}: i \in \mathbf{Z}^+\rangle$ is finitely generated. Similarly, if $\langle a^{y^i}: i \in \mathbf{Z}^-\rangle$ is finitely generated, then $\langle a^{\omega y^i}: i \in \mathbf{Z}^-\rangle$ is finitely generated. It follows that y acts semi-polycyclically on $\langle a^{\langle x\rangle}\rangle$, so that $\langle a, x\rangle$ is semi-polycyclic, contrary to hypothesis. Thus a has infinite order.

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We can now consider the infinite sequence

$$xa, xa^2, \ldots, xa^n, \ldots,$$

and in the usual way we find that $\langle a^\beta, xa^n \rangle$ is semi-polycyclic for some $\beta \neq 0$. Then $M = \langle a^\beta \rangle^{\langle x \rangle} \lhd G$ and x acts semi-polycyclically, so that by Lemma 2 of Groves (1978), G is semi-polycyclic. This final contradiction establishes Theorem D.

Problems of the sort we have been considering can be multiplied endlessly. Most of them seem very hard indeed. One that we have tackled with no success is to try to identify the members of $\mathfrak{N}_2^\#$, $\mathfrak{N}_2$ being the class of all nilpotent groups of class 2. We have no feeling whatever for what the infinitely generated and/or soluble groups of this type could be like. Is it true that $\mathfrak{N}_2^\# \subseteq \mathfrak{F}\mathfrak{N}$, for instance? We have no idea. As the wreath product Z_2 wr $Z_2^{(\aleph_0)}$ shows, $\mathfrak{S} \cap \mathfrak{N}_3^\# \not\subseteq \mathfrak{F}\mathfrak{N}$; no such easy way out for $\mathfrak{N}_2^\#$.

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