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RAMANUJAN CONGRUENCES FOR $p_{-k}(n)$ MODULO POWERS OF 17

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1. **Introduction.** For each integer r we define the sequence $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n)x^n = \prod_{m=1}^{\infty}(1-x^m)^r \text{ for } |x| < 1.$$

We note that $p_{-1}(n) = p(n)$, the ordinary partition function. On account of this some authors set $r = -k$ to make positive values of k correspond to positive powers of the generating function for $p(n)$:

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n.$$

We follow this convention here. In [3], Atkin proved the following theorem.

THEOREM 1. *Let k be a positive integer and $q = 2, 3, 5, 7$ or 13. If $24n \equiv k \pmod{q^r}$, then $p_{-k}(n) \equiv 0 \pmod{q^{\alpha r/2+\varepsilon}}$, where $\varepsilon = \varepsilon(q,k) = O(\log k)$ and $\alpha = \alpha(q,k)$ depends on q and the residue of $k \pmod{24}$ according to Table 1. Where there are blank entries, nothing is asserted.*

	1	2	3	4	5	6	7	8	9	10	11	12
$q = 2$								3				
$q = 3$			3			2			3			2
$q = 5$	2	1	1	1	2	2	1	1	1	1	1	0
$q = 7$	1	1	1	2	1	1	1	0	0	0	1	0
$q = 13$	0	0	0	0	0	0	0	1	0	0	0	0

	13	14	15	16	17	18	19	20	21	22	23	24
$q = 2$				3								0
$q = 3$			1			2			1			0
$q = 5$	0	0	1	1	0	0	0	1	1	0	0	0
$q = 7$	0	1	0	0	0	1	0	0	1	0	0	0
$q = 13$	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 1

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In [5], Gordon extended Theorem 1 to the case where $k < 0$. In this case, the conclusion of Theorem 1 holds with $\varepsilon = O(\log|k|)$, the last column of the above table changed to 6, 4, 2, 2, 0, and the rest of the table unchanged. He also obtained results analogous to Theorem 1 for $q = 11$ and $k \neq 0$. These results are given next.

THEOREM 2. *Let k be a nonzero integer. If* $24n \equiv k \pmod{11^r}$, *then* $p_{-k}(n) \equiv 0 \pmod{11^{\alpha r/2+\varepsilon}}$, *where* $\varepsilon = \varepsilon(k) = O(\log|k|)$ *and* $\alpha = \alpha(k)$ *depends on the residue of* $k \pmod{120}$. *The dependence of α on k is given in Table 2 for $k > 0$. Here the entry in the row labelled $24i$ and the column labelled j is $\alpha(24i+j)$. For $k < 0$, the dependence of α on k is the same except that the last column of the table is changed to 2, 2, 2, 0, 2.*

	1	2	3	4	5	6	7	8	9	10	11	12
0	2	1	2	1	1	1	2	2	1	1	2	2
24	1	1	1	1	2	2	1	1	2	2	1	0
48	1	1	2	2	1	1	1	0	1	0	1	0
72	2	1	1	1	2	1	2	1	2	1	2	2
96	0	0	1	0	1	0	1	0	1	1	0	0

	13	14	15	16	17	18	19	20	21	22	23	24
0	1	2	1	0	0	1	1	0	0	1	1	0
24	0	0	0	1	1	0	0	1	1	1	0	0
48	0	1	1	0	0	1	0	1	0	1	0	0
72	1	1	1	2	1	2	1	2	1	1	1	0
96	0	1	0	1	0	1	0	1	1	0	0	0

TABLE 2

Here we apply the method used by Gordon in his proof of Theorem 2 to obtain an analogous result for $q = 17$. We prove the following result.

THEOREM 3. *Let k be a nonzero integer. If* $24n \equiv k \pmod{17^r}$, *then* $p_{-k}(n) \equiv 0 \pmod{17^{\alpha r/2+\varepsilon}}$, *where* $\varepsilon = \varepsilon(k) = O(\log|k|)$ *and* $\alpha = \alpha(k)$ *depends on the residue of* $k \pmod{96}$. *The dependence of α on k is given in Table 3 for $k > 0$. Here the entry in the row labelled $24i$ and the column labelled j is $\alpha(24i+j)$. For $k < 0$, the dependence of α on k is the same except that the last column of the table is changed to 0, 2, 0, 0.*

The general plan of this paper is as follows. In § 2 we set up the notation and phrase the result of Theorem 3 in a manner more suitable for proof. In particular, we will see that it largely amounts to the proof of a certain key lemma (Lemma 4); this lemma will be the topic of § 3. Included in § 3 is an item of independent interest: a modular equation between functions on $\Gamma_0(289)$ and functions on $\Gamma_0(17)$. In § 4 we prove Theorem 3. We make some concluding remarks in § 5. Finally, there are two appendices concerned with certain lengthy numerical calculations.

	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0
48	1	1	1	1	1	2	1	0	0	0	0	0
72	0	0	0	0	0	0	0	0	0	1	0	0

	13	14	15	16	17	18	19	20	21	22	23	24
0	0	0	0	0	0	0	0	1	0	0	0	0
24	1	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	1	0	0	0	0	0	1	0
72	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 3

2. **Preliminary discussion and lemmas.** Let $\mathcal{L}$ be the complex vector space of meromorphic Laurent series $\sum_{n\geq n_0} a_n x^n$, convergent in some deleted neighborhood of 0. Define the dissection operator $U = U_{17}$ on $\mathcal{L}$ by

(1) $$U(\sum_{n\geq n_0} a_n x^n) = \sum_{17n\geq n_0} a_{17n} x^n.$$

One readily verifies that U is linear and has the following property:

(2) $$U\big(f(x^{17})g(x)\big) = f(x)U\big(g(x)\big) \text{ for } f(x) \text{ and } g(x) \text{ in } \mathcal{L}.$$

Let k be a fixed nonzero integer. In using the dissection operator to isolate the desired subsequences of $p_{-k}(n)$, it is necessary to make a preliminary index shift half of the time. For this reason we define the auxiliary sequence λ_r by

$$\lambda_r = \begin{cases} 0 & \text{if } r \text{ is odd;} \\ k & \text{if } r \text{ is even.} \end{cases}$$

Using λ_r we define a recursive sequence of 17-dissections of $P(x)^k$ as follows:

(3) $$\begin{cases} D_0(x) = P(x)^k \\ D_r(x) = U\big(x^{12\lambda_{r-1}} D_{r-1}(x)\big) & \text{for } r \geq 1. \end{cases}$$

It is readily verified as in [5] that for each $r \geq 0$ we have

(4) $$D_r(x) = \sum_{m\geq \mu_r} p_{-k}(17^r m + n_r) x^m,$$

where $n_r = -k(17^{2\lfloor (r+1)/2 \rfloor} - 1)/24$ and $\mu_r = \lceil -n_r/17^r \rceil$; here $\lfloor a \rfloor$ and $\lceil a \rceil$ respectively denote the floor and ceiling of a. It is apparent from (3) that

(5) $$\mu_r = \left\lceil \frac{12\lambda_{r-1} + \mu_{r-1}}{17} \right\rceil.$$

If we set

$$\omega(k) = \begin{cases} 1 & \text{if } k < 0 \text{ and } k \equiv 0 \pmod{24}; \\ 0 & \text{otherwise,} \end{cases}$$

we have additionally from the above

$$\mu_r = \begin{cases} \lceil 17k/24 \rceil + \omega(k) & \text{if } r > \log_{17}|k| \text{ and } r \text{ is odd;} \\ \lceil k/24 \rceil + \omega(k) & \text{if } r > \log_{17}|k| \text{ and } r \text{ is even.} \end{cases}$$

Further, $24n_r \equiv k \pmod{17^r}$. From this last observation, it is clear that the coefficients of $D_r(x)$ are those $p_{-k}(n)$ with $24n \equiv k \pmod{17^r}$. Therefore, to prove Theorem 3 it suffices to show that

(6) $$D_r(x) \equiv 0 \pmod{17^{\alpha r/2+\varepsilon}}$$

where α and ε are as described there, and where

$$\sum a_n x^n \equiv \sum b_n x^n \pmod{M}$$

means that $a_n \equiv b_n \pmod{M}$ for all n.

To establish (6), it is convenient to introduce a certain sequence of modular functions which are related to the $D_r(x)$. We first introduce some preliminary notation and results concerning modular functions. We set $x = e^{2\pi i\tau}$ and recall the Dedekind eta function

$$\eta(\tau) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n), \quad \operatorname{Im}\tau > 0.$$

From Theorem 1 of [7], $\phi(\tau) = \frac{\eta(289\tau)}{\eta(\tau)}$ is a meromorphic function on $\Gamma_0(289)$, holomorphic for $\operatorname{Im}\tau > 0$, and having orders 12 and -12 at 0 and $i\infty$ respectively. For any positive integer N, let $X_0(N)$ be the compactified Riemann surface of the group $\Gamma_0(N)$ and let $K_0(N)$ be the field of meromorphic functions on $X_0(N)$. For any N, $x = e^{2\pi i\tau}$ is a local uniformizing paramater at the cusp $i\infty$ of $X_0(N)$. Thus $\phi(\tau)$ has a Fourier expansion at $i\infty$ with lowest term x^{12}. For $f(\tau)$ in $K_0(17)$ we denote the order at a point p on $X_0(17)$ by $\operatorname{ord}_p f$, and for $F(\tau)$ in $K_0(289)$ we denote the order at a point P on $R_0(289)$ by $\operatorname{Ord}_P F$. All modular fuctions of interest here are holomorphic in the upper half plane, and so have poles only at cusps. The cusps of $X_0(17)$ are at $i\infty$ and 0. The cusp $i\infty$ of $X_0(17)$ has width 1, and lying above it on $X_0(289)$ are the cusps $i\infty$ and $h/17$, $1 \le h \le 16$, each having width 1. The cusp 0 of $X_0(17)$ has width 17, and the cusp 0 of $X_0(289)$ lies above it, the width being 289. We have, therefore, the following lemma.

LEMMA 1. *Suppose $f(\tau) \in K_0(17)$. Regarding $f(\tau)$ dually as an element of $K_0(289)$, we have*

$$\operatorname{Ord}_{h/17} f = \operatorname{Ord}_{i\infty} f = \operatorname{ord}_{i\infty} f, \quad 1 \le h \le 16$$

and

$$\operatorname{Ord}_0 f = 17 \operatorname{ord}_0 f.$$

If we let U act as in (1) on the Fourier expansions of elements of $K_0(289)$, then U acts as a Hecke operator, since for $f(\tau)$ in $K_0(289)$ with Fourier expansion $f(\tau) = \sum_{n \ge n_0} a_n x^n$ we have

(7) $$Uf(\tau) = \sum_{17n \ge n_0} a_{17n} x^n = \frac{1}{17} \sum_{h=0}^{16} f\big((\tau + h)/17\big).$$

We have the following lemma, proved in Corollary 1.10 of [6].

LEMMA 2. *If* $f(\tau) \in K_0(289)$, *then* $Uf(\tau) \in K_0(17)$. *If* $f(\tau)$ *is holomorphic for* $\operatorname{Im}\tau > 0$, *then so is* $Uf(\tau)$. *Furthermore,*

$$\operatorname{ord}_{i\infty} Uf \geq \frac{1}{17} \operatorname{Ord}_{i\infty} f$$

and

$$\operatorname{ord}_0 Uf \geq \min_{0 \leq h \leq 16} \operatorname{Ord}_{h/17} f.$$

Now the $D_r(x)$ are not themselves modular functions. However, we do have

$$P(x)^k = x^{k/24}\eta(\tau)^{-k} \text{ and } P(x^{289})^k = x^{289k/24}\eta(289\tau)^{-k},$$

so

$$\frac{P(x)^k}{P(x^{289})^k} = x^{-12k}\left(\frac{\eta(289\tau)}{\eta(\tau)}\right)^k$$

and

$$x^{12k}\frac{P(x)^k}{P(x^{289})^k} = \left(\frac{\eta(289\tau)}{\eta(\tau)}\right)^k = \phi(\tau)^k \in K_0(289).$$

From (2) we have

$$U\big(\phi(\tau)^k\big) = \frac{U\big(x^{12k}P(x)^k\big)}{P(x^{17})^k} = \frac{D_1(x)}{P(x^{17})^k};$$

and more generally for $s \geq 0$ we have from (2) and (3):

$$(8) \qquad \begin{cases} U\left(\frac{D_{2s-1}(x)}{P(x^{17})^k}\right) = \frac{U\left(D_{2s-1}(x)\right)}{P(x)^k} = \frac{D_{2s}(x)}{P(x)^k}, \\ U\left(\phi(\tau)^k \frac{D_{2s}(x)}{P(x)^k}\right) = \frac{U\left(x^{12k}D_{2s}(x)\right)}{P(x^{17})^k} = \frac{D_{2s+1}(x)}{P(x^{17})^k}. \end{cases}$$

We now define a new sequence of functions $L_r(x)$ by

$$(9) \qquad \begin{cases} L_{2s}(x) = \frac{D_{2s}(x)}{P(x)^k} \\ L_{2s+1} = \frac{D_{2s+1}(x)}{P(x^{17})^k} \end{cases}, \text{ for } s \geq 0.$$

Therefore

$$(10) \qquad \begin{cases} L_0(x) = 1, \\ L_r(x) = U\big(\phi(\tau)^{\lambda_{r-1}} L_{r-1}(x)\big), \quad \text{for } r \geq 1. \end{cases}$$

By Lemma 2, the $L_r(x)$ are in $K_0(17)$ and are holomorphic for $\operatorname{Im}\tau > 0$. From (4) we have

$$(11) \qquad \operatorname{ord}_{i\infty} L_r \geq \mu_r.$$

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In order to prove (6), it suffices to show that

$$L_r(x) \equiv 0 \pmod{17^{\alpha r/2+\varepsilon}} \tag{12}$$

where α and ε are as before. This is the object of the rest of this paper.

Let $\mathcal{V}$ be the algebra of functions in $K_0(17)$ which are holomorphic for $\operatorname{Im}\tau > 0$. By Lemma 2, $\mathcal{V}$ is invariant under the linear operators T_λ defined by $T_\lambda g = U(\phi^\lambda g)$. We have $L_{r+1} = T_{\lambda_r} L_r$, so each $L_r \in \mathcal{V}$. In the next lemma we give a basis for $\mathcal{V}$, so as to obtain matrices $C^{(\lambda)} = (c_{\mu,\nu}^{(\lambda)})$ for the T_λ. We will show that divisibility by powers of 17 of the entries $c_{\mu,\nu}^{(\lambda)}$ implies divisibility by powers of 17 of the Fourier coefficients of the L_r; included in this lemma are facts which we will use to show this implication. We note that Atkin [2] has obtained some similar results for $K_0(q)$, with $17 \le q \le 37$, q prime.

LEMMA 3. *There is a basis* $\mathcal{B} = \{J_\nu \mid -\infty < \nu < \infty\}$ *for* $\mathcal{V}$ *with the following properties:*

(i) $J_{\nu+4} = J_\nu J_4$;

(ii) $\operatorname{ord}_{i\infty} J_\nu = \nu$;

(iii) $\operatorname{ord}_0 J_\nu = \psi(\nu) = \begin{cases} -\nu - 1 & \text{if } \nu \equiv 1 \text{ or } 2 \pmod 4, \\ -\nu - 2 & \text{if } \nu \equiv 3 \pmod 4, \\ -\nu & \text{if } \nu \equiv 0 \pmod 4, \end{cases}$

(iv) the Fourier series for $J_\nu(\tau)$ *has 17-integral coefficients and leading coefficient 1;*

(v) the J_ν *satisfy the multiplication table below, which determines the structure constants of* $\mathcal{B}$.

	J_1	J_2	J_3
J_1	$J_2 + (17/4)\cdot J_4$	J_3	$J_4 + 8J_5 + (17/4)\cdot J_6$
J_2		$J_4 + 8J_5$	$J_5 + 8J_6 + 2\cdot 17J_8$
J_3			$J_6 + 8J_7 + (17/4)\cdot J_8 + 2\cdot 17J_9$

TABLE 4

PROOF OF LEMMA 3. To define J_1, J_2, J_3 and J_4, we use some results of Newman [7,8]. We begin by setting $J_4(\tau) = [\eta(17\tau)/\eta(\tau)]^6$, which by Theorem 1 of [7] is in $K_0(17)$. One can readily verify that $J_4 \in \mathcal{V}$, that $\operatorname{ord}_{i\infty} J_4 = 4 = -\operatorname{ord}_0 J_4$, and that J_4 has integral Fourier coefficients and leading coefficient 1. From Table (2.8) of [8], we have (using Newman's notation) a function S_5 in $\mathcal{V}$ with a triple pole at 0 and with a Fourier series of the form $17^{-2}\big(1 - 10x^2A(x)\big)$, where $A(x)$ has 17-integral coefficients and leading coefficient 1. We set $J_2 = (1 - 17^2S_5)/10$. Clearly $J_2 \in \mathcal{V}$, $\operatorname{ord}_{i\infty} J_2 = 2$, $\operatorname{ord}_0 J_2 = -3$, and J_2 has 17-integral Fourier coefficients and leading coefficient 1. Our next goal is to obtain J_{-3}. The function $f(\tau) = J_2(-1/17\tau)$ is in $\mathcal{V}$. It has a triple pole at $i\infty$ and a double pole at 0, and by equation (2.7) of [8] it has a Fourier series of the form $\big(1 - 17^{-2}B(x)\big)/10$, where $B(x)$ has integral coefficients and leading coefficient -20. We set $J_{-3} = (17^2/2)f$, so $J_{-3} \in \mathcal{V}$, $\operatorname{ord}_{i\infty} J_{-3} = -3$, $\operatorname{ord}_0 J_{-3} = 2$, and J_{-3} has 17-integral

	$\nu=0$	$\nu=1$	$\nu=2$	$\nu=3$
$\mu=0$	-1	10	13	20
$\mu=1$	0	7	10	11
$\mu=2$	1	4	11	12
$\mu=3$	-2	5	12	15

TABLE 5

Fourier coefficients and leading coefficient 1. Setting $J_1 = J_{-3}J_4$ and $J_3 = J_1J_2$, we see that J_1 and J_3 are in $\mathcal{V}$ with $\operatorname{ord}_{i\infty} J_1 = 1$, $\operatorname{ord}_{i\infty} J_3 = 3$, $\operatorname{ord}_0 J_1 = -2$ and $\operatorname{ord}_0 J_3 = -5$. Moreover, J_1 and J_3 have 17-integral Fourier coefficients and leading coefficients 1.

We now define J_ν for any integer ν. For $\nu = 4\nu_1+\nu_2$, $0 \le \nu_2 \le 3$, we set $J_\nu = J_4^{\nu_1} J_{\nu_2}$. Clearly properties (i) through (iv) hold for all J_ν. That $\{J_\nu \mid -\infty < \nu < \infty\}$ is a basis for $\mathcal{V}$ follows from an elementary argument using Liouville's theorem. The entries in Table 4 can be computed using the first few terms of the Fourier series of the J_ν and their orders given in (ii) and (iii). The structure constants are determined by this table and (i). This completes the proof of Lemma 3.

As discussed above, we let $C^{(\lambda)} = (c^{(\lambda)}_{\mu,\nu})$ denote the matrix of T_λ relative to $\mathcal{B}$, where, following [3,5], we let matrices act from the right, expressing elements of $\mathcal{V}$ as row vectors. Since $\operatorname{ord}_{i\infty} L_r \ge \mu_r$, we can write

$$L_r = \sum_{\nu \ge \mu_r} a_{r,\mu} J_\nu \tag{13}$$

(where the sum is finite). Application of the $C^{(\lambda_r)}$ gives the recurrence

$$a_{r+1,\nu} = \sum_{\mu \ge \mu_r} a_{r,\mu} c^{(\lambda_r)}_{\mu,\nu} \tag{14}$$

with initial conditions $a_{0,0} = 1$ and $a_{0,\nu} = 0$ for $\nu > 0$. Denoting the 17-adic order of a rational number a by $\pi(a)$, we will prove (12) by showing that all the $a_{r,\nu}$ are 17-integers with

$$\pi(a_{r,\nu}) \ge \alpha r + \varepsilon. \tag{15}$$

Equation (15) follows from the facts that the $c^{(\lambda)}_{\mu,\nu}$ are 17-integers with a certain lower bound on $\pi(c^{(\lambda)}_{\mu,\nu})$, and that iteration of the T_λ causes an accretion in the values $\pi(a_{r,\nu})$ as r increases. An explicit description of α and ε will emerge in the course of the proof. We now give the lower bound for $\pi(c^{(\lambda)}_{\mu,\nu})$.

LEMMA 4. *All the $c^{(\lambda)}_{\mu,\nu}$ are 17-integers and*

$$\pi(c^{(\lambda)}_{\mu,\nu}) \ge \left\lfloor \frac{17\nu - \mu - 12\lambda + \delta(\mu,\nu)}{24} \right\rfloor, \tag{16}$$

where $\delta(\mu,\nu)$ depends only on the residues of μ and ν (mod 4) *according to Table 5.*

The proof of Lemma 4 is fairly lengthy, involving a certain amount of numerical calculation. It is also the key to proving (15), and hence Theorem 3. The proof of Lemma 4 is the topic of the next section.

3. **Proof of Lemma 4.** We obtain two recurrences for the $T_\lambda J_\mu$ together with initial conditions. This completely determines all $T_\lambda J_\mu$, and hence all $c_{\mu,\nu}^{(\lambda)}$. Lemma 4 will follow by induction on λ and μ after writing the corresponding recurrences and initial conditions for the $c_{\mu,\nu}^{(\lambda)}$.

For the first (and simpler) recurrence, we observe, reasoning as in [5], that

$$T_\lambda J_\mu = U(\phi^\lambda J_\mu) = J_{-4} U(\phi^{\lambda+6} J_{\mu-4}) = J_{-4} T_{\lambda+6} J_{\mu-4}.$$

Thus, we have

$$\sum_\nu c_{\mu,\nu}^{(\lambda)} J_\nu = \sum_\nu c_{\mu-4,\nu}^{(\lambda+6)} J_{\nu-4} = \sum_\nu c_{\mu-4,\nu+4}^{(\lambda+6)} J_\nu.$$

Equating coefficients yields the corresponding recurrence for the $c_{\mu,\nu}^{(\lambda)}$:

$$c_{\mu,\nu}^{(\lambda)} = c_{\mu-4,\nu-4}^{(\lambda+6)} \tag{17}$$

To obtain the second recurrence, we observe that

$$T_\lambda J_\mu = U(\phi^\lambda J_\mu) = \frac{1}{17}\sum_{h=0}^{16} \phi\big((\tau+h)/17\big)^\lambda J_\mu\big((\tau+h)/17\big).$$

Each $\phi\big((\tau+h)/17\big) = t_h, 0 \le h \le 16$, is a root of the modular equation with coefficients σ_k in $\mathcal{V}$,

$$t^{17} + \sum_{k=1}^{17} (-1)^k \sigma_k t^{17-k} = 0, \tag{18}$$

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where the σ_k are given below:

$$\sigma_1 = 7\cdot 17J_1 + (115/2)\cdot 17^2J_2 + 84\cdot 17^3J_3 + 849\cdot 17^3J_4 + 640\cdot 17^4J_5 + 158\cdot 17^5J_6 + 20\cdot 17^6J_7 + 55\cdot 17^6J_8 + 13\cdot 17^7J_9 + (1/2)\cdot 17^8J_{10} + 17^8J_{12},$$

$$\sigma_2 = 176\cdot 17J_2 - 142\cdot 17^2J_3 - 4888\cdot 17^2J_4 - 4371\cdot 17^3J_5 - (2851/2)\cdot 17^4J_6 - 224\cdot 17^5J_7 - 595\cdot 17^5J_8 - 171\cdot 17^6J_9 - (15/2)\cdot 17^7J_{10} - 17^8J_{12},$$

$$\sigma_3 = -14\cdot 17^2J_3 + 900\cdot 17^2J_4 + 987\cdot 17^3J_5 + (847/2)\cdot 17^4J_6 + 80\cdot 17^5J_7 + 215\cdot 17^5J_8 + 71\cdot 17^6J_9 + (7/2)\cdot 17^7J_{10} + 9\cdot 17^7J_{12},$$

$$\sigma_4 = -10\cdot 17J_3 - 2858\cdot 17J_4 - 3287\cdot 17^2J_5 - (3107/2)\cdot 17^3J_6 - 312\cdot 17^4J_7 - 939\cdot 17^4J_8 - 323\cdot 17^5J_9 - (35/2)\cdot 17^6J_{10} - 3\cdot 17^7J_{12},$$

$$\sigma_5 = 163\cdot 17J_4 + 311\cdot 17^2J_5 + (435/2)\cdot 17^3J_6 + 46\cdot 17^4J_7 + 188\cdot 17^4J_8 + 63\cdot 17^5J_9 + (7/2)\cdot 17^6J_{10} + 11\cdot 17^6J_{12},$$

$$\sigma_6 = -247\cdot 17J_5 - (659/2)\cdot 17^2J_6 - 54\cdot 17^3J_7 - 590\cdot 17^3J_8 - 167\cdot 17^4J_9 - (15/2)\cdot 17^5J_{10} - 17^6J_{12},$$

$$\sigma_7 = 17J_5 + (33/2)\cdot 17^2J_6 - 2\cdot 17^3J_7 + 120\cdot 17^3J_8 + 29\cdot 17^4J_9 + (1/2)\cdot 17^5J_{10} - 5\cdot 17^5J_{12},$$

$$\sigma_8 = 6\cdot 17J_6 + 12\cdot 17^2J_7 - 401\cdot 17^2J_8 - 104\cdot 17^3J_9 + 3\cdot 17^5J_{12},$$

$$\sigma_9 = 63\cdot 17^2J_8 + 20\cdot 17^3J_9 - 15\cdot 17^4J_{12},$$

$$\sigma_{10} = -103\cdot 17J_8 - 44\cdot 17^2J_9 + 3\cdot 17^4J_{12},$$

$$\sigma_{11} = 5\cdot 17J_8 + 4\cdot 17^2J_9 - 5\cdot 17^3J_{12},$$

$$\sigma_{12} = -4\cdot 17J_9 - 17^3J_{12},$$

$$\sigma_{13} = 11\cdot 17^2J_{12},$$

$$\sigma_{14} = -3\cdot 17^2J_{12},$$

$$\sigma_{15} = 9\cdot 17J_{12},$$

$$\sigma_{16} = -17J_{12},$$

$$\sigma_{17} = J_{12}.$$

The σ_k can be obtained from the $U(\phi^k)$, since we have $17U(\phi^k) = \sum_{h=0}^{16} t_h^k = \pi_k$, a power sum; the Newton identities relate the σ_k and the π_k, and with the π_k known we can obtain the σ_k. The $U(\phi^k)$ arise also in computing initial conditions, so we discuss their computation there. Since each t_h satisfies (18), we deduce that for any integer λ

$$t_h^\lambda = \sum_{k=1}^{17}(-1)^{k+1}\sigma_k t_h^{\lambda-k}. \tag{19}$$

Multiplication of (19) by $J_\mu\big((\tau + h)/17\big)$ and summation on h gives

$$T_\lambda J_\mu = \sum_{k=1}^{17}(-1)^{k+1}\sigma_k T_{\lambda-k}J_\mu.$$

We get the corresponding recurrence for the $c^{\lambda}_{\mu,\nu}$ by equating coefficients of the J_ν, obtaining equations

$$\sum_{\rho,\sigma} \beta^{(\rho)}_{\sigma}(\nu) c^{(\lambda-\rho)}_{\mu,\nu-\sigma}. \tag{20}$$

The result depends on the residue class of $\nu \pmod 4$ and it is very long, so we do not reproduce all of it here. We give details only for the case $\nu \equiv 0 \pmod 4$ as an example:

$$\begin{aligned}
c^{\lambda}_{\mu,\nu} = {} & 7 \cdot 17 c^{(\lambda-1)}_{\mu,\nu-1} + (115/2) \cdot 17^2 c^{(\lambda-1)}_{\mu,\nu-2} + (5719/4) \cdot 17^2 c^{(\lambda-1)}_{\mu,\nu-3} \\
& + 849 \cdot 17^3 c^{(\lambda-1)}_{\mu,\nu-4} + 11352 \cdot 17^3 c^{(\lambda-1)}_{\mu,\nu-5} + 2656 \cdot 17^4 c^{(\lambda-1)}_{\mu,\nu-6} \\
& + 500 \cdot 17^5 c^{(\lambda-1)}_{\mu,\nu-7} + 55 \cdot 17^6 c^{(\lambda-1)}_{\mu,\nu-8} + 622 \cdot 17^6 c^{(\lambda-1)}_{\mu,\nu-9} \\
& + (97/2) \cdot 17^7 c^{(\lambda-1)}_{\mu,\nu-10} + (13/4) \cdot 17^8 c^{(\lambda-1)}_{\mu,\nu-11} + 17^8 c^{(\lambda-1)}_{\mu,\nu-12} \\
& + 17^9 c^{(\lambda-1)}_{\mu,\nu-13} - 176 \cdot 17 c^{(\lambda-2)}_{\mu,\nu-2} + 142 \cdot 17^2 c^{(\lambda-2)}_{\mu,\nu-3} \\
& + 4888 \cdot 17^2 c^{(\lambda-2)}_{\mu,\nu-4} + (149117/2) \cdot 17^2 c^{(\lambda-2)}_{\mu,\nu-5} + (49035/2) \cdot 17^3 c^{(\lambda-2)}_{\mu,\nu-6} \\
& + (19703/4) \cdot 17^4 c^{(\lambda-2)}_{\mu,\nu-7} + 595 \cdot 17^5 c^{(\lambda-2)}_{\mu,\nu-8} + 6710 \cdot 17^5 c^{(\lambda-2)}_{\mu,\nu-9} \\
& + (1151/2) \cdot 17^6 c^{(\lambda-2)}_{\mu,\nu-10} + (171/4) \cdot 17^7 c^{(\lambda-2)}_{\mu,\nu-11} + 17^8 c^{(\lambda-2)}_{\mu,\nu-12} \\
& + 15 \cdot 17^8 c^{(\lambda-2)}_{\mu,\nu-13} - 14 \cdot 17^2 c^{(\lambda-3)}_{\mu,\nu-3} + 900 \cdot 17^2 c^{(\lambda-3)}_{\mu,\nu-4} \\
& + (1967/2) \cdot 17^3 c^{(\lambda-3)}_{\mu,\nu-5} + (14343/2) \cdot 17^3 c^{(\lambda-3)}_{\mu,\nu-6} + (6427/4) \cdot 17^4 c^{(\lambda-3)}_{\mu,\nu-7} \\
& + 215 \cdot 17^5 c^{(\lambda-3)}_{\mu,\nu-8} + 2394 \cdot 17^5 c^{(\lambda-3)}_{\mu,\nu-9} + (439/2) \cdot 17^6 c^{(\lambda-3)}_{\mu,\nu-10} \\
& + (71/4) \cdot 17^7 c^{(\lambda-3)}_{\mu,\nu-11} + 9 \cdot 17^7 c^{(\lambda-3)}_{\mu,\nu-12} + 7 \cdot 17^8 c^{(\lambda-3)}_{\mu,\nu-13} \\
& + 10 \cdot 17 c^{(\lambda-4)}_{\mu,\nu-3} + 2858 \cdot 17 c^{(\lambda-4)}_{\mu,\nu-4} + (6579/2) \cdot 17^2 c^{(\lambda-4)}_{\mu,\nu-5} \\
& + (52859/2) \cdot 17^2 c^{(\lambda-4)}_{\mu,\nu-6} + (24503/4) \cdot 17^3 c^{(\lambda-4)}_{\mu,\nu-7} + 939 \cdot 17^4 c^{(\lambda-4)}_{\mu,\nu-8} \\
& + 4433 \cdot 17^4 c^{(\lambda-4)}_{\mu,\nu-9} + (1843/2) \cdot 17^5 c^{(\lambda-4)}_{\mu,\nu-10} + (323/4) \cdot 17^6 c^{(\lambda-4)}_{\mu,\nu-11} \\
& + 3 \cdot 17^7 c^{(\lambda-4)}_{\mu,\nu-12} + 35 \cdot 17^7 c^{(\lambda-4)}_{\mu,\nu-13} + 163 \cdot 17 c^{(\lambda-5)}_{\mu,\nu-4} \\
& + 311 \cdot 17^2 c^{(\lambda-5)}_{\mu,\nu-5} + (435/2) \cdot 17^3 c^{(\lambda-5)}_{\mu,\nu-6} + (3439/4) \cdot 17^3 c^{(\lambda-5)}_{\mu,\nu-7} \\
& + 188 \cdot 17^4 c^{(\lambda-5)}_{\mu,\nu-8} + 2332 \cdot 17^4 c^{(\lambda-5)}_{\mu,\nu-9} + 303 \cdot 17^5 c^{(\lambda-5)}_{\mu,\nu-10} \\
& + (63/4) \cdot 17^6 c^{(\lambda-5)}_{\mu,\nu-11} + 11 \cdot 17^6 c^{(\lambda-5)}_{\mu,\nu-12} + 7 \cdot 17^7 c^{(\lambda-5)}_{\mu,\nu-13} \\
& + 247 \cdot 17 c^{(\lambda-6)}_{\mu,\nu-5} + (659/2) \cdot 17^2 c^{(\lambda-6)}_{\mu,\nu-6} + (3919/4) \cdot 17^2 c^{(\lambda-6)}_{\mu,\nu-7} \\
& + 590 \cdot 17^3 c^{(\lambda-6)}_{\mu,\nu-8} + 2308 \cdot 17^3 c^{(\lambda-6)}_{\mu,\nu-9} + (471/2) \cdot 17^4 c^{(\lambda-6)}_{\mu,\nu-10} \\
& + (167/4) \cdot 17^5 c^{(\lambda-6)}_{\mu,\nu-11} + 17^6 c^{(\lambda-6)}_{\mu,\nu-12} + 15 \cdot 17^6 c^{(\lambda-6)}_{\mu,\nu-13} \\
& + 17 c^{(\lambda-7)}_{\mu,\nu-5} + (33/2) \cdot 17^2 c^{(\lambda-7)}_{\mu,\nu-6} - (135/4) \cdot 17^2 c^{(\lambda-7)}_{\mu,\nu-7} \\
& + 120 \cdot 17^3 c^{(\lambda-7)}_{\mu,\nu-8} + (1029/2) \cdot 17^3 c^{(\lambda-7)}_{\mu,\nu-9} + (9/2) \cdot 17^4 c^{(\lambda-7)}_{\mu,\nu-10} \\
& + (29/4) \cdot 17^5 c^{(\lambda-7)}_{\mu,\nu-11} - 5 \cdot 17^5 c^{(\lambda-7)}_{\mu,\nu-12} + 17^6 c^{(\lambda-7)}_{\mu,\nu-13} \\
& - 6 \cdot 17 c^{(\lambda-8)}_{\mu,\nu-6} - 12 \cdot 17^2 c^{(\lambda-8)}_{\mu,\nu-7} + 401 \cdot 17^3 c^{(\lambda-8)}_{\mu,\nu-8}
\end{aligned}$$

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$$+1765\cdot 17^2c_{\mu,\nu-9}^{(\lambda-8)} - 24\cdot 17^3c_{\mu,\nu-10}^{(\lambda-8)} + 26\cdot 17^4c_{\mu,\nu-11}^{(\lambda-8)}$$
$$-3\cdot 17^5c_{\mu,\nu-12}^{(\lambda-8)} + 63\cdot 17^2c_{\mu,\nu-8}^{(\lambda-9)} + 20\cdot 17^3c_{\mu,\nu-9}^{(\lambda-9)}$$
$$+5\cdot 17^4c_{\mu,\nu-11}^{(\lambda-9)} - 15\cdot 17^4c_{\mu,\nu-12}^{(\lambda-9)} + 103\cdot 17c_{\mu,\nu-8}^{(\lambda-10)}$$
$$+44\cdot 17^2c_{\mu,\nu-9}^{(\lambda-10)} + 11\cdot 17^3c_{\mu,\nu-11}^{(\lambda-10)} - 3\cdot 17^4c_{\mu,\nu-12}^{(\lambda-10)}$$
$$+5\cdot 17c_{\mu,\nu-8}^{(\lambda-11)} + 4\cdot 17^2c_{\mu,\nu-9}^{(\lambda-11)} + 17^3c_{\mu,\nu-11}^{(\lambda-11)}$$
$$-5\cdot 17^3c_{\mu,\nu-12}^{(\lambda-11)} + 4\cdot 17c_{\mu,\nu-9}^{(\lambda-12)} + 17^2c_{\mu,\nu-11}^{(\lambda-12)}$$
$$+17^3c_{\mu,\nu-12}^{(\lambda-12)} + 11\cdot 17^2c_{\mu,\nu-12}^{(\lambda-13)} + 3\cdot 17^2c_{\mu,\nu-12}^{(\lambda-14)}$$
$$+9\cdot 17c_{\mu,\nu-12}^{(\lambda-15)} + 17c_{\mu,\nu-12}^{(\lambda-16)} + c_{\mu,\nu-12}^{(\lambda-17)}.$$

We do not actually need the full details of the second recurrence for the $c_{\mu,\nu}^{(\lambda)}$. Rather, we use only the following consequences, which can be verified from the above for $\nu \equiv 0 \pmod 4$.

(a) For each residue class of $\nu \pmod 4$, the recurrence gives $c_{\mu,\nu}^{(\lambda)}$ as a linear combination of the $c_{\mu,\nu-\sigma}^{(\lambda-\rho)}$, $1 \le \rho \le 17$ and $1 \le \sigma \le 13$, with 17-integral coefficients. For each residue class of $\nu \pmod 4$, we can also write $c_{\mu,\nu}^{(\lambda)}$ as a linear combination of the $c_{\mu,\nu+\sigma}^{(\lambda+\rho)}$, $1 \le \rho \le 17$ and $-1 \le \sigma \le 12$, with 17-integral coefficients.

(b) For any λ, μ and ν we have

$$\pi(c_{\mu,\nu}^{(\lambda)}) \ge \min_{\substack{1\le\rho\le17\\1\le\sigma\le13}} \left(\pi(c_{\mu,\nu-\sigma}^{(\lambda-\rho)}) + e_\sigma^{(\rho)}(\nu)\right) \tag{21}$$

and

$$\pi(c_{\mu,\nu}^{(\lambda)}) \ge \min_{\substack{1\le\rho\le17\\1\le\sigma\le12}} \left(\pi(c_{\mu,\nu-\sigma}^{(\lambda-\rho)}) + e_{12-\sigma}^{(17-\rho)}(\nu)\right) \tag{22}$$

where the $e_\sigma^{(\rho)}(\nu)$ depend on the residue class of $\nu \pmod 4$ and are given in Tables 6–9 of Appendix 1. In particular, we note that in all cases

$$c_{\mu,\nu}^{(\lambda)} \equiv c_{\mu,\nu-12}^{(\lambda-17)} \pmod{17}. \tag{23}$$

If the $c_{\mu,\nu}^{(\lambda)}$ are 17-integers for a fixed μ and any 17 consecutive integers λ, then they are 17-integers for this fixed μ and all λ. This follows from a two-way induction on λ using (a). We conclude from this that all the $c_{\mu,\nu}^{(\lambda)}$ are 17-integers if there are 4 consecutive integers μ for each of which there are 17 consecutive integers λ so that the $c_{\mu,\nu}^{(\lambda)}$ are 17-integers for these μ and λ. This follows from a two-way induction on μ using (17). We assert that $\mu = -3, -2, -1$ and 0 have this property with $-13 \le \lambda \le 3$, $-14 \le \lambda \le 2$, $-17 \le \lambda \le -1$ and $-16 \le \lambda \le 0$, respectively. The corresponding initial values $c_{\mu,\nu}^{(\lambda)}$ are quite unwieldy to state and derive. We do not reproduce all of them here; a brief account of their derivation is given in Appendix 2. For example,

$$T_\lambda J_0 = U(\phi^\lambda) = \sum_\nu c_{0,\nu}^{(\lambda)} J_\nu, \tag{24}$$

where the $c_{0,\nu}^{(\lambda)}$ are as follows:

$$\begin{aligned}
U(1) &= 1J_0,\\
U(\phi^{-1}) &= -1J_0,\\
U(\phi^{-2}) &= -1J_0,\\
U(\phi^{-3}) &= 17J_0,\\
U(\phi^{-4}) &= -17J_0,\\
U(\phi^{-5}) &= -20J_{-3} + 17^2J_0,\\
U(\phi^{-6}) &= -30J_{-4} - 17^2J_0,\\
U(\phi^{-7}) &= -126J_{-4} - 56\cdot 17J_{-3} + 17^3J_0,\\
U(\phi^{-8}) &= -17^3J_0,\\
U(\phi^{-9}) &= 54J_{-6} + 108\cdot 17J_{-5} + 144\cdot 17J_{-4} + 108\cdot 17^2J_{-3} - 17^4J_0,\\
U(\phi^{-10}) &= -10J_{-7} - 145\cdot 17J_{-6}\\
&\quad - 100\cdot 17^2J_{-5} - 250\cdot 17^2J_{-4} - 130\cdot 17^3J_{-3} - 5\cdot 17^4J_{-2} - 17^4J_0,\\
U(\phi^{-11}) &= 1210J_{-7} - 11\cdot 17J_{-6} + 88\cdot 17^2J_{-5}\\
&\quad + 88\cdot 17^2J_{-4} + 154\cdot 17^3J_{-3} + 11\cdot 17^4J_{-2} + 17^5J_0,\\
U(\phi^{-12}) &= 594J_{-8} - 17^5J_0,\\
U(\phi^{-13}) &= -130J_{-9} + 18759J_{-8} + 9594\cdot 17J_{-7} + 1053\cdot 17^2J_{-6}\\
&\quad - 78\cdot 17^3J_{-5} - 247\cdot 17^3J_{-4} - 234\cdot 17^4J_{-3} - 13\cdot 17^5J_{-2} - 17^6J_0,\\
U(\phi^{-14}) &= -17^6J_0,\\
U(\phi^{-15}) &= -3990J_{-10} - 4690\cdot 17J_{-9} + 49605\cdot 17J_{-8}\\
&\quad + 2050\cdot 17^3J_{-7} + 7185\cdot 17^3J_{-6} + 90\cdot 17^4J_{-5}\\
&\quad + 945\cdot 17^4J_{-4} - 330\cdot 17^5J_{-3} - 15\cdot 17^6J_{-2} - 17^7J_0,\\
U(\phi^{-16}) &= 1248J_{-11} + 18096\cdot 17J_{-10} + 12480\cdot 17^2J_{-9}\\
&\quad + 31200\cdot 17^2J_{-8} + 16224\cdot 17^3J_{-7} + 624\cdot 17^4J_{-6} - 17^7J_0.
\end{aligned}$$

Equations (24) not only illustrate that the initial values $c_{\mu,\nu}^{(\lambda)}$ are 17-integers (in fact, ordinary integers for $\mu = 0$), but they also provide the necessary data for computing the σ_k in (18). In proving Lemma 4, we do not actually need the full details of the initial values for the $c_{\mu,\nu}^{(\lambda)}$. Rather, we only use the following consequences which can be verified from (24) for $\mu = 0$.

(c) For $\mu = -3, -2, -1$ and 0 and λ in the ranges given above, each $T_\lambda J_\mu$ is a 17-integral linear combination of the J_ν.

(d) For $\mu = -3, -2, -1$ and 0 and λ in the ranges given above, $\pi(c_{\mu,\nu}^{(\lambda)})$ has the values given in Tables 10–13 in Appendix 1.

From (c), we conclude now that all $c_{\mu,\nu}^{(\lambda)}$ are 17-integers.

It remains to discuss the proof of (16). One easily verifies from Tables 10–13 in Appendix 1 that (16) holds for the λ and μ shown there. A lengthy but straightforward two-way induction on λ using (b) shows that if (16) holds for a fixed μ and a range of 17 consecutive λ, then it holds for that fixed μ and all λ. That (16) holds for all μ and all λ follows from a two-way induction on μ using (17) and the fact that the right side of (16) is invariant under the index shift $(\lambda, \mu, \nu) \longrightarrow (\lambda + 6, \mu + 4, \nu + 4)$. Including the derivation of the initial conditions, this completes the proof of Lemma 4.

4. **Proof of Theorem 3.** We introduce some notation in terms of which we give a bound on the growth of $\pi(a_{r,\nu})$ for increasing r and thus prove Theorem 3. From (14) we have

$$\pi(a_{r,\nu}) \geq \min_{\mu \geq \mu_{r-1}} \left(\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})})\right). \tag{25}$$

Set

$$\theta(\lambda, \mu) = \begin{cases} 1 & \text{if } \pi(c_{\mu+i,\nu}^{(\lambda)}) > 0 \text{ for } 0 \leq i \leq 3 \text{ and all } \nu; \\ 0 & \text{otherwise.} \end{cases}$$

Using (17) and (23), we see that $\theta(\lambda + 6, \mu - 4) = \theta(\lambda, \mu)$ and $\theta(\lambda + 17, \mu) = \theta(\lambda, \mu)$. Thus θ is completely determined by its values for $0 \leq \lambda \leq 16$ and $0 \leq \mu \leq 3$, and these can be determined from the initial conditions in Appendix 1. In this range $\theta(\lambda, \mu) = 1$ if and only if $(\lambda, \mu) = (3, 0), (3, 1), (3, 2), (3, 3)$ or $(14, 3)$. Put $A_0 = 0$ and $A_r = A_{r-1} + \theta(\lambda_{r-1}, \mu_{r-1})$ for $r > 0$. We will prove by induction that

$$\pi(a_{r,\nu}) \geq A_r + \max\left(0, \left\lfloor \frac{17(\nu - \mu_r) - \delta_\nu}{24} \right\rfloor\right) \text{ for } r \geq 0, \tag{26}$$

where $\delta_\nu = 29, 23, 17$ or 16 according as $\nu \equiv 0, 1, 2$, or $3 \pmod 4$. A closer examination of the A_r will then yield Theorem 3.

To prove (26), it suffices by (11) to assume $\nu \geq \mu_r$. Clearly (26) holds for $r = 0$. Let $r > 0$ and suppose (26) holds for $r - 1$. We see from (25) that to complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq A_r + \max\left(0, \left\lfloor \frac{17(\nu - \mu_r) - \delta_\nu}{24} \right\rfloor\right) \tag{27}$$

for $\mu \geq \mu_{r-1}$ and $\nu \geq \mu_r$. In proving (27), we will first suppose $\nu = \mu_r$ or $\mu_r + 1$, and then $\nu \geq \mu_r + 2$.

CASE 1. $\nu = \mu_r$ or $\mu_r + 1$. Proving (27) here reduces to showing that its left side is at least A_r. This holds for $\mu_{r-1} \leq \mu \leq \mu_{r-1} + 3$, since for these μ, $\pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1}, \mu_{r-1})$ by the definition of θ and $\pi(a_{r-1,\mu}) \geq A_{r-1}$ by the induction hypothesis. It also holds for $\mu \geq \mu_{r-1} + 4$, since by induction

$$\pi(a_{r-1,\mu}) \geq A_{r-1} + \left\lfloor \frac{17(\mu - \mu_{r-1}) - \delta_\mu}{24} \right\rfloor \geq A_{r-1} + 1 \geq A_r.$$

Thus (27) holds for $\nu = \mu_r$ or $\mu_r + 1$.

CASE 2. $\nu \geq \mu_r + 2$. By (16) and induction, the left side of (27) is at least

$$A_{r-1} + \max\left(0, \left\lfloor \frac{17(\mu - \mu_{r-1}) - \delta_\mu}{24} \right\rfloor\right) + \left\lfloor \frac{17\nu - \mu - 12\lambda_{r-1} + \delta(\mu, \nu)}{24} \right\rfloor. \tag{28}$$

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Now (28) cannot decrease if μ is increased by 4, so its minimum occurs for some $\mu = \mu_{r-1}+i$, $0 \le i \le 3$; and since from (5) we have $\mu_r \ge (\mu_{r-1}+12\lambda_{r-1})/17$, the expression in (28) is at least

$$A_{r-1}+1+\left\lfloor \frac{17(\nu-\mu_r)+\delta(\mu_{r-1}+i,\nu)-27}{24}\right\rfloor.$$

From Lemma 4 we have $\delta(\mu_{r-1}+i,\nu)-27 \ge -\delta_\nu$, where δ_ν is as in (26). Hence

$$A_{r-1}+1+\left\lfloor \frac{17(\nu-\mu_r)+\delta(\mu_{r-1}+i,\nu)-27}{24}\right\rfloor \ge A_r+\left\lfloor \frac{17(\nu-\mu_r)-\delta_\nu}{24}\right\rfloor,$$

so (27) holds for $\nu \ge \mu_r+2$.

Having established (26), we know that $\pi(a_{r,\nu}) \ge A_r$ for all $r \ge 0$. We now identify α and ε so that A_r has the form $\alpha r+\varepsilon$ with α and ε as described in Theorem 3. For $r > 1+\log_{17}|k|$ we have

$$\begin{aligned} A_r &= \sum_{i=1}^{r-1}\theta(\lambda_i,\mu_i) \\ &= \sum_{0\le i\le \log_{17}|k|}\theta(\lambda_i,\mu_i)+N_1\theta\big(0,\lceil 17k/24\rceil+\omega(k)\big)+N_2\theta\big(k,\lceil k/24\rceil+\omega(k)\big), \end{aligned}$$

where N_1 and N_2 are respectively the numbers of odd and even integers i in the interval $\log_{17}|k| < i < r$. Further, for such r we always have

$$(29)\qquad |N_1-(1/2)(r-1-\log_{17}|k|)|+|N_2-(1/2)(r-1-\log_{17}|k|)| \le 1,$$

and also

(30)
$$\begin{aligned} A_r = &\sum_{0\le i\le \log_{17}|k|}\theta(\lambda_i,\mu_i) \\ &+(1/2)(r-1-\log_{17}|k|)\big[\theta(0,\lceil 17k/24\rceil+\omega(k))+\theta(k,\lceil k/24\rceil+\omega(k))\big] \\ &+\big[N_1-(1/2)(r-1-\log_{17}|k|)\big]\theta\big(0,\lceil 17k/24\rceil+\omega(k)\big) \\ &+\big[N_2-(1/2)(r-1-\log_{17}|k|)\big]\theta\big(k,\lceil k/24\rceil+\omega(k)\big) \end{aligned}$$

We now set

$$(31)\qquad \alpha=\alpha(k)=\theta\big(0,\lceil 17k/24\rceil+\omega(k)\big)+\theta\big(k,\lceil k/24\rceil+\omega(k)\big),$$

and (for all r)

$$(32)\qquad \varepsilon=\varepsilon(k,r)=A_r-\alpha r/2,$$

so that $A_r = \alpha r/2+\varepsilon$. By (29)–(32) and the definition of A_r, we have $|\varepsilon| \le 3+2\log_{17}|k|$, so $\varepsilon = O(\log|k|)$. A straightforward argument using the definition of α in (31) and the properties of θ shows that $\alpha(k+96) = \alpha(k)$ for $k > 0$ and $\alpha(k-96) = \alpha(k)$ for $k < 0$. The values of α can be computed from (31) and are as given in Table 3. This completes the proof of Theorem 3.

5. **Conclusion.** The author has checked by machine computation that the congruences asserted in Theorem 3 actually hold for $|k| \leq 96$ and $n \leq 12,000$. These computations show also that Theorem 3 can at least be sharpened for some k and small r. Atkin and Gordon have shown that Theorems 1 and 2 are best possible in the sense that every residue class (mod $24t$), where $t = 1$ and 5 in Theorems 1 and 2 respectively, contains both positive and negative values of k for which the constants $\alpha(k,q)$ cannot be increased without rendering the congruences false. Theorem 3 may not be best possible in this sense, although we assert nothing at present.

It would be desirable to determine the extent to which the congruences in Theorems 1, 2 and 3 can be generalized. In addition to the question of finding further values of q for which we have results of the form

$$p_{-k}\big(q^r m + n(q,r,k)\big) \equiv 0 \pmod{q^{\alpha r+\varepsilon}} \tag{33}$$

where α and ε behave as in these theorems, there are questions concerning the existence of more general congruences such as those discussed by Atkin and O'Brien in [4]. The results presented here provide first steps in establishing congruences of the form

$$p_{-k}\big(17^{r+2}m + n(r+2,k)\big) \equiv K(r,k)p_{-k}(17^r m + n(r,k)\big) \pmod{17^{r\alpha(k)+\varepsilon(k)}}. \tag{34}$$

(In the present paper $K(r,k) = 0$.) As described in [1] and [2], working modulo various primes up to 67, Atkin has obtained results and developed conjectures involving the coefficients of Klein's function $j(\tau)$, some of these results and conjectures having a yet more general nature than (34). Analogous congruences may hold for the $p_{-k}(n)$.

APPENDIX 1

Here we give tables of $e_\sigma^{(\rho)}(\nu)$ and initial conditions for $\pi\left(c_{\mu,\nu}^{(\lambda)}\right)$. Blank entries are to be taken as ∞.

$\sigma\backslash\rho$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0																	
1		1																
2		2	1															
3		2	2	2	1													
4		3	2	2	1	1												
5		3	2	3	2	2	1	1										
6		4	3	3	2	3	2	2	1									
7		5	4	4	3	3	2	2	2									
8		6	5	5	4	4	3	3	3	2	1	1						
9		6	5	5	4	4	3	3	2	3	2	2	1					
10		7	6	6	5	5	4	4	3									
11		8	7	7	6	6	5	5	4	4	3	3	2					
12		8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
13		9	8	8	7	7	6	6										

TABLE 6: $e_\sigma^{(\rho)}(0)$

$\sigma\backslash\rho$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0																	
1		1																
2		1	1															
3		2	2	2	1													
4		3	2	2	1	1												
5		4	3	3	2	2	1	1										
6		4	3	3	2	2	1	1	1									
7		5	4	4	3	3	2	2	1									
8		6	5	5	4	4	3	3	2	2	1	1						
9		7	6	6	5	5	4	4	3	3	2	2	1					
10		7	6	6	5	5	4	4	3	3	2	2	1					
11		8	7	7	6	6	5	5										
12		8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
13																		

TABLE 7: $e_{\sigma}^{(\rho)}(1)$

$\sigma\backslash\rho$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0																	
1		1																
2		2	1															
3		2	1	2	1													
4		3	2	2	1	1												
5		4	3	3	2	2	1	1										
6		5	4	4	3	3	2	2	1									
7		5	4	4	3	3	2	2	1									
8		6	5	5	4	4	3	3	2	2	1	1						
9		7	6	6	5	5	4	4	3	3	2	2	1					
10		8	7	7	6	6	5	5										
11		8	7	7	6	6	5	5	4	4	3	3	2					
12		8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
13																		

TABLE 8: $e_{\sigma}^{(\rho)}(2)$

$\sigma\backslash\rho$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0																	
1		1																
2		2	1															
3		3	2	2														
4		3	2	2	1	1												
5		4	3	3	2	2	1	1										
6		5	4	4	3	3	2	2	1									
7		6	5	5	4	4	3	3	2									
8		6	5	5	4	4	3	3	2	2	1	1						
9		7	6	6	5	5	4	4	3	3	2	2	1					
10		8	7	7	6	6	5	5										
11																		
12		8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
13																		

TABLE 9: $e_{\sigma}{}^{(\rho)}(3)$

$\lambda\backslash\nu$	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
0												0
−1												0
−2												0
−3												1
−4												1
−5									0			2
−6								0				2
−7								0	1			3
−8												3
−9						0	1	1	2			4
−10					0	1	2	2	3	4		4
−11					0	1	2	2	3	4		5
−12				0								5
−13			0	0	1	2	3	3	4	5		6
−14												6
−15		0	1	1	3	3	4	4	5	6		7
−16	0	1	2	2	3	4						7

TABLE 10: $\pi(c_{0,\nu}^{(\lambda)})$

$\lambda \backslash \nu$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
3												0
2												0
1											0	1
0										0		1
-1										0	1	2
-2												2
-3								0	1	1	2	3
-4							0	1	2	2	3	3
-5							0	1		2		4
-6						0						4
-7					0	0	1	2	3	3		5
-8												5
-9				0	1	1	2	3	4	4		6
-10			0	1	2	2	3	4				6
-11			0	1	3	2	3	4	5	5		7
-12		0										7
-13	0	0	1	2	3	3	4	5	6	7		8

TABLE 11 $\pi(c^{(\lambda)}_{-3,\nu})$

$\lambda \backslash \nu$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
2													0	1	1	2	3
1												0					
0											0	0					
-1												1					
-2												1					
-3									0	1	1	2					
-4												2					
-5								0	1		2	3					
-6							0					3					
-7						0	0	1	2	3	3	4					
-8												4					
-9					0	1	2	2	3	4	4	5					
-10				0	1	2	2	3	4			5					
-11				0	1	2	2	3	4	5	5	6					
-12			0									6					
-13		0	0	1	2	3	3	4	5	6	6	7					
-14	0	1	1	2	3	4	4	5	6	7	7	7					

TABLE 12: $\pi(c^{(\lambda)}_{-2,\nu})$

$\lambda \backslash \nu$	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
-1													0	1	2	3	3
-2												0					
-3											0	1	1				
-4												1					
-5										1	1	2	2				
-6									0			2					
-7								0	0	1	2		3				
-8											3	3	3				
-9							0	1	1	2		4					
-10						0	1	2	2	3	4	4					
-11						0	1	2	2	3	4	5	5				
-12					0							5					
-13				0	0	1	2	3	3	5	5	6	7				
-14												6					
-15			0	1	1	2	3	4	4	5	6	7					
-16		0	1	2	2	3	4					7					
-17	0	0	1	2	2	3	4	5	5	7	8	8	8				

TABLE 13: $\pi(c_{-1,\nu}^{(\lambda)})$

APPENDIX 2

Here we briefly discuss derivation of the initial conditions for the recurrences (17) and (20). Consider the functions J_μ on the Riemann surface $X_0(17)$. By Lemmas 1, 2 and 3 we have $T_\lambda J_\mu \in K_0(17)$ with $\text{ord}_0\, T_\lambda J_\mu \geq \min(-12\lambda + 17\psi(\mu), \mu)$, $\text{ord}_{i\infty}\, T_\lambda J_\mu \geq \lceil (12\lambda + \mu)/17 \rceil$, and $\text{ord}_p\, T_\lambda J_\mu \geq 0$ for other p on $X_0(17)$. Hence $T_\lambda J_\mu \in \mathcal{V}$, and using the orders at 0 and $i\infty$ we can determine for given λ and μ a maximal range of values ν for which $c_{\mu,\nu}^{(\lambda)}$ can be nonzero. We can minimize the computation by selecting λ and μ for which these ranges are as small as possible, and this can be accomplished by taking the valence of $T_\lambda J_\mu$ small. Also, the case $\mu = 0$ is of special importance since certain $T_\lambda J_0$ are necessary for computation of the modular equation. Therefore we include $\mu = 0$, first considering $T_\lambda J_0$ and then $T_\lambda J_\mu$ for other values of μ.

For $\mu = 0$, the λ for which the $T_\lambda J_0$ have the smallest valences turn out to be in the range $-16 \leq \lambda \leq 0$. For these λ we find $c_{0,\nu}^{(\lambda)}$ by equating the first $1 - \lceil 12\lambda/17 \rceil$ coefficients in the Fourier series of $T_\lambda J_0$ and the Fourier series of $c_{0,0}^{(\lambda)} J_0 + \cdots + c_{0,\lceil 12\lambda/17 \rceil}^{(\lambda)} J_{\lceil 12\lambda/17 \rceil}$. We get triangular systems of equations whose solutions yield (24). For $\mu = -3, -2$ and -1 the ranges for λ indicated in § 3 turn out to produce small valences for $T_\lambda J_\mu$. We again are led to triangular systems of equations.

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