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POINTWISE SEQUENTIALLY CLOSED IDEALS IN $C^*(X)$

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The purpose of this paper is to determine the conditions under which the maximal ideals of the ring $C^*(X)$—the bounded real-valued continuous functions on a completely regular Hausdorff space X—are closed under pointwise convergence of sequences. Whereas the maximal ideals of $C^*(X)$ are closed under pointwise convergence of nets if and only if X is compact, it is shown that a necessary and sufficient condition for their pointwise sequential closure is that X be pseudocompact (i.e. that all real-valued continuous functions of X be bounded). In the process it is shown that the pointwise convergence of sequences in $C(X)$ is equivalent to the pointwise convergence of their extensions to vX (the real compactification of X).

The topological space consisting of a set S with a topology t will be denoted by (S, t) (or simply by S if no confusion is possible). All topological spaces are assumed to be completely regular and Hausdorff. The notation is the same as that used in [2].

A subset $A \subset X$ is said to be *sequentially open* if every sequence converging to a point of A is eventually in A. Given a Hausdorff space (Y, s) it is possible to define a new topology t_s on Y by taking as a base for the topology t_s all the sequentially open sets of (Y, s). The topology t_s is a sequential topology (see [1]) and in addition is the weakest sequential topology stronger than s.

Considered as a subset of R^X, $C^*(X)$ is a topological space with the relative product topology p. The weakest sequential topology on $C^*(X)$ stronger than p will be denoted by t_p. Thus a sequence $\{f_n\}$ of functions in $C^*(X)$ converges to $f \in C^*(X)$ in the t_p-topology if and only if $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X$. The t_p-topology has been studied by Dudley [1] and Meyer [4]. It was shown in [1, Theorem 6.6] that in general $(C(X), t_p)$ is not a topological vector space, however it is an open problem to determine possible conditions under which $(C^*(X), t_p)$ and $(C(X), t_p)$ are topological vector spaces.

Since $C^*(X)$ is isomorphic to $C^*(\beta X)$ (where βX is the Stone-Cech compactification of X), the t_p-topology on $C^*(\beta X)$ induces a topology t_p' on $C^*(X)$. Thus a sequence $\{f_n\}$ is t_p'-convergent to $f \in C^*(X)$ if and only if $f_n^\beta(x)$ converges to $f^\beta(x)$ for all $x \in \beta X$. It is easy to see that t_p' is a sequential topology.

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LEMMA 1. *X is pseudocompact if and only if* $t_p = t'_p$.

Proof. Suppose that X is not pseudocompact, then for each $p \in \beta X - vX$, there exists a G_δ-set, U say, such that $p \in U$ and $U \cap X = \varnothing$ (see [2], 8.8). Without loss of generality it may be assumed that $U = \bigcap_n U_n$ where U_n is open for each $n \in N$ and $U_{n+1} \subset U_n$. Since βX is completely regular, there exist f_n^β in $C^*(\beta X)$ such that $f_n^\beta(p) = 0$ for all $n \in N$ and $f_n^\beta(y) = 1$ for all $y \in \beta X - U_n$. Denoting f_n^β / X by f_n it is clear that $\{f_n\}$ is t_p-convergent but not t'_p-convergent to 1.

Conversely suppose that X is pseudocompact, then X is G_δ-dense in βX. Since t_p and t'_p are both sequential topologies it suffices to prove that a sequence $\{f_n\}$ in $C^*(X)$ is t_p-convergent to f if and only if $\{f_n\}$ is t'_p-convergent to f (see [1]). Suppose that $\{f_n\}$ is t_p-convergent to f and let f_n^β (respectively f^β) denote the extension of f_n (respectively f) to βX. For each $y \in \beta X$ there exists a G_δ-set Z_n with the property that $y \in Z_n$ and f_n^β is constant on $Z_n (n \in N)$. (If $g(y) = r$, then $g^{-1}[r]$ is a zero-set—hence a G_δ—containing y on which g is constant.) Suppose f^β is constant on a G_δ-set Z with $y \in Z$. Then $(\bigcap_n Z_n) \cap Z = U$ is a G_δ in βX with the property that f^β and each f_n^β are simultaneously constant on it. Since U is a G_δ, $U \cap X \neq \varnothing$. Choose $x \in X \cap U$. Then

$$f_n^\beta(y) = f_n^\beta(x) = f_n(x) \to f(x) = f^\beta(x) = f^\beta(y).$$

In other words, the sequence $\{f_n\}$ is t'_p-convergent to f. The converse implication is trivial and the result follows.

Using a similar argument it is possible to prove:

THEOREM. *A sequence* $\{f_n\}$ *in* $C(X)$ *is pointwise convergent to* $f \in C(X)$ *if and only if the sequence* $\{f_n^v\}$ *is pointwise convergent to* $f^v \in C(vX)$. (*Here* f_n^v *and* f^v *denote the extensions of* f_n *and* f *to* vX.)

The corresponding statement for nets is clearly false, the first uncountable ordinal providing a counterexample.

LEMMA 2. *An ideal of* $C^*(X)$ *is* t'_p*-closed if and only if it is an intersection of maximal ideals of* $C^*(X)$.

Proof. It is clear that all the maximal ideals of $C^*(\beta X)$ are t_p-closed and hence so is any ideal which is an intersection of maximal ideals of $C^*(\beta X)$. Furthermore, the closure of an ideal of $C^*(\beta X)$ in the uniform norm topology is the intersection of all the maximal ideals containing it [6, Theorem 85]. Since the t_p-topology is a weaker topology than the uniform topology and the intersection of maximal ideals is always t_p-closed, it follows that the t_p-closure of an ideal is the intersection of all the maximal ideals containing it. Thus an ideal I of $C^*(\beta X)$ is t_p-closed if and only if it is an intersection of maximal ideals.

The map

$$\beta:(C^*(X), t_p') \to (C^*(\beta X), t_p)$$

defined by $\beta(f)=f^\beta$ is clearly a homeomorphism. It follows that an ideal of $C^*(X)$ is t_p'-closed if and only if it is an intersection of maximal ideals.

The following theorem characterizes the spaces for which the maximal ideals of $C^*(X)$ are t_p-closed.

THEOREM. *X is pseudocompact if and only if it has the property: An ideal of $C^*(X)$ is t_p-closed if and only if it is an intersection of maximal ideals.*

Proof. If X is pseudocompact, the result follows from Lemmas 1 and 2.

Conversely suppose that X is not pseudocompact. It suffices to exhibit a maximal ideal of $C^*(X)$ which is not t_p-closed. However, the functions $\{f_n\}$ constructed in the first part of Lemma 1 can be embedded in the free maximal ideal M^{*p} (see [2], 7.2). Thus M^{*p} is not t_p-closed.

It is interesting to note that, mutatis mutandis, the above theorems and lemmas remain valid if $C^*(X)$ is replaced by a uniformly closed, separating subalgebra B of $(R^X)^*$ which contains the constant functions. X is assumed to have the weak B topology. Details may be found in [3] or [5].

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