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Isomorphisms of some convolution algebras and their multiplier algebras

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Let G_1 and G_2 be two locally compact abelian groups and let $1 \leq p < \infty$. We prove that G_1 and G_2 are isomorphic as topological groups provided there exists a bipositive or isometric algebra isomorphism of $M(A_p(G_1))$ onto $M(A_p(G_2))$. As a consequence of this, we prove that G_1 and G_2 are isomorphic as topological groups provided there exists a bipositive or isometric algebra isomorphism of $A_p(G_1)$ onto $A_p(G_2)$. Similar results about the algebras $L_1 \cap L_p$ and $L_1 \cap C_0$ are also established.

1. Introduction

Let G be a locally compact abelian group and let $1 \leq p < \infty$. $(L_1 \cap L_p)(G)$ is the Banach algebra $L_1(G) \cap L_p(G)$ with the norm

$$\|f\|_{1,p} = \|f\|_1 + \|f\|_p \quad (f \in L_1(G) \cap L_p(G))$$

and the convolution as multiplication. Similarly, $(L_1 \cap C_0)(G)$ is the Banach algebra $L_1(G) \cap C_0(G)$ with the norm

$$\|f\|_{1,\infty} = \|f\|_1 + \|f\|_\infty \quad (f \in L_1(G) \cap C_0(G))$$

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and the convolution as multiplication. $A_p(G)$ is the Banach algebra consisting of all those functions $f \in L_1(G)$ such that $\hat{f} \in L_p(\hat{G})$ where $\hat{G}$ denotes the dual group of G. The multiplication in $A_p(G)$ is the convolution and the norm is given by

$$\|f\|^p = \|f\|_1 + \|\hat{f}\|_p \quad (f \in A_p(G)) .$$

A multiplier T on a commutative semi-simple Banach algebra A is a function on A to A such that $(Tx)y = T(xy) = x(Ty)$ for all $x, y \in A$. It is well known that a multiplier T of A is a continuous linear operator on A and the set $M(A)$ of all multipliers of A forms a commutative Banach algebra with multiplication as composition and the norm as operator norm. The properties of multipliers are discussed in Larsen [5] and for any definitions and results not mentioned in this paper we refer the reader to [5].

In this paper we are concerned with some subalgebras A of $L_1(G)$. For such an algebra A, a multiplier T of A is said to be *positive* if $Tf \geq 0$ almost everywhere whenever $f \geq 0$ almost everywhere and $f \in A$. Let G_1 and G_2 be two locally compact abelian groups and let A_1 and A_2 be linear subspaces of $L_1(G_1)$ and $L_1(G_2)$ respectively. A linear transformation $S : A_1 \to A_2$ is called *bipositive* whenever $Sf \geq 0$ almost everywhere if and only if $f \geq 0$ almost everywhere. The bipositive mappings between spaces of multipliers are defined analogously.

The main theorem of this paper is the following:

THEOREM 1. *Let G_1, G_2 be locally compact abelian groups and $1 \leq p < \infty$. G_1 and G_2 are isomorphic as topological groups provided there exists a bipositive or isometric algebra isomorphism of $M(A_p(G_1))$ onto $M(A_p(G_2))$.*

As a consequence of Theorem 1 we shall prove that G_1 and G_2 are isomorphic as topological groups provided there exists a bipositive or isometric algebra isomorphism of $A_p(G_1)$ onto $A_p(G_2)$.

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Similar results about the algebras $L_1 \cap L_p$ and $L_1 \cap C_0$ are also established.

The proof of our Theorem 1 heavily depends on the techniques of Gaudry in proving his Theorems 1 and 2 in [1].

2. Multipliers of $A_p(G)$

In this section we characterize norm preserving and positive multipliers of $A_p(G)$. We prove two other propositions which are used in proving Theorem 1.

PROPOSITION 1. *Let G be a compact abelian group and let T be a norm preserving multiplier of $A_p(G)$. Then there exists $a \in G$ and a complex number λ of absolute value 1 such that $T = \lambda\tau_a$ where τ_a denotes the operator of translation by amount a.*

Proof. Let $\gamma \in \hat{G}$. Then $\gamma * \gamma = \gamma$ and hence $T(\gamma) = T(\gamma*\gamma) = T\gamma * \gamma$. Therefore $T(\gamma) = \phi(\gamma)\gamma$ where $\phi(\gamma)$ is a complex number. Since T is norm preserving it follows that $|\phi(\gamma)| = 1$.

Now, for any trigonometric polynomial $\sum_1^n a_i\gamma_i$, we have

$$\left\|T\left(\sum_1^n a_i\gamma_i\right)\right\|^p = \left\|\sum_1^n a_iT(\gamma_i)\right\|_1 + \left(\sum_1^n |a_i\phi(\gamma_i)|^p\right)^{1/p}$$
$$= \left\|T\left(\sum_1^n a_i\gamma_i\right)\right\|_1 + \left(\sum_1^n |a_i|^p\right)^{1/p} .$$

On the other hand, since T is norm preserving

$$\left\|T\left(\sum_1^n a_i\gamma_i\right)\right\|^p = \left\|\sum_1^n a_i\gamma_i\right\|^p$$
$$= \left\|\sum_1^n a_i\gamma_i\right\|_1 + \left(\sum_1^n |a_i|^p\right)^{1/p} .$$

Therefore

$$\left\|T\left(\sum_1^n a_i\gamma_i\right)\right\|_1 = \left\|\sum_1^n a_i\gamma_i\right\|_1 .$$

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Since trigonometric polynomials are norm dense in $L_1(G)$ we conclude that there exists a unique norm preserving multiplier T' of $L_1(G)$ such that $T'f = Tf$ for each $f \in A_p(G)$. Hence by Theorem 3 of Wendel [7] it follows that there exist λ and a as desired such that

$$Tf = \lambda\tau_a f \quad \left(f \in A_p(G)\right) .$$

REMARK. It can be easily seen that Proposition 1 is true for arbitrary locally compact abelian groups G. The analogues of the proposition for the algebras $L_1 \cap L_p$ and $L_1 \cap C_0$ are also valid.

The case of $A_p(G)$ for noncompact G is exactly similar to that of $L_1 \cap L_p$ and $L_1 \cap C_0$. We give a proof for the case $L_1 \cap L_p$. The crux of the matter is that in each of these cases the multiplier algebra is isomorphic to the measure algebra $M(G)$ (see Larsen [5]).

Let T be a norm preserving multiplier of $\left(L_1 \cap L_p\right)(G)$. Then there exists a measure $\mu \in M(G)$ such that

$$Tf = \mu * f \quad \left(f \in \left(L_1 \cap L_p\right)(G)\right)$$

and $\|\mu\| = 1$. Now

$$\|\mu*f\|_{1,p} = \|\mu*f\|_1 + \|\mu*f\|_p .$$

Also $\|\mu*f\|_{1,p} = \|f\|_{1,p}$. Therefore

$$(1) \qquad \|\mu*f\|_1 + \|\mu*f\|_p = \|f\|_1 + \|f\|_p .$$

Since $\|\mu\| \le 1$, (1) implies

$$\|\mu*f\|_1 = \|f\|_1 \quad \left(f \in \left(L_1 \cap L_p\right)(G)\right) .$$

Since $\left(L_1 \cap L_p\right)(G)$ is norm dense in $L_1(G)$ we get the desired result, once again by Wendel's Theorem 3 in [7].

PROPOSITION 2. *Let G be a compact abelian group and let T be a positive multiplier of $A_p(G)$. Then there exists a unique positive measure $\mu \in M(G)$ such that $Tf = \mu * f$ for every $f \in A_p(G)$.*

Proof. From Theorem 6.2.2 of [5] it follows that there exists a unique pseudomeasure σ such that $Tf = \sigma * f$ for each f in $A_1(G)$. To show that σ is actually a positive measure we shall show that σ defines a positive multiplier of $L_2(G)$, that is $\sigma * f \geq 0$ almost everywhere for every $f \in L_2(G)$ such that $f \geq 0$ almost everywhere, and then an application of Theorem 3.6.1 of [5] will imply that σ is a positive measure.

Let now $f \in L_2(G)$ such that $f \geq 0$ almost everywhere. From Theorem 33.12 of [4] it follows that $L_1(G)$ admits an approximate unit $\{u_\alpha\}_{\alpha \in I}$ such that $u_\alpha \geq 0$ almost everywhere and $\hat{u}_\alpha$ has compact support. Clearly $u_\alpha \in A_1(G)$ for each $\alpha \in I$. Since $L_2(G) \subset L_1(G)$ therefore $u_\alpha * f \in A_1(G)$ and $u_\alpha * f \geq 0$ almost everywhere. Therefore $\sigma * u_\alpha * f \geq 0$ almost everywhere for each $\alpha \in I$. Since $\sigma * f \in L_2(G) \subset L_1(G)$ it follows that $\sigma * u_\alpha * f$ converges to $\sigma * f$ in the norm of L_1. Therefore there exists a sequence $\left\{u_{\alpha_n}\right\}_{\alpha_n \in I}$ such that $\sigma * u_{\alpha_n} * f$ converges to $\sigma * f$ almost everywhere. Hence $\sigma * f \geq 0$ almost everywhere and σ is a positive measure. Since $\sigma * f = Tf$ for each f in $A_1(G)$ and $A_1(G)$ is norm dense in $A_p(G)$ it follows that $\sigma * f = Tf$ for each $f \in A_p(G)$. This completes the proof of the proposition.

REMARK. It can be easily seen that Proposition 2 is true for arbitrary locally compact abelian groups G. The analogues of the proposition for the algebras $L_1 \cap L_p$ and $L_1 \cap C_0$ are also valid. Once again the case of $A_p(G)$ for noncompact G is exactly similar to that of $L_1 \cap L_p$ and $L_1 \cap C_0$ and the crux of the matter lies in the fact that in each of these cases the multiplier algebra is isomorphic to the measure algebra $M(G)$. The proof is easy and hence omitted.

PROPOSITION 3. *Let* G *be an infinite compact abelian group and*

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$1 \le p < \infty$. *Let* $\{m_i\}_{i \in I}$ *be a net in* $M(A_p(G))$ *such that* $\{m_i\}_{i \in I}$ *is bounded in the norm of* $M(A_p(G))$. *Then there exists a subnet* $\{m_{k_i}\}$ *of* $\{m_i\}$ *such that*

$$\lim_i \int \left(m_{k_i}(f)\right)^\wedge(\gamma)d\gamma = \int (m(f))^\wedge(\gamma)d\gamma$$

for each $f \in A_1(G)$. $d\gamma$ *denotes the Haar measure on* $\hat{G}$.

Proof. Case 1. $1 \le p \le 2$. In this case there exists a continuous algebra isomorphism of $M(A_p(G))$ onto $P(G)$, where $P(G)$ denotes the set of all pseudomeasures on G (see Corollary 6.4.1 of [5]). If $m \in A_p(G)$, then σ is the unique pseudomeasure such that

$$\int (m(f))^\wedge(\gamma)d\gamma = \langle f, \sigma \rangle \text{ for every } f \in A_1(G)$$

and

$$\|\sigma\|_{P(G)} \le \|m\|^p ,$$

where $\| \ \|^p$ denotes the multiplier norm of m. Let σ_i correspond to m_i. Then $\{\sigma_i\}$ is a net of pseudomeasures bounded in the norm of pseudomeasures. Therefore, since $A_1(G)^* = P(G)$, there exists a subnet $\{\sigma_{k_i}\}$ of $\{\sigma_i\}$ and $\sigma \in P(G)$ such that

$$(2) \qquad \lim_i \left\langle f, \sigma_{k_i} \right\rangle = \langle f, \sigma \rangle \quad (f \in A_1(G)) .$$

Let σ be the pseudomeasure corresponding to the multiplier m of $A_p(G)$. Then (2) implies

$$\lim_i \int \left(m_{k_i}(f)\right)^\wedge(\gamma)d\gamma = \int (m(f))^\wedge(\gamma)d\gamma$$

for every $f \in A_1(G)$.

Case 2. $2 < p < \infty$. In this case, by Theorem 6.4.2 of [5] there exists a continuous linear isomorphism β of $M(A_p(G))$ onto $B_p(G)^*$, defined by

$$\beta(T)(f) = \int_G (Tf)^{\wedge}(\gamma)d\gamma \quad (f \in B_p(G)) .$$

$B_p(G)$ is the normed linear space whose elements are those of $A_1(G)$ and the norm is defined by

$$\|f\|_B = \sup_T \left\{|\beta(T)(f)| : T \in M(A_p(G)), \|T\|^p \le 1\right\} .$$

Thus $\{m_i\}$ can be considered as a net in $B_p(G)^*$ and the boundedness of $\{m_i\}$ in the norm of $M(A_p(G))$ implies that it is also bounded in the norm of $B_p(G)^*$. Therefore there exists a subnet $\left\{m_{k_i}\right\}$ of $\{m_i\}$ and $m \in M(A_p(G))$ such that

$$\lim_i \int \left[m_{k_i}(f)\right]^{\wedge}(\gamma)d\gamma = \int (m(f))^{\wedge}(\gamma)d\gamma$$

for every $f \in A_1(G)$.

PROPOSITION 4. *Let G_1, G_2 be locally compact abelian groups and let $1 \le p < \infty$. If there exists an algebra isomorphism Ψ of $M(A_p(G_1))$ onto $M(A_p(G_2))$ then either both of the groups G_1 and G_2 are compact or both of them are noncompact.*

Proof. To prove the proposition we shall show that if one of the groups, say G_1, is compact then G_2 is also compact. Suppose G_2 is noncompact. Then $M(A_p(G_2))$ is isomorphic to $M(G_2)$. Thus Ψ can be considered as an algebra isomorphism of $M(A_p(G_1))$ into $M(G_2)$. Identifying the pseudomeasures on G_1 which define multipliers of $A_p(G_1)$ with the corresponding multipliers of $A_p(G_1)$ we see that the restriction of Ψ to $L_1(G_1)$ is an algebra isomorphism of $L_1(G_1)$ onto $M(G_2)$. By Theorem 4.1.3 of [6] it follows that there exists a subset Y of $\hat{G}_2$ and

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a piecewise affine map α of Y into $\hat{G}_1$ such that for every $f \in L_1(G_1)$,

$$(\Psi f)^\wedge(\gamma) = \begin{cases} \hat{f}(\alpha(\gamma)) & \text{if } \gamma \in Y , \\ 0 & \text{if } \gamma \notin Y . \end{cases}$$

Let $\sigma \in M(A_p(G_1))$ and $f \in A_1(G_1)$. Then $\sigma * f \in A_1(G_1)$ and we have

$$\begin{aligned}(\Psi(\sigma * f))^\wedge(\gamma) &= (\sigma * f)^\wedge(\alpha(\gamma)) \\ &= \hat{\sigma}(\alpha(\gamma))\hat{f}(\alpha(\gamma)) ,\end{aligned}$$

for every $\gamma \in Y$. On the other hand $\Psi(\sigma * f) = \psi(\sigma) * \psi(f)$. Therefore

$$\begin{aligned}(\psi(\sigma * f))^\wedge(\gamma) &= (\psi(\sigma))^\wedge(\gamma)(\Psi(f))^\wedge(\gamma) \\ &= (\psi(\sigma))^\wedge(\gamma)\hat{f}(\alpha(\gamma))\end{aligned}$$

for every $\gamma \in Y$. Hence

(3) $$\hat{\sigma}(\alpha(\gamma))\hat{f}(\alpha(\gamma)) = (\psi(\sigma))^\wedge(\gamma)\hat{f}(\alpha(\gamma)) \quad \text{for } \gamma \in Y .$$

Since (3) holds for every $f \in A_1(G_1)$ we obtain

(4) $$(\psi(\sigma))^\wedge(\gamma) = \hat{\sigma}(\alpha(\gamma)) \quad \text{for } \gamma \in Y .$$

Now we prove that α is one to one on Y. Let $\gamma_1, \gamma_2 \in Y$ such that $\gamma_1 \neq \gamma_2$. Choose $\mu \in M(G_2)$ such that $\hat{\mu}(\gamma_1) \neq \hat{\mu}(\gamma_2)$. Next, choose $\sigma \in M(A_p(G_1))$ such that $\Psi(\sigma) = \mu$. Then $\hat{\mu}(\gamma_1) = \hat{\sigma}(\alpha(\gamma_1))$ and $\hat{\mu}(\gamma_2) = \hat{\sigma}(\alpha(\gamma_2))$. Hence $\hat{\sigma}(\alpha(\gamma_1)) \neq \hat{\sigma}(\alpha(\gamma_2))$ and therefore $\alpha(\gamma_1) \neq \alpha(\gamma_2)$.

Next we show that $\alpha(Y) = \hat{G}_1$. Since $\hat{G}_1$ is discrete, $\alpha(Y)$ is closed in $\hat{G}_1$. If $\alpha(Y) \neq \hat{G}_1$, there exists $f \in A_1(G_1)$ such that $\hat{f} = 0$ on $\alpha(Y)$, but $\hat{f}$ is not identically zero. Since $\hat{f} \circ \alpha = 0$ we have $\Psi(f) = 0$; but this contradicts that Ψ is one to one.

Finally we prove that $Y = \hat{G}_2$. If $Y \neq \hat{G}_2$, since Y is a closed subset of $\hat{G}_2$ there exists $\mu \in M(G_2)$, $\mu \neq 0$, such that $\hat{\mu} = 0$ on Y. Choose $\sigma \in M(A_p(G_1))$ such that $\Psi(\sigma) = \mu$. By (4), $\hat{\sigma} = 0$ on $\hat{G}_1$ and therefore $\sigma = 0$; but $\Psi(\sigma) = \mu \neq 0$, a contradiction.

Thus we have shown that α is a piecewise affine homeomorphism of $\hat{G}_2$ onto $\hat{G}_1$. Since $\hat{G}_1$ is discrete it follows that $\hat{G}_2$ is discrete and hence G_2 is compact. This completes the proof of the proposition.

REMARK. The proof of Proposition 4 is based on the idea of the proof of Theorem 4.6.4 of [6].

3. Proof of Theorem 1

If G_1 and G_2 are noncompact then $M(A_p(G_i))$ is isometrically isomorphic to $M(G_i)$ (see Theorem 6.3.1 [5]); therefore the result follows immediately from Theorem 1 and 2 of [1].

We shall now assume that G_1 and G_2 are compact.

Case 1. *Suppose there exists an isometric algebra isomorphism* T *of* $M(A_p(G_1))$ *onto* $M(A_p(G_2))$.

For each $a \in G_1$ consider the translation operator τ_a . This multiplier is norm preserving and has norm preserving inverse τ_{-a} . Thus $T\tau_a$ has norm one and so does its inverse. It follows that $T\tau_a$ is, in fact, norm preserving. It follows from Proposition 1 that $T\tau_a$ is of the form

$$T\tau_a = \lambda(a)\tau_{a'} ,$$

where $|\lambda(a)| = 1$ and $a' \in G_2$. It is easy to see that a' is uniquely determined by a . From the fact that T is an isomorphism it is easily seen that the mapping $\phi : a \to a'$ is an algebraic isomorphism of G_1 onto G_2 . Since G_1 and G_2 are compact and Hausdorff, to prove that ϕ is a homeomorphism we need only show that ϕ is continuous. Let e and e' denote the identities of the groups G_1 and G_2 respectively. To prove the continuity of ϕ it is enough to show that if $\{a_i\}$ is a net in G_1 such that $a_i \to e$, then $\phi(a_i) \to e'$ in G_2 .

Suppose $\{\phi(a_i)\}$ does not converge to e'. Then there exists a neighbourhood U of e' and a subnet of $\{\phi(a_i)\}$ whose elements remain outside U for large i. Without loss of generality we shall assume that $\phi(a_i) \in CU$ (the complement of U) for all i. Consider then the net $\left\{T\tau_{a_i}\right\}$. This net is bounded in the norm of $M(A_p(G_2))$. By Proposition 3 it follows that there exists a subnet of $\left\{T\tau_{a_i}\right\}$ which, for the simplicity of notations, we again denote by $\left\{T\tau_{a_i}\right\}$, and a multiplier $m \in M(A_p(G_2))$ such that

(5) $$\lim_i \int \left(T\tau_{a_i}\right)(f)^\wedge(\gamma)d\gamma = \int (m(f))^\wedge(\gamma)d\gamma$$

for every $f \in A_1(G_2)$; $d\gamma$ denotes the Haar measure on $\hat{G}_2$.

For $h \in A_1(G_1)$ let m_h be the multiplier defined by the convolution by h. Then $\tau_{a_i}h \to h$ in $M(G_1)$, and since the topology of $M(G_1)$ is stronger than that induced by $M(A_p(G_1))$, we have $m_{\tau_{a_i}h} \to m_h$ in $M(A_p(G_1))$. Since T is continuous, we get

$$T\left(m_{\tau_{a_i}h}\right) \to T(m_h) .$$

But $m_{\tau_{a_i}h} = \tau_{a_i}\cdot m_h$. Therefore

$$T\left(\tau_{a_i}\right)\cdot T(m_h) \to T(m_h) \quad \text{in} \quad M(A_p(G_2)) .$$

Hence for each $f \in A_1(G_2)$ we have

(6) $$\int \left[T\left(\tau_{a_i}\right)\cdot T(m_h)(f)\right]^\wedge(\gamma)d\gamma \to \int (T(m_h)(f))^\wedge(\gamma)d\gamma .$$

From (5) the left hand side of (6) tends to

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$$\int \left(m \cdot T\left(m_h\right)(f)\right)^{\wedge}(\gamma)d\gamma \ .$$

Therefore

(7) $$\int \left(T\left(m_h\right)(f)\right)^{\wedge}(\gamma)d\gamma = \int \left(m \cdot T\left(m_h(f)\right)^{\wedge}\right)(\gamma)d\gamma$$

for every $f \in A_1\left(G_2\right)$.

From (7) it follows that $T\left(m_h\right)$ and $m \cdot T\left(m_h\right)$, considered as elements of $B_p\left(G_2\right)^*$, are identical. Therefore

(8) $$T\left(m_h\right) = m \cdot T\left(m_h\right) \ .$$

Applying T^{-1} to (8), we get

(9) $$m_h = T^{-1}(m) \cdot m_h \ .$$

From (9) it follows that $T^{-1}(m)(h*g) = h * g$ for all $h, g \in A_1\left(G_1\right)$. Since $A_1\left(G_1\right) * A_1\left(G_1\right)$ is dense in $A_1\left(G_1\right)$ it follows that

$$T^{-1}(m)(g) = g \quad \text{for all} \quad g \in A_1\left(G_1\right) \ .$$

Therefore $T^{-1}(m) = \text{identity}$. Hence $m = \text{identity}$. Thus we have shown that $\left\{T\tau_{a_i}\right\}$ considered as a net in $B_p\left(G_2\right)^*$ converges to $\tau_{e'}$ in the weak-star topology.

Also $\{\lambda\left(a_i\right)\}$ has a subnet converging to a complex number λ since $\left|\lambda\left(a_i\right)\right| = 1$ for all i . Further, since G_2 is compact, $\{\phi\left(a_i\right)\}$ has a subnet converging to some element a' of G_2 . Obviously $a' \neq e'$. To save renaming, suppose, without loss of generality, that $\lambda\left(a_i\right) \to \lambda$ and $\phi\left(a_i\right) \to a'$. Then $\lambda\left(a_i\right)\tau_{\phi\left(a_i\right)} \to \lambda\tau_{a'}$ in the weak-star topology of $B_p\left(G_2\right)^*$. Since $\lambda\left(a_i\right)\tau_{\phi\left(a_i\right)} = T\tau_{a_i}$ it follows that $\lambda\tau_{a'} = \tau_{e'}$. This is possible only if $a' = e'$ and $\lambda = 1$. Since $a' \neq e'$, we have reached a contradiction.

Case 2. *Suppose there exists a bipositive isomorphism of* $M(A_p(G_1))$ *onto* $M(A_p(G_2))$.

We begin with $a \in G_1$ and the multipliers τ_a and τ_{-a} . Then $T\tau_a$ and $T\tau_{-a}$ are both positive multipliers. From Proposition 2 it follows that there exist positive measures μ and ν such that

$$(T\tau_a)(f) = \mu * f$$

and

$$(T\tau_{-a})(f) = \nu * f$$

for $f \in A_1(G_2)$. Now it can be established, along the lines of the argument used in the proof of Case 1 of Theorem 2 of [1], that μ and ν are both Dirac measures and that $T\tau_a = \tau_{a'}$, $T\tau_{-a} = \tau_{b'}$, say, where $b' = -a'$. Further it can be easily seen that the mapping $\phi : a \to a'$ is an algebraic isomorphism of G_1 onto G_2 .

To complete the proof of the Theorem it remains only to show that ϕ is continuous. To this end we observe that we can copy the argument used in Case 1 once we observe that T is continuous and $\{T\tau_{a_i}\}$ is a net bounded in the norm of $M(A_p(G_2))$. The continuity of T follows from the fact that it is an algebra isomorphism of the commutative Banach algebra $M(A_p(G_1))$ onto the commutative semisimple Banach algebra $M(A_p(G_2))$. The boundedness of the net $\{T\tau_{a_i}\}$ follows from the continuity of T .

REMARK. It is easily seen that G_1 and G_2 are topologically isomorphic if there exists a bipositive or norm decreasing algebra isomorphism of $M((L_1 \cap L_p)(G_1))$ onto $M((L_1 \cap L_p)(G_2))$ or of $M((L_1 \cap C_0)(G_1))$ onto $M((L_1 \cap C_0)(G_2))$. Also if G_1 and G_2 are noncompact, "isometric" can be replaced by "norm decreasing" in Theorem 1. All these results follow from Theorem 3.1 of [2] and the fact that the multiplier algebras involved are all isometrically isomorphic to $M(G_i)$ as Banach algebras.

We need the following proposition in order to derive some consequences of Theorem 1 and the above remark.

PROPOSITION 5. *Let* $F(G_1)$ *and* $F(G_2)$ *be ideals of* $L_1(G_1)$ *and* $L_1(G_2)$ *, respectively, which are Banach algebras in their own norm and let* $M(F(G_i))$ *denote the multiplier algebra of* $F(G_i)$ *. If* S *is a bipositive or isometric algebra isomorphism of* $F(G_1)$ *onto* $F(G_2)$ *then* S *induces a bipositive or isometric algebra isomorphism of* $M(F(G_1))$ *onto* $M(F(G_2))$ *.*

Proof. The induced isomorphism $\phi : M(F(G_1)) \to M(F(G_2))$ is given by $\phi : T \to STS^{-1}$. It is routine to check that ϕ is bipositive or isometric depending on whether S is bipositive or isometric.

REMARK. The assumption that $F(G_i)$ is an ideal in $L_1(G_i)$ is made to ensure that $F(G_i)$ is a semi-simple algebra, so that it is meaningful to talk about the multiplier algebra of $F(G_i)$.

COROLLARY 1. *Let* G_1 *and* G_2 *be locally compact abelian groups and let* $1 \le p < \infty$ *. Then* G_1 *and* G_2 *are isomorphic as topological groups provided there exists a bipositive or isometric algebra isomorphism of* $A_p(G_1)$ *onto* $A_p(G_2)$ *.*

Proof. The result follows immediately from Proposition 5 and Theorem 1.

COROLLARY 2. *Let* G_1, G_2 *and* p *be as above. Then* G_1 *and* G_2 *are isomorphic as topological groups provided there exists a bipositive or isometric algebra isomorphism of* $(L_1 \cap L_p)(G_1)$ *onto* $(L_1 \cap L_p)(G_2)$ *or of* $(L_1 \cap C_0)(G_1)$ *onto* $(L_1 \cap C_0)(G_2)$ *.*

Proof. The result follows immediately from Proposition 5 and the remark following the proof of Theorem 1.

REMARK. *If* G_1 *and* G_2 *are compact then "isometric" can be replaced by "norm decreasing" in Corollary 1.*

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Proof of the Remark. Suppose G_1 and G_2 are compact and T is a norm decreasing algebra isomorphism of $A_p(G_1)$ onto $A_p(G_2)$. If $\gamma \in \hat{G}_1$, we shall show that $T\gamma \in \hat{G}_2$. Since T is an isomorphism we get

$$T(\gamma) * T(\gamma) = T(\gamma * \gamma) = T(\gamma) .$$

Now $(T\gamma)\hat{} = 1$ or 0. If $(T\gamma)\hat{}$ takes value 1 at two distinct characters then

$$\|T\gamma\|^p \geq 1 + 2^{1/p} > 2 .$$

This contradicts the fact that T is norm decreasing. Therefore $(T\gamma)\hat{}$ takes value 1 at one and only one character because T is a norm decreasing isomorphism. This implies that $T\gamma \in \hat{G}_2$.

Consider now an arbitrary trigonometric polynomial $\sum_1^n a_i\gamma_i$ on G_1. Then

$$\left\|T\left(\sum_1^n a_i\gamma_i\right)\right\|^p = \left\|\sum_1^n a_i T\gamma_i\right\|_1 + \left(\sum_1^n |a_i|^p\right)^{1/p} \tag{10}$$

$$\leq \left\|\sum_1^n a_i\gamma_i\right\|_1 + \left(\sum_1^n |a_i|^p\right)^{1/p} .$$

The inequality follows because T is norm decreasing. From (10) we conclude that

$$\left\|T\left(\sum_1^n a_i\gamma_i\right)\right\|_1 \leq \left\|\sum_1^n a_i\gamma_i\right\|_1 .$$

This shows that T can be extended as a norm decreasing isomorphism of $L_1(G_1)$ onto $L_1(G_2)$. The result now follows from Theorem 3 of [3].

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