# ON THE DISCRIMINANT $x'Ax.y'Ay - (x'Ay)^2$.

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(received March 15, 1958)

1. Let x, y be column matrices of n real homogeneous coordinates $x_j$, $y_j$ ($j = 1, \ldots, n$) representing points in $(n-1)$-dimensional real projective space $P_{n-1}$. Let A be an $n \times n$ real symmetric matrix. The equation $x'Ax = 0$ represents a quadric in $P_{n-1}$ and the equation

$$x'Ax \cdot y'Ay - (x'Ay)^2 = 0$$

represents in variable y the tangential cone with x as vertex, a pair of straight lines if $n = 3$. The left-hand difference may be written as a quadratic form in y, viz. $y'Sy$, whose matrix

$$S = x'Ax \cdot A - Axx'A$$

is seen to be singular since $Sx = 0$. (As to the notation see [1].) Moreover if A is regular and $x'Ax \neq 0$, then the equation $Sz = 0$ has no linearly independent solution except $z = x$; thus S has rank $n - 1$.

If A is positive definite, then it represents an imaginary quadric and by Cauchy-Schwarz's inequality

$$(1) \qquad x'Ax \cdot y'Ay - (x'Ay)^2 \geqslant 0$$

for all x, y, with the sign of equality if and only if the two points x and y in $P_{n-1}$ coincide. Thus there is no real tangential cone to this quadric, which fact may be expressed by saying that every real point x is an inner point of the imaginary quadric.

From now on let A denote a regular real symmetric matrix and let x be a fixed point in $P_{n-1}$ such that $x'Ax > 0$. It will be shown that the following two properties of A are equivalent:

(i) A is of the congruence type $[+, -, \ldots, -]$, i.e. A has the signature $2 - n$;

(ii) $y'Sy \leqslant 0$ for all y (i.e. S is non-positively semi-definite) equality holding if and only if the points x and y coincide.



https://doi.org/10.4153/CMB-1958-018-0 Published online by Cambridge University Press

The fact that (ii) follows from (i) has been pointed out recently by J. Aczel [2]. The inequality $y'Sy < 0$ indicates that those points x for which $x'Ax > 0$ are the inner points of the quadric A because they are not vertices of tangential cones.

2. For the proof it will be sufficient to assume A in its congruence normal form. In the positive definite case this is the unit matrix E so that $S = x'x \cdot E - xx'$. The characteristic polynomial of S is found to be

$$|\lambda E - S| = |(\lambda - x'x)E + xx'| = \lambda(\lambda - x'x)^{n-1}.$$

All eigen values of S being non-negative one has $y'Sy \geqslant 0$ whence follows Cauchy-Schwarz's inequality in its primitive form: $x'x \cdot y'y - (x'y)^2 \geqslant 0$ with equality if and only if the two points x and y in $P_{n-1}$ coincide.

3. In the same way, namely by calculating the characteristic roots of S, the inequality (ii) will be proved in its primitive form if A has the signature $2 - n$. Let

$$J = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix} = [1, -1, \ldots, -1]$$

be the congruence normal form of A and $S = \sigma J - Jxx'J$ where

$$\sigma = x'Jx = x_1^2 - x_2^2 - \ldots - x_n^2 > 0.$$

Consider the eigen value problem $Sz = \lambda z$ which can also be written in the form $(\sigma J - \lambda E)z = x'Jz \cdot Jx$ or

$$(2) \qquad (\sigma E - \lambda J)z = x'Jz \cdot x.$$

It is equivalent to the following system:

$$(3) \qquad \left\{ \begin{array}{lclcc} & & & \text{I} & \text{II} \\ (\sigma - \lambda)z_1 & = & x'Jz \cdot x_1 & x_1 & x_1 \\ (\sigma + \lambda)z_2 & = & x'Jz \cdot x_2 & -x_2 & x_2 \\ \cdots & & \cdots & \cdots & \cdots \\ (\sigma + \lambda)z_n & = & x'Jz \cdot x_n & -x_n & x_n \end{array} \right.$$

whence



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$$\text{I.}\quad (\sigma - \lambda)x_1 z_1 - (\sigma + \lambda)(x_2 z_2 + \ldots + x_n z_n) = x'Jz \cdot \sigma$$

which means $\lambda x'z = 0$. Therefore

(4) either $\lambda = 0$ or $x'z = 0$ ;

$$\text{II.}\quad (\sigma - \lambda)x_1 z_1 + (\sigma + \lambda)(x_2 z_2 + \ldots + x_n z_n) = x'Jz \cdot x'x$$

so that by (4) if $\lambda \neq 0$ either

(a) $\lambda = - x'x$ or (b) $x'Jz = 0$.

In the case (a) the eigen value equation (2) will be $(\sigma E + x'xJ)z = x'Jz \cdot x$ and instead of (3) one has the system

$$2x_1^2 z_1 = x'Jz \cdot x_1$$

$$-2(x_2^2 + \ldots + x_n^2)z_i = x'Jz \cdot x_i \quad (i = 2, \ldots, n)$$

whose solution z is uniquely defined if $x_2^2 + \ldots + x_n^2$ is different from zero. For since $\sigma > 0$ one has $x_1 \neq 0$ and it may be assumed that $z_1 = x_1$ ; then $x'Jz = 2x_1^2$ so that the $z_i$ are readily expressible in terms of the $x_j$.

Thus $\lambda = - x'x$ is another simple eigen value.

If $x_2^2 + \ldots + x_n^2 = 0$ (i.e. all $x_i = 0$ $(i = 2, \ldots\ldots, n)$ ) then obviously S equals the diagonal matrix $[0, -x_1^2, \ldots, -x_1^2]$.

In the case (b) the system (3) will be

$$(5)\qquad (\sigma - \lambda)z_1 = 0, \quad (\sigma + \lambda)z_i = 0 \quad (i = 2, \ldots, n).$$

If $z_1 \neq 0$, then $\lambda = \sigma > 0$ and accordingly $z_i = 0$ for $i = 2, \ldots, n$. Since $x_1 \neq 0$ this is incompatible with (4). Thus $\sigma$ cannot be an eigen value and necessarily $z_1 = 0$ so that $\lambda = -\sigma$ appears as $(n - 2)$-fold eigen value of S. The corresponding eigen vectors are given by the solutions of the equation $x_2 z_2 + \ldots + x_n z_n = 0$. Thus all eigen values of S are negative except $\lambda = 0$ and therefore S is negative semi-definite and

$$(6)\qquad (x_1^2 - x_2^2 - \ldots - x_n^2)(y_1^2 - y_2^2 - \ldots - y_n^2) \leq (x_1 y_1 - x_2 y_2 - \ldots - x_n y_n)^2$$

if

$$x_1^2 - x_2^2 - \ldots - x_n^2 > 0 .$$



https://doi.org/10.4153/CMB-1958-018-0 Published online by Cambridge University Press

Since 0 is a simple eigen value of S it follows that the equality sign is valid if and only if x and y represent the same point in $P_{n-1}$.

4. It remains to be shown that in all other cases S cannot be semi-definite. It will be sufficient to investigate the case where A has the congruence normal form

$$J = [1, 1, -1, \ldots, -1].$$

Let

$$\sigma = x'Jx = x_1^2 + x_2^2 - x_3^2 - \ldots - x_n^2 > 0.$$

The eigen value equations are now the following:

$$(7) \quad \begin{cases} (\sigma - \lambda)z_1 = x'Jz \cdot x_1 & x_1 & x_1 \\ (\sigma - \lambda)z_2 = x'Jz \cdot x_2 & x_2 & x_2 \\ (\sigma + \lambda)z_3 = x'Jz \cdot x_3 & -x_3 & x_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\sigma + \lambda)z_n = x'Jz \cdot x_n & -x_n & x_n \end{cases} \qquad \begin{matrix} \text{I} & \text{II} \end{matrix}$$

In the case I the discussion is the same as in the preceding section and if $\lambda \neq 0$ one has the condition $x'z = 0$. In the case II there is again the alternative either (a) or (b). If $\lambda = -x'x$ and $x_3^2 + \ldots x_n^2 > 0$ the system (7) has a unique linearly independent solution and therefore $-x'x$ is a simple eigen value of S. If $x_3^2 + \ldots + x_n^2 = 0$, then

$$S = \begin{pmatrix} x_2^2 & -x_1x_2 & 0 & 0 \\ -x_1x_2 & x_1^2 & 0 & 0 \\ 0 & 0 & -x_1^2-x_2^2 & 0 \\ \vdots & \vdots & \vdots & \ddots\vdots \\ 0 & 0 & 0 \cdots & -x_1^2-x_2^2 \end{pmatrix}$$

which has the (n − 2) fold eigen value $-x_1^2 - x_2^2$ and the simple eigen value $x_1^2 + x_2^2$.

In the case (b) the system (7) becomes

$$(\sigma - \lambda)z_1 = 0, \ (\sigma - \lambda)z_2 = 0, \ (\sigma + \lambda)z_i = 0 \ (i > 2).$$

Thus the eigen value $\lambda = \sigma$ cannot be excluded as in section 3. In fact let $z_1 \neq 0$ and therefore $\lambda = \sigma$; then $z_i = 0$ ($i > 2$) and $z_2$ can be found such that the condition $x'z = x_1z_1 + x_2z_2 = 0$ is satisfied; i.e. g., $x_1 = 0$, then $z_2 = 0$. Thus the matrix S has a simple positive eigen value $\sigma$ ; hence S is not semi-definite. This precludes the existence of an inequality of the type (ii) in the present case.

So it is in all the other cases. If

$$J = [\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{n-p}]$$

it is found that

$$|\lambda E - S| = \lambda(\lambda + x'x)(\lambda - \sigma)^{p-1}(\lambda + \sigma)^{n-p-1}$$

where

$$\sigma = x'Jx.$$

5. Aczel's proof of the inequality (6) uses the method commonly applied in the proof of Cauchy-Schwarz's inequality. He observes that the quadratic function of the real variable $\xi$ :

$$\eta = f(\xi) = (x_1\xi + y_1)^2 - (x_2\xi + y_2)^2 - \ldots -(x_n\xi + y_n)^2$$
$$= \sigma\xi^2 + 2x'Jy \cdot \xi + y'Jy ,$$

has, because of $\sigma > 0$, as graph in the $\xi\eta$ -plane a parabola open above that cuts or touches the $\xi$ – axis whatever y may be. Thus the discriminant of the function $f(\xi)$ must be non-negative. In all the other cases the sign of the discriminant depends on the choice of the point y.

## REFERENCES


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