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SOME NEUTRAL EQUATIONS WITH A CONTROL PARAMETER

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This paper presents sufficient conditions, involving accretive operators, for the existence, uniqueness and continuous dependence on a control parameter of the solutions of some initial and boundary value problems for neutral functional differential equations.

The main purpose of the present paper is to obtain conditions for the existence, uniqueness and continuous dependence on a control parameter of the solutions of some initial and boundary value problems for neutral functional differential equations.

Let us consider the following initial value problem:

$$x'(t) = X\left(\mu,\ t,\ x\left(\Delta_1(t)\right),\ \ldots,\ x\left(\Delta_m(t)\right),\ x'\left(\tau_1(t)\right),\ \ldots,\ x'\left(\tau_n(t)\right)\right)\ ,\quad t > 0\ , \tag{1}$$

$$x(t) = \psi(t,\ \mu)\ ,\quad x'(t) = \frac{\partial\psi(t,\mu)}{\partial t}\ ,\quad t \le 0\ ,$$

where the unknown function $x(t)$ takes values in some Banach space B with a norm $\|\cdot\|$ and its derivative is in the strong sense. The control parameter μ takes values in B. The deviations $\Delta_i(t)$ $(i = 1, 2, \ldots, m)$ and $\tau_l(t)$ $(l = 1, 2, \ldots, n)$ are of a mixed type and in the general case - unbounded.

The author's recent paper [1] contains existence and uniqueness

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results for the strong absolutely continuous solution of the initial value problem (1) with L^p-derivative (without parameter) where $1 \le p < \infty$ and comparison with the previous related results. Here we are going to find a strong absolutely continuous solution of the same initial value problem with L^∞-derivative.

We shall consider also the following boundary value problem with control parameter: find a solution of the problem

$$(2)\quad \begin{aligned} x'(t) &= \mu + X\big(\mu, t, x(\Delta_1(t)), \ldots, x(\Delta_m(t)),\\ &\qquad x'(\tau_1(t)), \ldots, x'(\tau_n(t))\big), \quad t > 0, \\ x(t) &= \psi(t, \mu), \quad x'(t) = \frac{\partial\psi(t,\mu)}{\partial t}, \quad t \le 0, \end{aligned}$$

and a value of the parameter $\mu = \mu_0 \in B$ such that $x(T, \mu_0) = x_T$, where $x_T \in B$ is a value given in advance.

Analogous problems for some other classes of functional differential equations have been considered in [2]-[4].

The boundary value problem with control parameter allows the following physical interpretation: let us suppose that the initial state and velocity of some physical system depends on the control parameter μ. We look for a value $\mu = \mu_0$ such that at instant $t = T$ the system attains the given state x_T.

The operator $A : \mathrm{Dom}\, A \to \mathcal{B}$ ($\mathcal{B}$ is a Banach space) is said to be *accretive* if

$$\|(I+\lambda A)x-(I+\lambda A)y\|_{\mathcal{B}} \ge \|x-y\|_{\mathcal{B}}$$

for $\lambda > 0$ and $x, y \in \mathrm{Dom}\, A$ (I is identity map), and m-accretive if $\mathrm{Range}(I+\lambda A) = \mathcal{B}$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ correspondingly.

We shall first prove some auxiliary propositions, based on the results of Webb [6] and Martin [5].

PROPOSITION 1. *Let the following conditions hold:*

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1. *the nonlinear continuous operators* $N_i : B_i \times B \to B_i$ *satisfy the inequalities*

(3) $\|[(1+\lambda\gamma-\lambda\alpha)x+\lambda N_i(x, \mu)]-[(1+\lambda\gamma-\lambda\alpha)y+\lambda N_i(y, \mu)]\|_i \geq \|x-y\|_i \quad (i = 1, 2)$

for every $x, y \in B_i$ *and* $\mu \in B$, *where* $\gamma > 0$, $\alpha > 0$ *are constants;*

2. *the linear map* $j : B_1 \to B_2$ *satisfies the condition* $j(N_1(x, \mu)) = N_2(jx, \mu)$ *for every* $f \in B_1$ *and* $\mu \in B$.

Then for maps $x(\mu) : B \to B_1$ *and* $y(\mu) : B \to B_1$, *connected by the condition* $jx(\mu) = N_2(y(\mu), \mu) + \gamma y(\mu)$, *there exists a unique map* $z(\mu) : B \to B_1$ *for which* $N_1(z(\mu), \mu) + \gamma z(\mu) = x(\mu)$ *and* $jz(\mu) = y(\mu)$.

Proof. The operator $N_i(\cdot, \mu) + (\gamma-\alpha)I$ is continuous and inequality (3) shows that it is accretive. Since it is defined on the whole space B_i , the operator is m-accretive (Martin [5]). If we choose $\lambda = \frac{1}{\alpha}$ then the operator

$$I + \frac{1}{\alpha}[N_i(\cdot, \mu)+(\gamma-\alpha)I] = \frac{1}{\alpha}[N_i(\cdot, \mu)+\gamma I]$$

maps B_i onto B_i $(i = 1, 2)$ for $\mu \in B$. If in (3) we set $\lambda = \frac{1}{\alpha}$, then we obtain

$$\left\|\frac{1}{\alpha}[N_i(x, \mu)+\gamma x] - \frac{1}{\alpha}[N_i(y, \mu)+\gamma y]\right\|_i \geq \|x-y\|_i \quad (i = 1, 2) .$$

Therefore $N_i(\cdot, \mu) + \gamma I$ maps B_i into B_i one-to-one for every $\mu \in B$ and its inverse operator satisfies the inequality

$$\left\|[N_i(x, \mu)+\gamma x]^{-1}-[N_i(y, \mu)+\gamma y]^{-1}\right\|_i \leq \frac{1}{\alpha}\|x-y\|_i . \tag{4}$$

If $x(\mu) : B \to B_1$ and $y(\mu) : B \to B_2$ are such that $jx(\mu) = N_2(y(\mu), \mu) + \gamma y(\mu)$, then there exists a unique map $z(\mu) : B \to B_1$ for which $N_1(z(\mu), \mu) + \gamma z(\mu) = x(\mu)$. In addition, we have

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$$N_2(y(\mu),\mu) + \gamma y(\mu) = jx(\mu) = j[N_1(z(\mu),\mu)+\gamma z(\mu)]$$
$$= N_2(jz(\mu),\mu) + \gamma(jz(\mu)) ,$$

that is, $jz(\mu) = y(\mu)$.

PROPOSITION 2. *Let the conditions of Proposition* 1 *hold, let the maps* $x(\mu) : B \to B_1$, $y(\mu) : B \to B_2$ *be continuous and let* $\|N_i(x,\mu)-N_i(y,\mu)\|_i \le M\|x-y\|_i$ $(i = 1, 2)$, *where* $x, y \in B_i$, $M > 0$ *is a constant,* $\gamma > M$.

Then the map $z(\mu) : B \to B_1$, *satisfying the equation* $N_1(z(\mu),\mu) + \gamma z(\mu) = x(\mu)$, *is continuous.*

Proof. Let $z(\mu)$ and $z(\mu_0)$ be solutions of the following operator equations:

$$N_1(z(\mu),\mu) + \gamma z(\mu) = x(\mu) , \quad N_1(z(\mu_0),\mu_0) + \gamma z(\mu_0) = x(\mu_0) .$$

Then we have

$$\|z(\mu)-z(\mu_0)\|_1$$
$$\le \frac{1}{\gamma}\|N_1(z(\mu),\mu)-N_1(z(\mu_0),\mu_0)\|_1 + \frac{1}{\gamma}\|x(\mu)-x(\mu_0)\|_1$$
$$\le \frac{1}{\gamma}\|N_1(z(\mu),\mu)-N_1(z(\mu),\mu_0)\|_1 + \frac{M}{\gamma}\|z(\mu)-z(\mu_0)\|_1 + \frac{1}{\gamma}\|x(\mu)-x(\mu_0)\|_1 .$$

This last inequality implies

$$\|z(\mu)-z(\mu_0)\|_1 \le \frac{1}{\gamma-M}\|N_1(z(\mu),\mu)-N_1(z(\mu),\mu_0)\|_1 + \frac{1}{\gamma-M}\|x(\mu)-x(\mu_0)\|_1 .$$

Bearing in mind the continuity of operator N_1 and of the map $x(\mu)$, we conclude that $z(\mu) : B \to B_1$ is continuous.

PROPOSITION 3. *Let the conditions of Proposition* 1 *hold and let* $z_1(\mu)$, $z_2(\mu)$ *satisfy the equalities*

$$N_1(z_i(\mu),\mu) + \gamma z_i(\mu) = x_i(\mu) , \quad jz_i(\mu) = y_i(\mu) ,$$
$$N_2(y_i(\mu),\mu) + \gamma y_i(\mu) = jx_i(\mu) \quad (i = 1, 2) .$$

The the following estimate holds:

$$\|z_1(\mu)-z_2(\mu)\|_1 \le \frac{1}{\alpha}\|x_1(\mu)-x_2(\mu)\|_1 .$$

The proof is a consequence from inequality (4).

DEFINITION 1. The function $\tau(t) : R^1_+ \to R^1$ possesses the property (S) if the inverse image of every set with null measure is measurable $\left(R^1 = (-\infty, \infty)\right.$, $R^1_+ = [0, \infty)$, $\left.R^1_- = (-\infty, 0]\right)$.

DEFINITION 2. The function

$$X\left(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n\right) : B \times R^1_+ \times B^{m+n} \to B$$

satisfies the Caratheodory condition, if it is measurable in t and continuous in $\mu, u_1, \ldots, u_m, v_1, \ldots, v_n$.

DEFINITION 3. The function $X(u, v) : B \times B \to B$ is uniformly continuous in u with respect to v if for every $\varepsilon > 0$ there exist $\delta > 0$ such that $\|u-\bar{u}\| < \delta$ implies $\|X(u; v)-X(\bar{u}, v)\| < \varepsilon$ and the number δ does not depend on the variable v .

If we set $\varphi(t) = x'(t)$ for $t > 0$ and $\theta(t, \mu) = \frac{\partial\psi(t,\mu)}{\partial t}$ for $t \le 0$ then we obtain the equivalent initial value problem

$$\varphi(t) = X\left(\mu, t, \psi(0, \mu) + \int_0^{\Delta_1(t)} \varphi(s)ds, \ldots, \psi(0, \mu) + \int_0^{\Delta_m(t)} \varphi(s)ds, \varphi\left(\tau_1(t)\right), \ldots, \varphi\left(\tau_n(t)\right)\right) , \quad t > 0 ,$$

(1')

$$\varphi(t) = \theta(t, \mu) , \quad t \le 0 .$$

In analogous way from (2) we obtain

$$\varphi(t) = \mu + X\left(\mu, t, \psi(0, \mu) + \int_0^{\Delta_1(t)} \varphi(s)ds, \ldots, \psi(0, \mu) + \int_0^{\Delta_m(t)} \varphi(s)ds, \varphi\left(\tau_1(t)\right), \ldots, \varphi\left(\tau_n(t)\right)\right) , \quad t > 0 ,$$

(2')

$$\varphi(t) = \theta(t, \mu) , \quad t \le 0 .$$

DEFINITION 4. $\alpha\left(\mu, t, x_1, \ldots, x_m, y_1, \ldots, y_n\right)$,

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$\beta(\mu, t, x_1, \ldots, x_m, y_1, \ldots, y_n) : B \times R_+^{m+n+1} \to R_+^1$ are comparison functions if they satisfy the Caratheodory condition, are nondecreasing in x_i, y_l and $\bar{\alpha}(\mu, t, x_1, \ldots, x_m) = \alpha(\mu, t, x_1, \ldots, x_m, y, \ldots, y)$ is essentially bounded for any fixed $y \in R_+^1$ $(i = 1, \ldots, m$; $l = 1, \ldots, n)$.

THEOREM 1. *Let the following conditions hold:*

1. *the functions* $\Delta_i(t) : R_+^1 \to R^1$ $(i = 1, 2, \ldots, m)$ *and* $\tau_l(t) : R_+^1 \to R^1$ $(l = 1, 2, \ldots, n)$ *are measurable and* $\tau_l(t)$ *has property* (S);

2. *the function*

$$X(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n) : B \times R_+^1 \times B^{m+n} \to B$$

satisfies the Caratheodory condition and the conditions

$$\|X(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n)\| \le \frac{1}{\gamma} \alpha(\mu, t, \|u_1\|, \ldots, \|u_m\|, \|v_1\|, \ldots, \|v_n\|) ,$$

$$\|X(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n) - X(\mu, t, \bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_n)\| \le \frac{M}{\gamma} \beta(\mu, t, \|u_1-\bar{u}_1\|, \ldots, \|u_m-\bar{u}_m\|, \|v_1-\bar{v}_1\|, \ldots, \|v_n-\bar{v}_n\|) ,$$

where $\gamma > M > 0$ *are constants,* α *and* β *are comparison functions and*

$$\beta(\mu, t, |\Delta_1(t)|y, \ldots, |\Delta_m(t)|y, y, \ldots, y) \le y .$$

In addition $X(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n)$ *is uniformly continuous in* μ *with respect to the other variables;*

3. *the initial function* $\theta(\cdot, \mu) \in L^\infty(R_-^1; B)$, $\mu \in B$ *and is uniformly continuous in* μ *with respect to* t .

Then there exists a unique solution $\varphi(\cdot, \mu) \in L^\infty(R^1; B)$ *of the initial value problem* (1'), *which depends continuously on the parameter* μ .

Proof. Let B_1 be the Banach space $L^{\infty}(R^1; B)$ with a norm $\|f\|_1 = \text{ess sup}\{\|f(t)\| : t \in R^1\}$, and B_2 the Banach space $L^{\infty}(R^1_-; B)$ with norm $\|g\|_2 = \text{ess sup}\{\|g(t)\| : t \in R^1_-\}$.

We define the operator $N_1(f, \mu) : B_1 \times B \to B_1$ by the formula

$$N_1(f, \mu)(t) = \begin{cases} -\gamma X\left[\mu, t, \psi(0, \mu) + \int_0^{\Delta_1(t)} f(s)ds, \ldots, \psi(0, \mu) \right. \\ \qquad \left. + \int_0^{\Delta_m(t)} f(s)ds, f(\tau_1(t)), \ldots, f(\tau_n(t))\right] , & t > 0 , \\ 0 , \quad t \le 0 , \end{cases}$$

where $f \in B_1$, $\mu \in B$, and the operator $N_2(g, \mu) : B_2 \times B \to B_2$ by the formula $N_2(g, \mu)(t) = 0$, $t \le 0$, $g \in B_2$, $\mu \in B$.

The map $j : B_1 \to B_2$ is defined as a restriction of the function $f \in B_1$ on the semiaxis R^1_- . It is easily seen that $j(N_1(f, \mu)) = N_2(jf, \mu)$ for every $f \in B_1$ and $\mu \in B$.

We shall first show that if the function $f \in B_1$ then the function $N_1(f, \mu) \in B_1$.

Indeed, the function $N_1(f, \mu)(t)$ is strongly measurable and satisfies the inequality

$$\begin{aligned} \|N_1(f, \mu)(t)\| &\le \alpha(\mu, t, \|\psi(0, \mu)\|+|\Delta_1(t)|\|f\|_1, \\ &\qquad \ldots, \|\psi(0, \mu)\|+|\Delta_m(t)|\|f\|_1, \|f\|_1, \ldots, \|f\|_1) \\ &\le \text{ess sup}\Big\{\alpha(\mu, t, \|\psi(0, \mu)\|+|\Delta_1(t)|\|f\|_1, \|f\|_1, \ldots, \|f\|_1) : \\ &\qquad t \in R^1, \mu \in B\Big\} . \end{aligned}$$

Therefore $N_1(f, \mu)(t) \in B_1$.

In what follows, we shall prove the Lipschitz continuity of the

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operator N_1 . Let $f, g \in \mathcal{B}_1$. Then we have

$$\|N_1(f, \mu)(t)-N_1(g, \mu)\| \le M\beta\left(\mu, t, |\Delta_1(t)|\|f-g\|_1, \ldots, |\Delta_m(t)|\|f-g\|_1, \|f-g\|_1, \ldots, \|f-g\|_1\right) \le M\|f-g\|_1 ,$$

that is, $\|N_1(f, \mu)-N_1(g, \mu)\|_1 \le M\|f-g\|_1$.

If we set $\alpha = \gamma - M$, then

$$\|\left[(1+\lambda\gamma-\lambda\alpha)f+\lambda N_i(f, \mu)\right]-\left[(1+\lambda\gamma-\lambda\alpha)g+\lambda N_i(g, \mu)\right]\| \ge (1+\lambda M)\|f-g\|_i - \lambda M\|f-g\|_i = \|f-g\|_i \quad (i = 1, 2) .$$

Define the function $\sigma(t, \mu) : R \times B \to \mathcal{B}_2$ by the formula

$$\sigma(t, \mu) = \begin{cases} 0 , & t > 0 , \\ N_2(\theta, \mu) + \gamma\theta(t, \mu) , & t \le 0 . \end{cases}$$

Since $\sigma(t, \mu)$ is uniformly continuous in μ with respect to t , it defines a continuous map $\sigma(\mu) : B \to \mathcal{B}_1$. Then Proposition 1 implies an existence of a unique map $\varphi(\mu) : B \to \mathcal{B}_1$ for which

$$N_1\left(\varphi(\mu), \mu\right) + \gamma\varphi(\mu) = \sigma(\mu) , \quad j\varphi(\mu) = \theta(\mu) ,$$

that is, the function $\varphi(\cdot, \mu) \in L^\infty\left(R^1; B\right)$ is a solution of the problem (1').

Moreover the operator $N_1(f, \mu)$ is continuous in μ and so Proposition 2 implies that the map $\varphi(\mu) : B \to \mathcal{B}_1$ is continuous. This completes the proof of Theorem 1.

As an immediate consequence of Proposition 3, we obtain

THEOREM 2. *Let the conditions of Theorem* 1 *hold and let* $\varphi\left(\theta_1, \mu\right)(t)$ *and* $\varphi\left(\theta_2, \mu\right)(t)$ *be solutions of the problem* (1') *with initial functions* $\theta_1(t, \mu)$ *and* $\theta_2(t, \mu)$. *Then*

$$\|\varphi\left(\theta_1, \mu\right)(t)-\varphi\left(\theta_2, \mu\right)(t)\| \le \frac{\gamma}{\gamma-M} \operatorname{ess\,sup}\left\{\|\theta_1(t, \mu)-\theta_2(t, \mu)\| : t \in R^1_-\right\} .$$

THEOREM 3. *Let the following conditions hold:*

1. *the functions* $\Delta_i(t) : R^1_+ \to R^1$ $(i = 1, 2, \ldots, m)$ *and* $\tau_l(t) : R^1_+ \to R^1$ $(l = 1, \ldots, n)$ *are measurable and* $\tau_l(t)$ *possess the property* (S) $(i = 1, \ldots, m)$;
2. *the function*

$$X(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n) : B \times R^1_+ \times B^{m+n} \to B$$

satisfies condition 2 *of Theorem* 1 *and the condition*

$$\|X(\mu, t, u_1, \ldots, u_m, v_1, \ldots, v_n) - X(\bar{\mu}, t, u_1, \ldots, u_m, v_1, \ldots, v_n)\| \le \frac{\xi(t)}{\gamma} \|\mu-\bar{\mu}\|$$

where the function $\xi(t) : R^1_+ \to R^1_+$ *is measurable and* $\xi(t) \in L^\infty(R^1_+)$;

3. *the initial function* $\psi(t, \mu) : R^1_+ \times B \to B$ *is Lipschitz continuous in* μ , *that is,* $\|\psi(0, \mu)-\psi(0, \bar{\mu})\| \le \kappa\|\mu-\bar{\mu}\|$, $\kappa = \text{const.} > 0$, $\theta(t, \mu) : R^1_- \times B \to B$ *is uniformly continuous in* μ *with respect to* t , $\theta(\cdot, \mu) \in L^\infty(R^1_-; B)$ *and*

$$\gamma\kappa + \int_0^T \xi(t)dt < \gamma T , \quad T = \text{const.} > 0 .$$

Then there exists a unique value $\mu = \mu_0$ *such that* $\varphi(t, \mu_0)$ *is a solution of the boundary value problem with control parameter* (2'), *and* $x(T, \mu_0) = x_T$.

Proof. Let $\mathcal{B}_1$ be the Banach space $L^\infty(R^1; B)$ and $\mathcal{B}_2$ the Banach space $L^\infty(R^1_-; B)$.

Let us define the operators $N_1 : \mathcal{B}_1 \times B \to \mathcal{B}_1$, $N_2 : \mathcal{B}_2 \times B \to \mathcal{B}_2$, $N : B \times \mathcal{B}_1 \to B$ by the formulae

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$$N_1(f, \mu)(t) = \begin{cases} -\gamma\mu - \gamma X\Big(\mu,\ t,\ \psi(0, \mu) + \int_0^{\Delta_1(t)} f(s)ds,\ \ldots,\ \psi(0, \mu) \\ \qquad + \int_0^{\Delta_m(t)} f(s)ds,\ f(\tau_1(t)),\ \ldots,\ f(\tau_n(t))\Big)\ , \quad t > 0\ , \\ 0\ , \quad t \le 0\ , \end{cases}$$

$f \in B_1$, $\mu \in B$; $N_2(g, \mu)(t) = 0$, $g \in B_2$, $\mu \in B$;

$$N\mu = -\frac{\gamma}{T}x_T + \frac{\gamma}{T}\psi(0, \mu) + \frac{\gamma}{T}\int_0^T X\Big(\mu,\ t,\ \psi(0, \mu) + \int_0^{\Delta_1(t)} f(s)ds,$$
$$\ldots,\ \psi(0, \mu) + \int_0^{\Delta_m(t)} f(s)ds;\ f(\tau_1(t)),\ \ldots,\ f(\tau_n(t))\Big)dt\ .$$

The linear map $j : B_1 \to B_2$ is defined as in the proof of Theorem 1.

The estimate

$$\|N_1(f, \mu)(t)\| \le \gamma\|\mu\| + \alpha\big(\mu,\ t,\ \|\psi(0, \mu)\|+|\Delta_1(t)|\,\|f\|_1,$$
$$\ldots,\ \|\psi(0, \mu)\|+|\Delta_m(t)|\,\|f\|_1,\ \|f\|_1,\ \ldots,\ \|f\|_1\big)$$

shows that $N_1(f, \mu)(t) \in B_1$.

The Lipschitz continuity of the operator N_1 follows from inequalities

$$\|N_1(f, \mu)(t)-N_1(g, \mu)(t)\|$$
$$\le M\beta\big(\mu,\ t,\ |\Delta_1(t)|\,\|f-g\|_1,\ \ldots,\ |\Delta_m(t)|\,\|f-g\|_1,\ \|f-g\|_1,\ \ldots,\ \|f-g\|_1\big)$$
$$\le M\|f-g\|_1\ .$$

Then Proposition 1 implies an existence of uniqueness function $\varphi(t, \mu)$ such that

$$\varphi(t, \mu) = \mu + X\Big(\mu,\ t,\ \psi(0, \mu) + \int_0^{\Delta_1(t)} \varphi(s, \mu)ds,$$
$$\ldots,\ \psi(0, \mu) + \int_0^{\Delta_m(t)} \varphi(s, \mu)ds,\ \varphi(\tau_1(t), \mu),\ \ldots,\ \varphi(\tau_n(t), \mu)\Big)\ , \quad t > 0\ ,$$

$$\varphi(t, \mu) = \theta(t, \mu) , \quad t \le 0 .$$

For the operator N we obtain

$$\|N\mu - N\bar{\mu}\| \le \frac{\gamma\kappa}{T} \|\mu-\bar{\mu}\| + \frac{1}{T}\int_0^T \xi(t)dt\|\mu-\bar{\mu}\| .$$

If we set

$$m_T = \frac{\gamma\kappa}{T} + \frac{1}{T}\int_0^T \xi(t)dt , \quad \alpha = \gamma - m_T ,$$

then we can easily verify that the operator $N + (\gamma-\alpha)I$ is accretive. Therefore there exists a unique $\mu_0 \in B$ such that $N\mu_0 + \gamma\mu_0 = 0$.

We shall show that $x(T, \mu_0) = x_T$.

Indeed, bearing in mind the definition of the operator N we have

$$\begin{aligned} x(T, \mu_0) &= \psi(0, \mu_0) + \int_0^T \varphi(s, \mu_0)ds \\ &= \psi(0, \mu_0) - \frac{1}{\gamma}\Bigg[-T\frac{\gamma}{T}x_T + T\frac{\gamma}{T}\psi(0, \mu_0) + T\frac{\gamma}{T}\int_0^T X\Bigg(\mu_0, t, \psi(0, \mu_0) \\ &\quad + \int_0^{\Delta_1(t)} \varphi(s, \mu_0)ds, \dots, \psi(0, \mu_0) + \int_0^{\Delta_m(t)} \varphi(s, \mu_0)ds, \varphi(\tau_1(t), \mu_0), \\ &\quad \dots, \varphi(\tau_n(t), \mu_0)\Bigg)dt\Bigg] + \int_0^T X\Bigg(\mu_0, t, \psi(0, \mu_0) + \int_0^{\Delta_1(t)} \varphi(s, \mu_0)ds, \\ &\quad \dots, \psi(0, \mu_0) + \int_0^{\Delta_m(t)} \varphi(s, \mu_0)ds, \varphi(\tau_1(t), \mu_0), \\ &\quad \dots, \varphi(\tau_n(t), \mu_0)\Bigg)dt = x_T . \end{aligned}$$

Theorem 3 is thus proved.

REMARK. We note that the condition for the measurability of the functions $\Delta_i(t)$ is necessary. Zverkin [7] has proved that if the neutral equation possesses an absolutely continuous solution, the deviation is a measurable function.

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