# ON THE CRITICAL GALTON-WATSON PROCESS WITH IMMIGRATION

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## 1. Introduction

Consider a Galton-Watson process in which each individual reproduces independently of all others and has probability $a_j$ $(j = 0, 1, \cdots)$ of giving rise to $j$ progeny in the following generation and in which there is an independent immigration component where $b_j$ $(j = 0, 1, \cdots)$ is the probability that $j$ individuals enter the population at each generation. Then letting $X_n$ $(n = 0, 1, \cdots)$ be the population size of the $n$-th generation, it is known (Heathcote [4], [5]) that $\{X_n\}$ defines a Markov chain on the non-negative integers. Unless otherwise stated, we shall consider only those offspring and immigration distributions that make the Markov chain $\{X_n\}$ irreducible and aperiodic.

Heathcote [5] has shown that in the case $\alpha = \sum_j ja_j < 1$, a necessary and sufficient condition for $\{X_n\}$ to be positive-recurrent is that $\sum_{j=1}^{\infty} b_j \log j < \infty$. Seneta [7] has shown that if $\alpha = 1$ and if $2\gamma = \sum_j j(j-1)a_j = \infty$ then it is possible for $\{X_n\}$ to be positive-recurrent.

In this paper we consider the case $\alpha = 1$ and $\beta, \gamma < \infty$ where $\beta = \sum_j jb_j$ is the mean of the immigration distribution. We shall show that the Markov chain $\{X_n\}$ may be either null-recurrent or transient. In the case of null-recurrence we obtain some information on the occupation times of the zero state. Finally, in the last section we show that $X_n/n$ tends in distribution to a random variable having a gamma distribution.

## 2. Classification of the Markov chain $(X_n)$

Let $p_{ij}^{(n)}$ $(i, j, n = 0, 1, \cdots)$ be the $n$-step transition probability from state $i$ to $j$ and let $P_i^{(n)}(x) = \sum_{j=0}^{\infty} p_{ij}^{(n)} x^j$ $(|x| < 1)$. Then letting $A(x) = \sum_{j=0}^{\infty} a_j x^j$ and $B(x) = \sum_{j=0}^{\infty} b_j x^j$ $(|x| < 1)$ be the probability generating functions of the offspring and immigration distributions respectively, it is not difficult to show that

$$(1) \qquad P_i^{(n)}(x) = [A_n(x)]^i \prod_{m=0}^{n-1} B[A_m(x)]$$

where $A_0(x) = x$ and $A_{n+1}(x) = A(A_n(x))$ $(n = 0, 1, \cdots)$, so that in particular we have

$$p_{00}^{(n)} = B(0) \prod_{m=1}^{n-1} B(A_m(0)) \tag{2}$$

and it is clear that $p_{00}^{(n)}$ is a non increasing sequence.

Before stating theorem 1, we shall state a theorem which will play a key role in our work.

THEOREM A. (Kesten, Ney and Spitzer [6]). *If* $\alpha = 1$ *and* $0 < \gamma < \infty$ *and*

$$1/(1-x)+n\gamma-1/[1-A_n(x)] = h_n(x) \qquad (0 \leqq x < 1)$$

*then* $\lim_{n\to\infty} h_n(x)/n = 0$ *uniformly in* $0 \leqq x < 1$. *Furthermore,* $h_n(x) = \sum_{m=0}^{n-1} \delta(A_m(x))$ *where* $\delta(x)$ *satisfies the inequality*

$$\frac{-\gamma^2(1-x)}{1-a_0} \leqq \delta(x) \leqq \varepsilon(x) \qquad (0 \leqq x < 1) \tag{3}$$

*where* $0 \leqq \varepsilon(x) = \gamma-[A(x)-x]/(1-x)^2 \leqq \gamma$ *and* $\varepsilon(x)$ *is non-increasing in* $x$ *and* $\varepsilon(x) \downarrow 0$ $(x \uparrow 1)$.

Observe that our assumption of irreducibility implies that $\gamma > 0$.

THEOREM 1. *Let* $\alpha = 1$ *and* $\beta, \gamma < \infty$, *then the Markov chain* $\{X_n\}$ *is not positive-recurrent. Further, let* $\sigma = \beta/\gamma$, *then* $\{X_n\}$ *is null-recurrent if* $\sigma < 1$ *and transient if* $\sigma > 1$. *Define* $\varepsilon(x)$ *as in theorem A and let* $\varepsilon(x) = O[(1-x)^\delta]$ $(x \uparrow 1)$ *for some* $\delta > 0$. *If* $B''(1-) < \infty$ *then* $p_{00}^{(n)} \sim Cn^{-\sigma}$ *as* $n \to \infty$ *where* $C$ *is a finite, positive constant, and in particular, if* $\sigma = 1$ *then* $\{X_n\}$ *is null-recurrent.*

PROOF. By irreducibility and aperiodicity, the Markov chain $\{X_n\}$ is not positive-recurrent if $\lim_{n\to\infty} p_{00}^{(n)} = 0$, that is if the infinite product $\prod_{m=1}^{\infty} B(A_m(0))$ diverges to zero. (Observe that irreducibility implies that $B(0) > 0$.) It is known (Seneta [7]) that if $\beta < \infty$ then a necessary and sufficient condition for this is the divergence of the integral $\int_0^1 [(1-x)/(A(x)-x)]dx$. By Taylor's theorem, $A(x) = 1-(1-x)+(1-x)^2A''(\theta)/2$ $(x < \theta < 1)$ and so if $0 < A''(1-) < \infty$, we see that the integrand is bounded below by $[(1-x)\gamma]^{-1}$ and so the integral diverges.

Thus the Markov chain will be transient or null-recurrent according as the series $\sum_{n=0}^{\infty} p_{00}^{(n)}$ converges or diverges, and by Raabe's test (Ferrar [2]) the first alternative will occur if $\lim_{n\to\infty} n(1-p_{00}^{(n+1)}/p_{00}^{(n)}) = \lim_{n\to\infty} n[1-B(A_n(0))] > 1$ and the second alternative occurs if this limit $< 1$; the equality in the last expression follows from equation (2). The hypotheses and Taylor's theorem show that for $0 \leqq x < 1$, $B(x) = 1-\beta(1-x)+o(1-x)$ $(x \uparrow 1)$ and since $A_n(0) \uparrow 1$ $(n \to \infty)$ (Harris [3]) we have

$$n[1-B(A_n(0))] = \beta n(1-A_n(0))+no(1-A_n(0)) \quad (n \to \infty) \tag{4}$$

Theorem A, with $x = 0$, shows that as $n \to \infty$ the right hand side of expression (4) tends to $\sigma$.

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We shall now show that under all the conditions stated in the theorem, $0 < \lim_{n\to\infty} n^\sigma p_{00}^{(n)} < \infty$. From equation (2) we have

$$n^\sigma p_{00}^{(n)} = B(0)\prod_{m=1}^{n-1}\left(\frac{m+1}{m}\right)^\sigma B(A_m(0)) = B(0)\prod_{m=1}^{n-1} D_m$$

where $D_m = (1+1/m)^\sigma B(A_m(0))$. A necessary and sufficient condition for the required limit to exist and be finite and positive is that $-\infty < \sum_{m=1}^\infty (D_m-1) < \infty$. Theorem A shows that $1-A_m(0) = 1/(1-h_m+m\gamma)$ where $h_m = o(m)$ $(m\to\infty)$. Using this fact, a three term Taylor expansion of $B(x)$ to the left of $x = 1$, and the fact that $(1+1/m)^\sigma = 1+\sigma/m+O(1/m^2)$ enables us to write

$$\begin{aligned} D_m-1 &= \frac{\sigma}{m} - \frac{\beta}{m\gamma+1-h_m} + O(1/m^2) \\ &= \frac{\sigma(1-h_m)/\gamma}{m^2+m(1-h_m)/\gamma} + O(1/m^2) \end{aligned}$$

It is clear that $\sum_{m=1}^\infty (D_m-1)$ will converge to a finite limit if $\sum_{m=1}^\infty h_m/m^2$ does so. In fact Harris [3] shows that if $A'''(1-) < \infty$, that is $\delta \geqq 1$, $[h_m] = O(\log m)$ so in this case the series converges. We now consider the case $0 < \delta < 1$.
Using expression (3), the non-increasing nature of $\varepsilon(x)$ and the fact that $A_n(0)\uparrow 1$ $(n\to\infty)$, we obtain

$$\frac{-\gamma^2}{1-a_0}\sum_{m=1}^\infty \frac{1}{m^2}\sum_{k=0}^{m-1}(1-A_k(0)) \leqq \sum_{m=1}^\infty \frac{h_m}{m^2} \leqq \sum_{m=1}^\infty \frac{1}{m^2}\sum_{k=0}^{m-1}\varepsilon(A_k(0)) \tag{5}$$

For sufficiently large $n$ there exist positive constants $a$, $b$ such that $a/n < 1-A_n(0) < b/n$ and so we see that the terms of the series on the left of equation (5) are $O[(\log m)/m^2]$ for large $m$. Using the condition on $\varepsilon(x)$ given in the statement of the theorem and also that $0 \leqq \varepsilon(x) \leqq \gamma$ $(0 \leqq x < 1)$, it is not difficult to show that the terms on the right hand of equation (5) are $O(m^{-1-\delta})$ for large $m$. The proof is now complete.

## Remarks

1. By an argument very similar to that used in the first part of the proof of lemma 8 of Kesten et al. (1966), it follows that the condition on $\varepsilon(x)$ given in theorem 1 above may be replaced by $\sum_{j=1}^\infty a_j j^2 \log j < \infty$.

2. In obtaining the asymptotic form of $p_{00}^{(n)}$ we have not made use of irreducibility; we only require $A(0)$, $\gamma > 0$ and that $0 < B(0) < 1$, that is, there is positive probability of no immigrants in any generation.

3. By way of example, let $A(x) = 1/(2-x)$ and $B(x) = [1/(2-x)]^\nu$ $(\nu > 0)$. Using the fractional linear generating function in Harris [3] p. 9, we see that $p_{00}^{(n)} \sim n^{-\nu}$ $(n\to\infty)$. If instead we have a Poisson immigration component, that is $B(x) = e^{-\beta(1-x)}$, then $p_{00}^{(n)} \sim e^{-\eta\beta}n^{-\beta}$, where $\eta$ is Euler's constant.

Since the Markov chain is transient when $\sigma > 1$, the zero state is entered only finitely often with probability one, that is, with probability one, after a finite number of generations have passed the population size will always be positive. The situation when $\sigma \leqq 1$ is of course different, and the following theorem gives some information on this matter.

THEOREM 2. *Let* $\alpha = 1$, $0 < B(0), \beta, \gamma < \infty$, $B(0) < 1$ *and either let* $\varepsilon(x) = O[(1-x)^\delta]$ $(\delta > 0,\ 0 \leqq x < 1)$ *or* $\sum_{j=1}^\infty a_j j^2 \log j < \infty$, *so that* $p_{00}^{(n)} \sim Cn^{-\sigma}$ $(0 < C < \infty)$ *and let* $\sigma \leqq 1$. *Define the sequence* $\{U_n\}$ $(n = 1, 2, \cdots)$ *by* $U_n = C\Gamma(1-\sigma)n^{1-\sigma}$ *if* $\sigma < 1$ *and* $U_n = C \log n$ *if* $\sigma = 1$. *Finally let* $V(\cdot)$ *be the indicator function of the zero state, that is* $V(j) = 1$ *if* $j = 0$, *and* $V(j) = 0$ *otherwise. Then if* $0 < \sigma \leqq 1$, *we have*

$$\lim_{n\to\infty} \Pr\left\{\frac{1}{U_n}\sum_{m=0}^n V(X_m) \leqq x\right\} = G_{1-\sigma}(x)$$

*where* $G_\xi(x)$ *is the Mittag-Leffler distribution function given by*

$$G_\xi(x) = \frac{1}{\pi\xi}\int_0^x \sum_{j=0}^\infty \frac{(-)^{j-1}}{j!} \sin(\pi\xi j)\Gamma(\xi j+1)y^{j-1}dy$$

*if* $1 > \xi \geqq 0$.

PROOF. Use of the inequality

$$\int_\nu^{n+1} x^{-\sigma}dx < \sum_{m=\nu}^n m^{-\sigma} < \int_{\nu-1}^n x^{-\sigma}dx$$

where $\nu$, $n$ are positive integers and $n > \nu$ shows that

$$\sum_{m=0}^n p_{00}^{(m)} \sim \begin{cases} Cn^{1-\sigma}/(1-\sigma) & \text{if } \sigma < 1 \\ C\log n & \text{if } \sigma = 1 \end{cases} \quad (n \to \infty)$$

An Abelian theorem then shows that

$$\sum_{n=0}^\infty p_{00}^{(n)} x^n \sim \begin{cases} C\Gamma(1-\sigma)(1-x)^{-1+\sigma} & \text{if } \sigma < 1 \\ -C\log(1-x) & \text{if } \sigma = 1 \end{cases} \quad (x \uparrow 1)$$

The conditions of the occupation time theorem of Darling and Kac [1] are seen to be fulfilled and the theorem follows.

## 3. A limit theorem

It is easy to show from (1) that when $\alpha = 1$,

$$E(X_n|X_0 = i) = n\beta + i$$

and so it seems appropriate to investigate the limit in some sense of $X_n/n$ as $n \to \infty$. If we consider convergence in distribution, the following theorem shows that we get a non-trivial result.

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THEOREM 3. *Let* $\alpha = 1$ *and* $0 < \gamma, \beta, \sigma < \infty$. *Then the sequence* $\{X_n/n\}$ $(n = 1, 2, \cdots)$ *tends in distribution to the gamma variate having the density function*

$$f(t) = \frac{1}{\beta\Gamma(\sigma)}\left(\frac{t}{\beta}\right)^{\sigma-1} e^{-t/\beta} \qquad (t > 0)$$

PROOF. It follows from equation (1) that $(\theta > 0)$,

$$\psi_i^{(n)}(\theta) = E(e^{-\theta X_n/n}|X_0 = i) = [A_n(e^{-\theta/n})]^i \prod_{m=0}^{n-1} B[A_m(e^{-\theta/n})]$$

Since $A_n(x) \uparrow 1$ $(n \to \infty)$, we see that the first term on the right tends to unity. Writing $b_{mn}(\theta) = B[A_m(e^{-\theta/n})]$, and using $\log(1-x) \geqq -x-x^2/(1-x)$ $(0 \leqq x < 1)$, we have

$$(6) \qquad \phi^{(n)}(\theta) = \log \psi_0^{(n)}(\theta) = \sum_{m=0}^{n-1} \log[1-(1-b_{mn}(\theta))]$$
$$= -\sum_{m=0}^{n-1}(1-b_{mn}(\theta))+R_1^{(n)}(\theta)$$

where

$$0 \geqq R_1^{(n)}(\theta) \geqq -\sum_{m=0}^{n-1}[1-b_{mn}(\theta)]^2/b_{mn}(\theta)$$
$$\geqq -[(1-b_{0n}(\theta))/b_{0n}(\theta)]\sum_{m=0}^{n-1}(1-b_{mn}(\theta))$$

since $b_{mn}(\theta)$ is non-decreasing in $m$ for fixed $n$ and $\theta$. Since $1-b_{0n}(\theta) \to 0$ $(n \to \infty)$ it is clear that $R_1^{(n)}(\theta) \to 0$ if the first expression on the right hand side of equation (6) has a finite limit as $n \to \infty$.

For $0 \leqq x < 1$, we have $1-B(x) = \beta(1-x)-(1-x)f(x)$ where $0 \leqq f(x) = (1-x)B''(\xi)/2$ and $x < \xi < 1$ and $f(x) = o(1)$ as $x \uparrow 1$. Then letting $a_{mn}(\theta) = A_m(e^{-\theta/n})$, equation (6) becomes

$$\phi^{(n)}(\theta) = -\beta\sum_{m=0}^{n-1}(1-a_{mn}(\theta))+R_1^{(n)}(\theta)+R_2^{(n)}(\theta)$$

where

$$0 \leqq R_2^{(n)}(\theta) = \sum_{m=0}^{n-1}(1-a_{mn}(\theta))f[a_{mn}(\theta)]$$
$$\leqq (1-e^{-\theta/n})\sum_{m=0}^{n-1} f(a_{m1}(\theta))$$

since $f(x)$ is non-increasing. The last expression approaches zero $(n \to \infty)$ since $1-e^{-\theta/n} \sim \theta/n$ and $f[a_{n1}(\theta)] = o(1)$ $(n \to \infty)$.

Theorem A shows that we can write

$$1-A_m(x) = \frac{1-x}{1+\gamma m(1-x)}[1+g_m(x)]$$

where $g_m(x) \to 0$ uniformly in $0 \leqq x < 1$ $(m \to \infty)$ and $g_m(1) = 0$. Thus we have

$$\phi^{(n)}(\theta) = -\beta \sum_{m=0}^{n-1} \frac{1-e^{-\theta/n}}{1+\gamma m(1-e^{-\theta/n})} + R_1^{(n)}(\theta) + R_3^{(n)}(\theta) + o(1) \qquad (n \to \infty)$$

where

$$R_3^{(n)}(\theta) = -\beta \sum_{m=0}^{n} \frac{1-e^{-\theta/n}}{1+\gamma m(1-e^{-\theta/n})} g_m(e^{-\theta/n}) \tag{7}$$

It follows from the uniform convergence of the $g_m(\cdot)$ that there exists $M(\varepsilon)$ such that $|g_m(e^{-\theta/n})| < \varepsilon$ $(n = 1, 2, \cdots)$ if $m > M(\varepsilon)$. Breaking the summation in expression (7) into the form $\sum_{m=0}^{M(\varepsilon)} + \sum_{m=M(\varepsilon)+1}^{n}$ and using the fact that

$$[1-e^{-\theta/n}]/[1+\gamma m(1-e^{-\theta/n})] \leqq 1-e^{-\theta/n} \sim \theta/n \qquad (n \to \infty)$$

shows that $R_3^{(n)}(\theta) = o(1)$ $(n \to \infty)$.

It is easily seen that

$$0 \leqq \frac{\theta/n}{1+\gamma m\theta/n} - \frac{1-e^{-\theta/n}}{1+\gamma m(1-e^{-\theta/n})} \leqq \frac{\theta^2}{2n^2}$$

so that finally we have

$$\phi^{(n)}(\theta) = -\beta \sum_{m=0}^{n-1} \frac{\theta/n}{1+\gamma m\theta/n} + R_1^{(n)}(\theta) + o(1) \qquad (n \to \infty)$$

The sum in this expression can be recognized as an upper Darboux sum of the Riemann integral

$$\theta \int_0^1 (1+\gamma\theta x)^{-1} dx,$$

so that we obtain

$$\phi^{(n)}(\theta) = -\sigma \log(1+\gamma\theta) + o(1) \qquad (n \to \infty)$$

and thus $\lim_{n\to\infty} \psi_i^{(n)}(\theta) = (1+\gamma\theta)^{-\sigma}$ which is the Laplace transform of the density function given in the statement of the theorem. The convergence in distribution assertion follows from the continuity theorem for Laplace-Stieltjes transforms.

It is clear that the theorem is true under the conditions given in remark 2 following theorem 1.

*Added in proof.* Since submitting this paper, the author has learned that theorem 3 was obtained independently by E. Seneta in *J. Roy. Stat. Soc.* 32 B (1970), 149–52.

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