CLOSE ENCOUNTERS

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ABSTRACT

The mathematical treatment of encounters is discussed briefly, and it is pointed out that in a spherical system with an isotropic distribution function and equal-mass stars, the relevant equations can be analytically orbit-averaged. Even in small systems, close encounters have little effect on core collapse. Near a black hole, however, close encounters may cause an accumulation of stars on large, very eccentric orbits.

1. INTRODUCTION

It is well known that two-body relaxation of large stellar systems is dominated by the effects of distant encounters (in three spatial dimensions--in two dimensions, close encounters dominate [Rybicki 1970]). The Fokker-Planck equation, which is based on the approximation that scatterings are weak and frequent, is particularly appropriate for inverse-square forces and has been very successfully applied to the dynamical evolution of star clusters. Why should one be interested in close gravitational encounters?

First of all, the relaxation effects of close encounters are smaller than those of weak ones by a factor which scales only logarithmically with N , the number of particles in the system. Therefore it is important to check that omission of close encounters from evolutionary calculations does not make for important quantitative errors. Secondly, we may sometimes be interested in processes that occur only by large energy transfers in a single encounter: for example, the ejection of stars with a finite velocity at infinity. Thirdly, the formalism that has been invented to treat close encounteres is of intrinsic interest. This formalism has a surprising simplicity, which must be traceable to the fact that the keplerian two-body problem can be solved exactly.

After briefly indicating the principal formulae, I will pass on to

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some applications. This is strictly a review talk, everything here has appeared in the literature.

2. FORMALISM

The statistical treatment of close encounters starts from $R(\vec{v},\vec{v}')$, the probability per unit time that the test star suffers an encounter changing its velocity from $\vec{v}$ to $\vec{v}' = \vec{v} + \Delta\vec{v}$ per unit $d^3\Delta\vec{v}$. An explicit expression for R is

$$R(\vec{v},\vec{v}') = \frac{4G^2}{|\Delta v|^5} \int dm_F\, m_F^2 \int_{\vec{w}\cdot\Delta\vec{v}=0} d^2\vec{w}\, f_F\left(\vec{w} + \vec{v} + \Delta\vec{v}\,\frac{m_T+m_F}{2m_F},\, m_F\right), \tag{1}$$

where $f_F(\vec{v}_F,m_F)$ is the local distribution of field stars in velocity and mass, and is normalized to the total number density of field stars.

Equation (1) is exact in the sense that $\Delta\vec{v}$ is not assumed to be small compared to $\vec{v}$. Notice that the velocity integral is over a plane in velocity space perpendicular to $\Delta\vec{v}$ (but not, in general, passing through the origin). This is a consequence of energy and momentum conservation. It is, however, a remarkable peculiarity of the inverse-square law that the jacobian is independent of the integration variables.

Formulae equivalent to equation (1) have been given by Agekian (1959) and Hénon (1960a) for isotropic f_F and by Ipser and Semanzato (1983) and Goodman (1983) in the general case. Gryzinski (1959; 1965a,b) treated Coulomb scattering at great length, and his methods appear to be equivalent to equation (1).

The time evolution of a distribution $f_T(\vec{v},\vec{x},t;m_T)$ of test particles embedded in the field distribution f_F is governed by

$$\frac{Df_T(\vec{v},\vec{x},t;m_T)}{Dt} = \int d^3\vec{v}'\, f_T(\vec{v}',\vec{x},t;m_T)\, R(\vec{v}',\vec{v}) - f_T(\vec{v},\vec{x},t;m_T) \int d^3\vec{v}'\, R(\vec{v},\vec{v}'), \tag{2}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial\vec{x}} - \frac{\partial\Phi}{\partial\vec{x}}\cdot\frac{\partial}{\partial\vec{v}} \tag{3}$$

is the convective derivative along the unperturbed test-star orbits in the smooth mean cluster potential Φ . I will refer to (2) as the "master equation", but it is really just a form of the Boltzmann equation. The Fokker-Planck equation (cf. Rosenbluth, MacDonald and Judd 1957) can be derived from the master equation by expanding R through second order in $\Delta\vec{v}$.

The master equation (2) is cumbersome for evolutionary calculations because the collisional term on the righthand side fluctuates along an

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eccentric orbit, while the average strength of the operator is so small that many orbits must be followed before f_T changes significantly. In the parlance of numerical analysis, the master equation is "stiff": it contains two very different timescales. The remedy is to average over the short timescale and obtain a reduced equation for the long-term evolution. The possibility of doing this is the most important numerical advantage that statistical methods, whether based on the Fokker-Planck equation or on the master equation, have over direct N-body schemes.

Hénon (1961) and more recently Cohn (1980) showed that the orbit average can be done analytically for the Fokker-Planck equation if f_F and f_T are assumed to be functions of time and binding energy $E = \phi - v^2/2$ only, where $\phi \equiv -\Phi$ is defined to be positive in the cluster. It has been shown (Goodman 1983) that the same can be done for the scattering kernel and for the master equation if all stars have the same mass, m. The orbit-averaged rate of scatterings from E to $E' = E + \Delta E$ per unit E'; denoted by $K(E,E')$, is

$$p(E)K(E,E') = \frac{8\pi^2G^2m^2}{|\Delta E|^3}\Bigg\{[2q(E) - \Delta Ep(E)]\int_{E_{min}}^{E'} dE_F\, f(E_F) + \int_{E'}^{\phi(0)+\Delta E} dE_F\, f(E_F)[2q(E_F - \Delta E) - \Delta Ep(E_F - \Delta E)]\Bigg\} \quad \text{if } E' < E,$$

$$= \frac{8\pi^2G^2m^2}{|\Delta E|^3}\Bigg\{[2q(E') + \Delta Ep(E')]\int_{E_{min}}^{E'} dE_F\, f(E_F) + \int_{E'}^{\phi(0)} dE_F\, f(E_F)[2q(E_F) + \Delta Ep(E_F)]\Bigg\} \quad \text{if } E' > E, \tag{4}$$

in terms of the functions $q(E)$ and $p(E)$, where $4\pi^2q(E)$ is the volume of phase space at binding energies between E and $\phi(0)$, and $p(E) \equiv -dq(E)/dE$:

$$q(E) \equiv \frac{4}{3}\int_0^{\phi^{-1}(E)} dr\, r^2[2\phi(r) - 2E]^{3/2}, \tag{5}$$

and E_{min} is the minimum binding energy in the cluster (often 0). From now on it will be assumed that the test and field stars represent the same population, so the subscript "F" has been omitted from f. The orbit-averaged master equation is

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$$\left[p(E)\frac{\partial}{\partial t} + \frac{\partial q}{\partial t}\frac{\partial}{\partial E}\right] f(E,t)$$
$$= \int dE'\ p(E')f(E')K(E,E') - p(E)f(E)\int dE'K(E,E') \ . \qquad (6)$$

The remaining convective term on the left represents the adiabatic adjustment of the distribution function to the changes in the mean cluster potential.

3. EVAPORATION FROM AN ISOLATED CLUSTER

Hénon (1969) computed the rate of escape of stars by close encounters from a Plummer model. For equal-mass stars, he found

$$N^{-1}\frac{dN}{dt} = -\frac{4.26 \times 10^{-3}}{N}\left[\frac{GM}{R_0^3}\right]^{1/2} ,$$

$$E^{-1}\frac{dE}{dt} = -\frac{1.34 \times 10^{-3}}{N}\left[\frac{GM}{R_0^3}\right]^{1/2} , \qquad (7)$$

where R_0 is the Plummer core radius, M the total mass, and E the total energy of the system. Most importantly, he found that the escape rate is considerably higher for unequal mass stars. For example, if the stars are divided into two equinumerous groups, the mass of every star in the first group being m_1 , and in the second m_2 , then the escape rate for $m_2/m_1 = 2$ is approximately 3 times greater than for $m_2/m_1 = 1$, and approximately 30 times greater for $m_2/m_1 = \infty$.

Hénon thought that close encounters were the only way that stars could escape from an isolated system, on the grounds that as stars diffuse to weaker and weaker binding energies, their orbit-averaged diffusion rates tend to zero (Hénon 1960b). Spitzer and Shapiro (1972) have shown, however, that escape does occur diffusively via an extended halo of radial orbits that continue to pass through the core. The diffusive mechanism is faster than ejection by close encounters because it enjoys the $\ln\Lambda$ factor.

4. EQUILIBRIUM DISTRIBUTION IN A SQUARE WELL

Petrovskaya (1970a, 1970b), Kaliberda (1971), and Retterer (1979) calculated "equilibrium" test-star distribution functions in a square-well potential with finite escape velocity. The field star distribution was taken to be either a maxwellian or a lowered maxwellian. Under these idealized conditions, the convective terms in the master equation vanish and the distribution function is independent of position, so that it is not necessary to perform the orbit-average.

I use the term "equilibrium" somewhat loosely. There can be no true equilibrium because the test stars gradually evaporate. These workers computed the most slowly-decaying eigenmode of the master equation: because the field-star distribution is fixed, the evolutionary equations are linear. The results for the test-star distribution agree well with Fokker-Planck calculations, although for $N \sim 10^2$-10^5 there is a slight ($\sim$10%) excess of stars in the tail relative to the Fokker-Planck solutions. Both the Fokker-Planck and the close-encounter solutions agree with a lowered maxwellian; the agreement is poorer for $m_T/m_F \gtrsim 0.5$ than for equal-mass stars.

5. CLOSE ENCOUNTERS IN CORE COLLAPSE

I have applied equation (6) to the evolution of a cluster of equal-mass stars (Goodman 1983, 1984), using a modified version of a computer program kindly lent to me by Haldan Cohn. In this code (Cohn 1980), the potential is recalculated at each time step to make it consistent with the distribution function.

Even with the simplifications made in deriving equation (6) from (2), evolutionary calculations with the master equation are expensive. I used a uniform grid of 100 cells in energy space, of which 10 cells corresponded to the core of this system. Since all energy changes computed were multiples of the cell width, the effective value of $p_{max}/p_{min} \equiv \Lambda$ for this system was 10 to 100; hence $\ln\Lambda = 2.3$-4.6 . This corresponds to a system with 25 particles in the core, and the relative importance of close encounters compared to distant ones is greatest for small N .

Nevertheless the results of this calculation agree very well with those of Cohn (1980), who used the orbit-averaged Fokker-Planck equation. Cohn found that the late stages of core collapse proceed homologously, as predicted by Lynden-Bell and Eggleton (1980). In this phase the radial density profile is a power law in the inner parts of the system:

$$\rho(r) \propto r^{-\beta} \quad \text{with} \quad \beta = 2.23 \ . \tag{8a}$$

He also found that the central three-dimensional velocity dispersion $v_m^2(0)$, central density $\rho(0)$, and central relaxation time $t_r(0)$ (as defined by Spitzer and Hart (1971)) obey

$$v_m^2(0) \propto \rho(0)^{\gamma} \quad \text{with} \quad \gamma = 0.10 \ , \tag{8b}$$

$$t_r(0)\,\frac{d\,\ln\rho(0)}{dt} = \xi \quad \text{with} \quad \xi = 3.6 \times 10^{-3} \ . \tag{8c}$$

My calculations confirm (8a) and (8b) to two figures. In (8c) I found $\xi = 2.5 \times 10^{-3}$ to 4.6×10^{-3} , depending on what I took for the effective value of $\ln\Lambda$, which of course enters the definition of $t_r(0)$.

These results constitute strong confirmation of the accuracy of the Fokker-Planck equation in describing strictly two-body relaxation in core collapse. Neither Cohn's calculation nor my own, however, takes any account of three-body effects, which must become very important once the number of particles in the core is less than 10^2.

6. CLOSE ENCOUNTERS NEAR A BLACK HOLE

Lin and Tremaine (1980) have considered the rate of ejection of stars from a stellar cusp surrounding a black hole.

The calculation is very easy if one makes the following assumptions: (i) The masses of the stars are equal. (ii) The smooth mean potential ϕ in the cusp is dominated by the hole, $\phi = GM_h/r$. (iii) The Bahcall and Wolf (1976) formula

$$f(E) \approx 2 \frac{n_0}{(2\pi\sigma_0^2)^{3/2}} \left[\frac{E}{\sigma_0^2}\right]^{1/4} \tag{9}$$

adequately describes the distribution function between $E = \sigma_0^2/2$ and a few times this value, where σ_0 is the one-dimensional velocity dispersion and n_0 the stellar number density in the surrounding cluster core.

Under these assumptions, the number of stars ejected from the cusp per unit time with positive energies at infinity per unit mass greater than E_0 is

$$\dot{N}(E_0) = \int_{E_0}^{\infty} dE \int_0^{\infty} dE' \; 4\pi^2 p(E')f(E')K(E',-E) \; . \tag{10}$$

In the keplerian potential of the black hole,

$$q(E') = \frac{\pi}{3\sqrt{2}} \frac{G^3M_h^3}{E'^{3/2}} \; . \tag{11}$$

Because p , q , and f are power laws, the integrals involved in computing K and S are elementary, and the result is (Lin and Tremaine 1980)

$$\dot{N}(E_0) = 110 \frac{G^5M_h^3m_*^2n_0^2}{\sigma_0^{11}} \left|\frac{E_0}{\sigma_0^2}\right|^{-1} \; . \tag{12}$$

It is convenient to re-express this result in dimensionless form in terms of the total number of stars in the cusp

$$N_c = 4\pi^2 \int_{\sigma_0^2/2}^{\infty} dE \; p(E)f(E) \tag{13}$$

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and the reference relaxation time defined by Spitzer and Hart (1971) evaluated with the parameters appropriate to the core surrounding the cusp:

$$t_{rf} \frac{\dot{N}(E_0)}{N_c} = \frac{3.5}{\ln\Lambda} \left| \frac{E_0}{\sigma_0^2} \right|^{-1} . \qquad (14)$$

For situations of interest, $\ln\Lambda$ is ≤ 18 ; thus at least a fifth of the stars in the cusp will be ejected with $|E_0| \geq \sigma_0^2$, every relaxation time. (Relaxation times in dense galactic nuclei may be as short as a few $\times 10^9$ years.) Furthermore, Lin and Tremaine show that if we put $E_0 = \sigma_0^2$, the ejection rate (12) is $\gtrsim 3$ times larger than the consumption rate of the hole for typical globular cluster parameters with $M = 10^3 M_\odot$. The same can be true of a galactic nucleus.

Why is the ejection rate so large? Principally because, under the $E^{1/4}$ law, at any given point in the cusp most of the stars are already close to the escape velocity: 84% have $v > 0.5 v_{esc}$ and 19% have $v > 0.9 v_{esc}$. From the details of the calculation it can be seen that the main contribution to $\dot{N}(E_0)$ comes from stars with binding energies $E' \gtrsim |E_0|/12$. On the other hand, because the maximum energy that can be gained in a local close encounter is v_{esc}^2 , these stars suffer the encounters that eject them near pericenter where $v_{esc}^2/2 > E' + |E|$. Thus the potential well of the black hole serves as a catalyst for ejection by permitting the stars to gain large kinetic energies which are then occasionally exchanged in close encounters.

As stressed to me by Jerry Ostriker, the population of eccentric orbits by this mechanism tends to exaggerate the apparent mass of the black hole, because when one looks straight at the cusp, these stars are moving directly along the line of sight, and one sees their full space velocity. Indeed, Duncan and Wheeler (1980) and Binney and Mamon (1982) have argued that the cusp in M87 could be explained by velocity anisotropy alone without a black hole.

It should be pointed out, however, that for values of σ_0 that are typical of dense galactic nuclei, the 90° deflection distance is comparable to a stellar diameter; thus physical collisions will probably influence the ejection rate.

REFERENCES

Agekian, T.A. 1959, Sov. Astr.--A.J. 3, 46.
Bahcall, J.N. and Wolf, R.A. 1976, Ap. J. 209, 214.
Binney, J. and Mamon, G.A. 1982, M.N.R.A.S. 200, 361.
Cohn, H. 1980, Ap. J. 242, 765.
Duncan, M.J. and Wheeler, J.C. 1980, Ap. J. 237, L27.
Goodman, J. 1983, Ap. J. 270, 700; (errata) 1984, Ap. J. 278, 893.
Gryzinski, M. 1959, Phys. Rev. 115, 374.
Gryzinski, M. 1965a, Phys. Rev. 138, A305.
Gryzinski, M. 1965b, Phys. Rev. 138, A322.
Hénon, M. 1960a, Ann. d'Ap. 23, 467.

Hénon, M. 1960b, Ann. d'Ap. 23, 668.
Hénon, M. 1961, Ann. d'Ap. 24, 369.
Hénon, M. 1969, Astr. Ap. 2, 151.
Ipser, J. and Semanzato, R. 1983, Ap. J. 271, 294.
Kaliberda, V.S. 1971, Sov. Astr.--A.J. 14, 773.
Lin, D.N.C. and Tremaine, S. 1980, Ap. J. 242, 789.
Lynden-Bell, D. and Eggleton, P.P. 1980, M.N.R.A.S. 191, 483.
Petrovskaya, I.V. 1970a, Sov. Astr.--A.J. 13, 647.
Petrovskaya, I.V. 1970b, Sov. Astr.--A.J. 13, 957.
Retterer, J.M. 1979, A.J. 84, 370.
Rosenbluth, M.N., MacDonald, W.M. and Judd, D.L. 1957, Phys. Rev. 107,1.
Rybicki, G.B. 1970, in I.A.U. Coll. No. 10, Gravitational N-Body Problem, ed. M. Lecar, p. 22.
Spitzer, L. and Hart, M.H. 1971, Ap. J. 164, 399.
Spitzer, L. and Shapiro, S.L. 1972, Ap. J. 173, 529.

DISCUSSION

INAGAKI: You got a very good agreement with Haldan Cohn's results. Does the homologous evolution continue even though the number of particles in the core becomes less than 100?

GOODMAN: Yes, it does, in my numerical calculations. Many of the assumptions upon which the calculations are based break down, however, when $N_{core} < 100$. For example, the relaxation time becomes comparable to the dynamical time, so that the validity of orbit-averaged equations is questionable in this regime. More importantly, three-body interactions become important for $N_{core} < 100$, whereas both Cohn's calculations and my own neglect these effects. My calculation shows merely that the increasing importance of close two-body encounters as N_{core} decreases does not disturb the homological evolution.

SEVERNE: In comparing the master equation for close encounters and the Fokker-Planck equation, it would seem to me that the former has the sounder theoretical foundation and thus justifies the latter, rather than inversely. Indeed, both equations contain the basic assumption that velocity transfer is local and instantaneous. This assumption is however well satisfied only in the case of close encounters. Could you comment on this?

GOODMAN: The approximation of local and instantaneous encounters is very reasonable, even for the Fokker-Planck equation. In a self-gravitating system of N particles and half-mass radius R, the close-encounter distance is of order R/N, and the mean interparticle distance is of order $(R/N)^{1/3}$. Since the Chandrasekhar formula weights all logarithmic intervals in impact parameter equally, two thirds of the effects of encounters can therefore be attributed to impact parameters smaller than the mean interparticle distance; these encounters are certainly local. In fact, the inhomogeneity of the system is probably important only for impact parameters of order R/e or larger, and if these are incorrectly represented by the local impulse approximation, the relative error in the calculated relaxation rate is of order $(\ln N)^{-1}$.