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A NOTE ON GUNNINGHAM'S FORMULA

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Abstract

Gunningham ['Spin Hurwitz numbers and topological quantum field theory', *Geom. Topol.* **20**(4) (2016), 1859–1907] constructed an extended topological quantum field theory (TQFT) to obtain a closed formula for all spin Hurwitz numbers. In this note, we use a gluing theorem for spin Hurwitz numbers to re-prove Gunningham's formula. We also describe a TQFT formalism naturally induced by the gluing theorem.

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1. Introduction

Let X be a surface of general type with a smooth canonical divisor D. The complex curve D has genus $h = K_X^2 + 1$ and the normal bundle N to D is a theta characteristic on D (that is, $N^2 = K_D$) with $p \equiv h^0(N) \equiv \chi(\mathcal{O}_X) \pmod 2$. The pair (D, N) is called a *spin curve* of genus h with parity p. The Gromov–Witten (GW) invariants of X are the same as the local GW invariants of the spin curve (D, N) that depend only on (h, p). In particular, for $d > 0$ the dimension zero local GW invariant of the spin curve (D, N) is given by the formula

$$GT_d^{h,p} = \sum_f \frac{(-1)^{h^0(f^*N)}}{|\mathrm{Aut}(f)|}, \tag{1.1}$$

where the sum is over all degree d étale covers f (see [10, 11, 13]). One can calculate these local invariants by extending the (weighted) signed sum to certain ramified covers, which are the spin Hurwitz numbers.

A partition $\alpha \vdash d$ is odd if all parts in α are odd. We set

$$\mathrm{OP}(d) = \{\alpha \vdash d : \alpha \text{ is odd}\}.$$

Fix k points $y^1, \ldots, y^k$ in D and consider degree d holomorphic maps $f : C \to D$ from possibly disconnected curves C of Euler characteristic $\chi(C)$ that are ramified only

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over the fixed points y^i with ramification profile $\alpha^i = (\alpha^i_1, \dots, \alpha^i_{\ell_i}) \in \mathrm{OP}(d)$. By the Riemann–Hurwitz formula, the (ramified) covers f satisfy

$$d\chi(D) - \chi(C) + \sum_{i=1}^{k} (\ell(\alpha^i) - d) = 0, \tag{1.2}$$

where $\chi(D) = 2 - 2h$ and $\ell(\alpha^i)$ is the length of the partition α^i. By the Hurwitz formula, the twisted line bundle

$$N_f = f^*N \otimes O_C\Big(\sum_{i,j} \frac{1}{2}(\alpha^i_j - 1)x^i_j\Big) \tag{1.3}$$

is a theta characteristic on C where $f^{-1}(y^i) = \{x^i_j\}$ and f has multiplicity α^i_j at x^i_j. We define the parity $p(f)$ of a map f as

$$p(f) \equiv h^0(N_f) \pmod 2.$$

Given $\alpha^1, \dots, \alpha^k \in \mathrm{OP}(d)$, the spin Hurwitz number of genus h and parity p is defined as a (weighted) sum of covers f satisfying (1.2) with sign determined by the parity $p(f)$:

$$H^{h,\pm}_{\alpha^1,\dots,\alpha^k} = \sum_f \frac{(-1)^{p(f)}}{|\mathrm{Aut}(f)|} \tag{1.4}$$

where + or − denotes the parity of the spin curve (D, N). If $k = 0$ (or unramified) then this is the étale spin Hurwitz number that equals the local invariant (1.1). We will call $\chi(C)$ in (1.2) the *domain Euler characteristic* for the spin Hurwitz number (1.4).

Eskin, Okounkov and Pandharipande [5] first studied the spin Hurwitz numbers for genus $h = 1$ with trivial theta characteristic, that is, $(h, p) = (1, -)$. They related the parity of maps to combinatorics of the Sergeev group $\mathsf{C}(d)$. A partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of d is strict if $\lambda_1 > \cdots > \lambda_\ell$. Let $\mathrm{SP}(d)$ denote the set of strict partitions of d and set

$$\mathrm{SP}^+(d) = \{\lambda \in \mathrm{SP}(d) : \ell(\lambda) \text{ is even}\} \quad \text{and} \quad \mathrm{SP}^-(d) = \{\lambda \in \mathrm{SP}(d) : \ell(\lambda) \text{ is odd}\}.$$

The irreducible spin $\mathsf{C}(d)$-supermodules V^λ are indexed by strict partitions $\lambda \in \mathrm{SP}(d)$ and the conjugacy class corresponding to an odd partition $\alpha^i \in \mathrm{OP}(n)$ acts in V^λ as multiplication by a constant, which is the central character $\mathbf{f}_{\alpha^i}(\lambda)$ (see Section 2).

Theorem 1.1 [5]. *With the notation as above,*

$$H^{1,-}_{\alpha^1,\dots,\alpha^k} = 2^{\chi(C)/2}\Big(\sum_{\lambda\in\mathrm{SP}^+(d)} \prod_i \mathbf{f}_{\alpha^i}(\lambda) - \sum_{\lambda\in\mathrm{SP}^-(d)} \prod_i \mathbf{f}_{\alpha^i}(\lambda)\Big), \tag{1.5}$$

where $\chi(C)$ is the domain Euler characteristic.

Recently, Gunningham [6] constructed a fully extended (spin) topological quantum field theory (TQFT). His extended TQFT gives a formula for all spin Hurwitz numbers. For each strict partition $\lambda \in \mathrm{SP}(d)$, let V^λ be as above and set

$$c_\lambda = \frac{\dim V^\lambda}{|\mathsf{C}(d)|}. \tag{1.6}$$

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THEOREM 1.2 [6].

$$H^{h,\pm}_{\alpha^1,\ldots,\alpha^k} = 2^{(\chi(C)+\chi(D))/2}\Big(\sum_{\lambda\in \mathrm{SP}^+(d)} 2^{(\chi(D))/2} c_\lambda^{\chi(D)} \prod_i \mathbf{f}_{\alpha^i}(\lambda) \pm \sum_{\lambda\in \mathrm{SP}^-(d)} c_\lambda^{\chi(D)} \prod_i \mathbf{f}_{\alpha^i}(\lambda)\Big), \quad (1.7)$$

where $\chi(D) = 2 - 2h$ and $\chi(C)$ is the domain Euler characteristic.

Independently, Parker and the author [12] adapted the degeneration method of the GW theory to obtain a gluing theorem for spin Hurwitz numbers. For a partition $\gamma \vdash d$, let $\gamma(k)$ be the number of parts of size k in γ and set

$$z_\gamma = \prod_k k^{\gamma(k)}\gamma(k)!. \quad (1.8)$$

THEOREM 1.3 [12]. *Let $\alpha^1,\ldots,\alpha^s,\beta^1,\ldots,\beta^r \in \mathrm{OP}(d)$. Then*

$$H^{h,p}_{\alpha^1,\ldots,\alpha^s,\beta^1,\ldots,\beta^r} = \sum_{\gamma\in\mathrm{OP}(d)} z_\gamma \cdot H^{h_1,p_1}_{\alpha^1,\ldots,\alpha^s,\gamma} \cdot H^{h_2,p_2}_{\beta^1,\ldots,\beta^r,\gamma},$$

$$H^{h+1,p}_{\alpha^1,\ldots,\alpha^s} = \sum_{\gamma\in\mathrm{OP}(d)} z_\gamma \cdot H^{h,p}_{\alpha^1,\ldots,\alpha^s,\gamma,\gamma},$$

where $h = h_1 + h_2$ and $p \equiv p_1 + p_2$ (mod 2).

Our main goal is to re-prove Gunningham's formula (1.7). To that end, we need to calculate the $(h, p) = (0, +)$ spin Hurwitz numbers.

Section 2 gives a brief review of the representation theory of the Sergeev group $\mathsf{C}(d)$ and a key fact (Lemma 2.2) about the central characters of $\mathsf{C}(d)$.

Section 3 follows the approach of [5] to show

$$H^{0,+}_{\alpha^1,\ldots,\alpha^k} = 2^{(\chi(C)+2)/2}\Big(\sum_{\lambda\in \mathrm{SP}^+(d)} 2c_\lambda^2 \prod_i \mathbf{f}_{\alpha^i}(\lambda) + \sum_{\lambda\in \mathrm{SP}^-(d)} c_\lambda^2 \prod_i \mathbf{f}_{\alpha^i}(\lambda)\Big). \quad (1.9)$$

In Section 4 we use the gluing theorem with (1.5) and (1.9) to prove the formula (1.7). We also observe that the formula (1.7) gives the gluing theorem (see Remark 4.2).

The spin Hurwitz numbers are not defined for $(h, p) = (0, -)$ since the only theta characteristic on $\mathbb{P}^1$ is $\mathcal{O}(-1)$. In Section 5 we first extend the gluing theorem to include the case $(h, p) = (0, -)$ and then describe a TQFT formalism naturally induced by the (extended) gluing theorem.

2. Representations of the Sergeev group

This section reviews the representation theory of the Sergeev group relevant to our discussion. We generally follow the notation and terminology of [5]. For proofs and more details, we refer to [7, 8, 14] and [4, Ch. 3] and references therein.

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2.1. Sergeev group. The Sergeev group $\mathsf{C}(d)$ is the semidirect product

$$\mathsf{C}(d) = \mathsf{Cliff}(d) \rtimes \mathsf{S}(d),$$

where $\mathsf{Cliff}(d)$ is the Clifford group generated by $\xi_1, \ldots, \xi_d$ and a central element ϵ subject to the relations

$$\xi_i^2 = 1, \quad \epsilon^2 = 1, \quad \xi_i\xi_j = \epsilon\xi_j\xi_i \quad (i \neq j),$$

and the symmetric group $\mathsf{S}(d)$ on d letters acts on $\mathsf{Cliff}(d)$ by permuting the ξ_i.

The group $\mathsf{C}(d)$ is a double cover of the hyperoctahedral group $\mathsf{B}(d) = \mathbb{Z}_2^d \rtimes \mathsf{S}(d)$. Since $\mathsf{Cliff}(d)/\{1, \epsilon\} \cong \mathbb{Z}_2^d$, setting $\epsilon = 1$ gives a short exact sequence of groups

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathsf{C}(d) \overset{\theta}{\longrightarrow} \mathsf{B}(d) \longrightarrow 0.$$

The group $\mathsf{B}(d)$ embeds in the symmetric group $\mathsf{S}(2d)$ on the set $\{\pm 1, \ldots, \pm d\}$ via

$$\xi_i g(\pm k) = \begin{cases} \mp g(k) & \text{if } g(k) = i, \\ \pm g(k) & \text{if } g(k) \neq i. \end{cases}$$

Notice that $\mathsf{B}(d)$ is the centraliser of the involution $k \to -k$ in $\mathsf{S}(2d)$.

2.2. Conjugacy classes. The symmetric group $\mathsf{S}(d)$ embeds in $\mathsf{B}(d)$ and $\mathsf{C}(d)$. An element g of $\mathsf{B}(d)$ and $\mathsf{C}(d)$ is a *pure permutation* if $g \in \mathsf{S}(d)$. Define a $\mathbb{Z}_2$-grading on $\mathsf{C}(d)$ by setting

$$\deg \xi_i = 1, \quad \deg(g) = \deg(\epsilon) = 0, \quad g \in \mathsf{S}(d). \tag{2.1}$$

An even (respectively, odd) conjugacy class is a conjugacy class of an even (respectively, odd) element. For each conjugacy class C, either $C \cap \epsilon C = 0$ or $C = \epsilon C$.

(a) Let C_γ be the conjugacy class in $\mathsf{C}(d)$ of a pure permutation g of cycle type $\gamma \in \mathrm{OP}(d)$. Then C_γ and ϵC_γ are disjoint even conjugacy classes and

$$|C_\gamma| = |\epsilon C_\gamma| = \frac{|\mathsf{C}(d)|}{2^{\ell(\gamma)+1} z_\gamma} \tag{2.2}$$

where z_γ is the order of the centraliser of g in $\mathsf{S}(d)$ given by (1.8).

(b) We can write all conjugacy classes of $\mathsf{C}(d)$ as

$$\underbrace{C_1, \epsilon C_1, \ldots, C_m, \epsilon C_m,}_{\text{even}} \quad \underbrace{\hat{C}_1, \epsilon\hat{C}_1, \ldots, \hat{C}_q, \epsilon\hat{C}_q,}_{\text{odd}} \quad \underbrace{\tilde{C}_1, \ldots, \tilde{C}_s,}_{\epsilon\tilde{C}_i = \tilde{C}_i} \tag{2.3}$$

where $m = |\mathrm{OP}(d)| = |\mathrm{SP}(d)|$ and $q = |\mathrm{SP}^-(d)|$.

The denominator $2^{\ell(\gamma)+1} z_\gamma$ in (2.2) is the order of the centraliser of g in $\mathsf{C}(d)$.

Definition 2.1. For a partition $\gamma \vdash d$, we define

$$\vartheta_\gamma = 2^{\ell(\gamma)+1} z_\gamma.$$

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2.3. Spin $\mathsf{C}(d)$-supermodules. For a finite group G, let $G^\wedge$ denote the set of irreducible complex representations of G. The central element ϵ acts as multiplication by either $+1$ or -1 on each $V \in \mathsf{C}(d)^\wedge$. If ϵ acts on V as multiplication by 1, then $V \in \mathsf{B}(d)^\wedge$. Let $\mathsf{C}(d)^\wedge_-$ be the set of irreducible complex representations of $\mathsf{C}(d)$ on which ϵ acts as multiplication by -1. We have

$$\mathsf{C}(d)^\wedge = \mathsf{B}(d)^\wedge \cup \mathsf{C}(d)^\wedge_-. \tag{2.4}$$

The grading (2.1) makes the group algebra $\mathbb{C}[\mathsf{C}(d)]$ a semisimple associative superalgebra. A *spin* $\mathsf{C}(d)$*-supermodule* is a supermodule over $\mathbb{C}[\mathsf{C}(d)]$ on which ϵ acts as multiplication by -1. The irreducible (or simple) spin $\mathsf{C}(d)$-supermodules are indexed by strict partitions $\lambda \in \mathrm{SP}(d)$. For each $\lambda \in \mathrm{SP}(d)$, let V^λ be its corresponding irreducible spin $\mathsf{C}(d)$-supermodule.

(c) For $\lambda \in \mathrm{SP}^+(d)$, we have $V^\lambda \in \mathsf{C}(d)^\wedge_-$.
(d) For $\lambda \in \mathrm{SP}^-(d)$, we have $V^\lambda = V^\lambda_0 \oplus V^\lambda_1$ (as a module over $\mathbb{C}[\mathsf{C}(d)]$) such that $V^\lambda_0, V^\lambda_1 \in \mathsf{C}(d)^\wedge_-$ and they are not isomorphic.

2.4. Central characters. For $\lambda \in \mathrm{SP}(d)$, let ζ^λ denote the character of the irreducible $\mathbb{C}(d)$-supermodule V^λ. By (2.3), the character ζ^λ is determined by its values $\zeta^\lambda(C_\gamma) = -\zeta^\lambda(\epsilon C_\gamma)$ on even conjugacy classes C_γ and ϵC_γ where $\gamma \in \mathrm{OP}(d)$. For $\lambda, \mu \in \mathrm{SP}(d)$,

$$\langle \zeta^\lambda, \zeta^\mu \rangle = \sum_{\gamma \in \mathrm{OP}(d)} \frac{2}{\vartheta_\gamma} \zeta^\lambda(C_\gamma)\zeta^\mu(C_\gamma) = \begin{cases} \delta_{\lambda\mu} & \text{if } \lambda \in \mathrm{SP}^+(d), \\ 2\delta_{\lambda\mu} & \text{if } \lambda \in \mathrm{SP}^-(d), \end{cases} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the space of class functions of the finite group $\mathsf{C}(d)$.

For each $\gamma \in \mathrm{OP}(d)$, the class sum $\overline{C_\gamma} = \sum_{x \in C_\gamma} x$ has degree zero and lies in the centre of the superalgebra $\mathbb{C}[\mathsf{C}(d)]$, so it acts on V^λ as multiplication by a constant. This constant is the central character $\mathbf{f}_\gamma(\lambda)$ obtained from the formula

$$\mathbf{f}_\gamma(\lambda) = \frac{|C_\gamma|}{\dim V^\lambda} \zeta^\lambda(C_\gamma). \tag{2.6}$$

When $\lambda \in \mathrm{SP}^-(d)$, the central character $\mathbf{f}_\gamma(\lambda)$ of $V^\lambda = V^\lambda_0 \oplus V^\lambda_1$ is equal to the central characters of the irreducible representations $V^\lambda_0, V^\lambda_1 \in \mathsf{C}(d)^\wedge_-$.

Now (2.5) and (2.6) give a crucial fact for our later discussions.

LEMMA 2.2. *Let c_λ be as in (1.6). Then*

$$\sum_{\gamma \in \mathrm{OP}(d)} \vartheta_\gamma \mathbf{f}_\gamma(\lambda)\mathbf{f}_\gamma(\mu) = \begin{cases} \delta_{\lambda\mu}/2c_\lambda^2 & \text{if } \lambda \in \mathrm{SP}^+(d), \\ \delta_{\lambda\mu}/c_\lambda^2 & \text{if } \lambda \in \mathrm{SP}^-(d). \end{cases}$$

2.5. Centre. Let $C_1, \ldots, C_k$ be conjugacy classes in a finite group G and let $n(C_1, \ldots, C_k)$ be the number of solutions $(g_1, \ldots, g_k) \in C_1 \times \cdots \times C_k$ of the equation $g_1 \cdots g_k = 1$. Then

$$n(C_1, \ldots, C_k) = |G| \sum_{\lambda \in G^\wedge} \left(\frac{\dim V^\lambda}{|G|} \right)^2 \prod_i \mathbf{f}_{C_i}(\lambda), \tag{2.7}$$

where $\mathbf{f}_{C_i}(\lambda)$ are the central characters of V^λ (cf. [15, Theorem 7.2.1]).

The central element ϵ acts on the centre $\mathcal{Z}(\mathbb{C}[\mathsf{C}(d)])$ of the (ungraded) group algebra $\mathbb{C}[\mathsf{C}(d)])$ with ± 1 eigenvalues. We denote by

$$\mathcal{Z}_0^+ \subset \mathcal{Z} \tag{2.8}$$

the (-1)-eigenspace consisting of even-degree elements. This space has a basis

$$\{u_\gamma = \tfrac{1}{2}(\overline{C}_\gamma - \epsilon\overline{C}_\gamma) : \gamma \in \mathrm{OP}(d)\}.$$

For notational simplicity, we set $\mathbf{1} = (1^d)$. Since $u_\alpha u_\beta \in \mathcal{Z}_0^+$ and $u_{\mathbf{1}} u_\alpha = u_\alpha$ for all $\alpha, \beta \in \mathrm{OP}(d)$, the space $\mathcal{Z}_0^+$ is a commutative associative algebra with identity $u_{\mathbf{1}}$.

Now (c) and (d) in Section 2.3 and (2.7) give the following formula.

LEMMA 2.3. *If $u_\alpha u_\beta = \sum_\gamma a^\gamma_{\alpha\beta} u_\gamma$, then the structure constants $a^\gamma_{\alpha\beta}$ are given by*

$$a^\gamma_{\alpha\beta} = \vartheta_\gamma\Big(\sum_{\lambda\in\mathrm{SP}^+(d)} 2c_\lambda^2 \mathbf{f}_\alpha(\lambda)\mathbf{f}_\beta(\lambda)\mathbf{f}_\gamma(\lambda) + \sum_{\lambda\in\mathrm{SP}^-(d)} c_\lambda^2 \mathbf{f}_\alpha(\lambda)\mathbf{f}_\beta(\lambda)\mathbf{f}_\gamma(\lambda)\Big).$$

3. Calculation of genus-zero spin Hurwitz numbers

In this section, we calculate the spin Hurwitz numbers of genus $h = 0$, following the arguments of [5]. We generally follow the notation and terminology in [5].

3.1. Quadratic form. Consider a degree d map

$$f : C \to \mathbb{P}^1 \tag{3.1}$$

ramified only over fixed points $y^1, \dots, y^k \in \mathbb{P}^1$ with ramification profile $\alpha^i \in \mathrm{OP}(d)$ at y^i satisfying (1.2). Let $N = \mathcal{O}(-1)$ and let $L = N_f$ denote the theta characteristic on C defined by (1.3). For each (connected) component C_i of C where $1 \le i \le n$, the theta characteristic $L_i = L|_{C_i}$ on C_i determines a quadratic form q_{L_i} on the group $J_2(C_i)$ of elements of order two in the Jacobian of C_i by

$$q_{L_i}(\rho_i) \equiv h^0(L_i \otimes \rho_i) + h^0(L_i) \pmod 2$$

such that

$$(-1)^{h^0(L_i)} = 2^{-g(C_i)} \sum_{\rho_i \in J_2(C_i)} (-1)^{q_{L_i}(\rho_i)}.$$

For $\rho = (\rho_1, \dots, \rho_n)$ in $J_2(C) = J_2(C_1) \times \cdots \times J_2(C_n)$, let $q_L(\rho) = \sum\limits_i q_{L_i}(\rho_i)$. Then

$$(-1)^{p(f)} = \prod_i (-1)^{h^0(L_i)} = 2^{(\chi(C)/2)-n} \sum_{\rho\in J_2(C)} (-1)^{q_L(\rho)}. \tag{3.2}$$

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3.2. Canonical lift. Each $\rho \in J_2(C)$ defines an unramified double cover $C_\rho \to C$ which, when composed with f, gives a degree $2d$ cover

$$f_\rho : C_\rho \to \mathbb{P}^1.$$

Let σ be the fixed-point-free involution in the symmetric group $\mathsf{S}(2d)$ given by the covering transformation permuting the sheets of $C_\rho \to C$. By our construction, the monodromy group of f_ρ lies in the centraliser of the involution σ in $\mathsf{S}(2d)$, which is the group $\mathsf{B}(d)$ (see Section 2.1). The monodromy of f_ρ thus defines a homomorphism

$$M_{f_\rho} : \pi_1(\mathbb{P}^\times) \to \mathsf{B}(d), \tag{3.3}$$

where $\mathbb{P}^\times = \mathbb{P}^1 \setminus \{y^1, \ldots, y^k\}$.

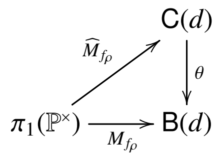
One can choose a small loop δ_i encircling only the branch point y_i such that

- $\pi_1(\mathbb{P}^\times) = \langle \delta_1, \ldots, \delta_k | \prod_i \delta_i = 1 \rangle$,
- $M_{f_\rho}(\delta_i)$ is conjugate to a pure permutation g_i of cycle type $\alpha^i \in \mathrm{OP}(d)$ in $\mathsf{B}(d)$.

Then by (a) in Section 2.2,

$$\theta^{-1}(M_{f_\rho}(\delta_i)) \subset C_{\alpha^i} \sqcup \epsilon C_{\alpha^i},$$

where C_{α^i} is the conjugacy class of the pure permutation g_i in the Sergeev group $\mathsf{C}(d)$. The monodromy of f_ρ is said to have a *canonical lift to* $\mathsf{C}(d)$ if there exists a homomorphism $\widehat{M}_{f_\rho} : \pi_1(\mathbb{P}^\times) \to \mathsf{C}(d)$ such that $\widehat{M}_{f_\rho}(\delta_i) \in C_{\alpha^i}$ for all i and the following diagram commutes:



The following fact is a special case of Theorem 1 of [5]: the case of $\mathbb{P}^1$.

Proposition 3.1. *$q_L(\rho) = 0$ if and only if the monodromy of f_ρ has a canonical lift to $\mathsf{C}(d)$.*

3.3. Weighted count. Let G be $\mathsf{S}(d)$, $\mathsf{B}(d)$ or $\mathsf{C}(d)$, and let C_{α^i} denote the conjugacy class of a pure permutation in G with cycle type $\alpha^i \in \mathrm{OP}(d)$. We set

$$M = \{\alpha^1, \ldots, \alpha^k\} \tag{3.4}$$

and denote by $H_G(M)$ the set of homomorphisms $\psi : \pi_1(\mathbb{P}^\times) \to G$ sending the conjugacy class of the loop δ_i into the conjugacy class C_{α^i}. Taking into account the action of G by conjugation, we set

$$h_G(M) = \frac{|H_G(M)|}{|G|}.$$

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The groups $\mathsf{B}(d)$ and $\mathsf{C}(d)$ have natural homomorphisms to $\mathsf{S}(d)$ by definition. Given a homomorphism $\phi \in H_{\mathsf{S}(d)}(M)$, let $H_G(M;\phi)$ be the set of homomorphisms $\psi \in H_G(M)$ with commutative diagram

$$\begin{array}{ccc} & & G \\ & \overset{\psi}{\nearrow} & \downarrow \\ \pi_1(\mathbb{P}^\times) & \xrightarrow[\phi]{} & \mathsf{S}(d) \end{array}$$

where $G \to \mathsf{S}(d)$ is the natural homomorphism. The weighted count of such homomorphisms is

$$h_G(M;\phi) = \frac{|H_G(M;\phi)|}{|G|}.$$

By (2.7) and the definition, $h_G(M;\phi)$ and $h_G(M)$ satisfy

$$\sum_{\phi \in H_{\mathsf{S}(d)}(M)} h_G(M;\phi) = h_G(M) = \sum_{\lambda \in G^\wedge} \left(\frac{\dim V^\lambda}{|G|}\right)^2 \prod_i \mathbf{f}_{C_i}(\lambda). \tag{3.5}$$

Remark 3.2. There is a bijection between ramified covers $f : C \to \mathbb{P}^1$ (as in (3.1)) and orbits of the action of $\mathsf{S}(d)$ on $H_{\mathsf{S}(d)}(M)$ by conjugation. This bijection is given by the monodromy $M_f : \mathbb{P}^\times \to \mathsf{S}(d)$ of the map f. The order of the stabiliser of M_f is $|\mathrm{Aut}(f)|$ and hence

$$h_{\mathsf{S}(d)}(M) = \frac{1}{|\mathsf{S}(d)|} \sum_{O_f} |O_f| = \sum_f \frac{1}{|\mathrm{Aut}(f)|} \tag{3.6}$$

where O_f is the orbit of M_f. This is the ordinary Hurwitz number that counts ramified covers of $\mathbb{P}^1$ with ramification data specified by M in (3.4).

Lemma 3.3. *Let $f : C \to \mathbb{P}^1$ and O_f be as in Remark 3.2 and let $\phi \in O_f$. If the domain C has n (connected) components $C_1, \ldots, C_n$, then*

$$|J_2(C)| = 2^{-d+n} |H_{\mathsf{B}(d)}(M;\phi)|.$$

Proof. The proof is identical to that of Lemma 3 in [5]. Assigning to each $\rho \in J_2(C)$ the monodromy M_{f_ρ} given in (3.3) defines a bijection between $J_2(C)$ and orbits of the action of $\mathbb{Z}_2^d \subset \mathsf{B}(d)$ on $H_{\mathsf{B}(d)}(M;\phi)$ by conjugation. Let $\rho = (\rho_1, \ldots, \rho_n)$ and $C_{\rho_i} \to C$ be the double cover determined by ρ_i in $J_2(C_i)$. Then the stabiliser of M_{f_ρ} is generated by $\sigma_1, \ldots, \sigma_n$, where σ_i is the involution permuting the sheets of $C_{\rho_i} \to C_i$. So, every orbit of the action of $\mathbb{Z}_2^d$ on $H_{\mathsf{B}(d)}(M;\phi)$ has 2^{d-n} elements and hence

$$|H_{\mathsf{B}(d)}(M;\phi)| = \sum_{\text{orbits}} 2^{d-n} = \sum_{\rho \in J_2(C)} 2^{d-n}.$$

This completes the proof of the lemma. □

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3.4. Proof of (1.9). Let $\phi \in H_{\mathsf{S}(d)}(M)$ be as in Lemma 3.3. By our choice of the conjugacy classes C_{α^i} in G, the homomorphism $\theta : \mathsf{C}(d) \to \mathsf{B}(d)$ induces a one-to-one function

$$H_{\mathsf{C}(d)}(M;\phi) \to H_{\mathsf{B}(d)}(M;\phi). \tag{3.7}$$

Moreover, by Proposition 3.1, $q_L(\rho) = 0$ if and only if the monodromy M_{f_ρ} lies in the image of (3.7). Using $|\mathsf{C}(d)| = 2|\mathsf{B}(d)| = 2^{d+1}d!$, (3.2) and Lemma 3.3,

$$\begin{aligned}(-1)^{p(f)} &= 2^{(\chi(C)/2)-n}\sum_{\rho\in J_2(C)}(-1)^{q_L(\rho)} = 2^{(\chi(C)/2)-d}(2|H_{\mathsf{C}(d)}(M;\phi)| - |H_{\mathsf{B}(d)}(M;\phi)|)\\ &= 2^{\chi(C)/2}d![4h_{\mathsf{C}(d)}(M;\phi) - h_{\mathsf{B}(d)}(M;\phi)].\end{aligned} \tag{3.8}$$

Now it follows that

$$\begin{aligned}H^{0,+}_{\alpha^1,\cdots,\alpha^k} &= \sum_f \frac{(-1)^{p(f)}}{|\mathrm{Aut}(f)|} = \sum_{\phi\in H_{\mathsf{S}(d)}(M)} 2^{\chi(C)/2}[4h_{\mathsf{C}(d)}(M;\phi) - h_{\mathsf{B}(d)}(M;\phi)]\\ &= 2^{\chi(C)/2}\sum_{\lambda\in C(d)^\wedge_-} 2^2\Big(\frac{\dim V^\lambda}{|\mathsf{C}(d)|}\Big)^2\prod_i \mathbf{f}_{\alpha^i}(\lambda)\\ &= 2^{(\chi(C)+2)/2}\Big(\sum_{\lambda\in\mathrm{SP}^+(d)} 2c_\lambda^2\prod_i \mathbf{f}_{\alpha^i}(\lambda) + \sum_{\lambda\in\mathrm{SP}^-(d)} c_\lambda^2\prod_i \mathbf{f}_{\alpha^i}(\lambda)\Big),\end{aligned}$$

where the second equality follows from (3.6) and (3.8), the third from (2.4) and (3.5), and the last from (c) and (d) in Section 2.3. This completes the proof of (1.9).

4. A proof of Gunningham's formula (1.7)

For $\alpha^1,\ldots,\alpha^k \in \mathrm{OP}(d)$, we set

$$H(h,p)_{\alpha^1,\ldots,\alpha^k} = 2^{(\chi(C)+\chi(D))/2}H^{h,p}_{\alpha^1,\ldots,\alpha^k},$$

where $\chi(D) = 2-2h$ and $\chi(C)$ is the domain Euler characteristic.

Using the Einstein summation convention, we raise indices by the formula

$$H(h,p)^{\beta^1,\ldots,\beta^r}_{\alpha^1,\ldots,\alpha^s} = \vartheta_{\beta^1}\cdots\vartheta_{\beta^r}H(h,p)_{\alpha^1,\ldots,\alpha^s,\beta^1,\ldots,\beta^s}.$$

For notational convenience, we denote multi-indices $\alpha^1,\ldots,\alpha^s$ by boldface index $\boldsymbol{\alpha}$. Then the gluing theorem (Theorem 1.3) can be written as

$$\begin{aligned}H(h_1+h_2,p_1+p_2)^{\boldsymbol{\beta},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\eta}} &= H(h_1,p_1)^{\boldsymbol{\beta},\gamma}_{\boldsymbol{\alpha}}H(h_2,p_2)^{\boldsymbol{\delta}}_{\boldsymbol{\eta},\gamma},\\ H(h+1,p)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} &= H(h,p)^{\boldsymbol{\beta},\gamma}_{\boldsymbol{\alpha},\gamma}.\end{aligned} \tag{4.1}$$

Remark 4.1. By additivity of the Euler characteristic, if $\chi(C)$, $\chi(C_1)$ and $\chi(C_2)$ are the domain Euler characteristics for $H(h_1+h_2,p_1+p_2)^{\boldsymbol{\beta},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\eta}}$, $H(h_1,p_1)^{\boldsymbol{\beta},\gamma}_{\boldsymbol{\alpha}}$ and $H(h_2,p_2)^{\boldsymbol{\delta}}_{\boldsymbol{\eta},\gamma}$, then

$$\chi(C) = \chi(C_1)+\chi(C_2)-2\ell(\gamma).$$

The first formula in (4.1) follows from Theorem 1.3 and the definition $\vartheta_\gamma = 2^{\ell(\gamma)+1}z_\gamma$. The proof of the second formula is the same.

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Now observe that

$$
\begin{aligned}
H(1,+)_{\alpha^1,\ldots,\alpha^k} &= H(0,+)^{\gamma}_{\alpha^1,\ldots,\alpha^k,\gamma} \\
&= \sum_{\gamma} \vartheta_\gamma \Big(\sum_{\lambda\in \mathrm{SP}^+(d)} 2c_\lambda^2 \prod_i \mathbf{f}_{\alpha^i}(\lambda)\mathbf{f}_\gamma(\lambda)\mathbf{f}_\gamma(\lambda) + \sum_{\lambda\in \mathrm{SP}^-(d)} c_\lambda^2 \prod_i \mathbf{f}_{\alpha^i}(\lambda)\mathbf{f}_\gamma(\lambda)\mathbf{f}_\gamma(\lambda)\Big) \\
&= \sum_{\lambda\in \mathrm{SP}^+(d)} \prod_i \mathbf{f}_{\alpha^i}(\lambda) + \sum_{\lambda\in \mathrm{SP}^-(d)} \prod_i \mathbf{f}_{\alpha^i}(\lambda),
\end{aligned}
$$

where the first equality follows from the gluing theorem, the second from (1.9) and the last from Lemma 2.2. In this way, the gluing theorem, (1.5) and (1.9) inductively give

$$
H(h,\pm)_{\alpha^1,\ldots,\alpha^k} = \sum_{\lambda\in \mathrm{SP}^+(d)} 2^{1-h} c_\lambda^{2-2h} \prod_i \mathbf{f}_{\alpha^i}(\lambda) \pm \sum_{\lambda\in \mathrm{SP}^-(d)} c_\lambda^{2-2h} \prod_i \mathbf{f}_{\alpha^i}(\lambda). \tag{4.2}
$$

This completes the proof of (1.7).

Remark 4.2. By Lemma 2.2, one can easily obtain (4.1) from (4.2). Therefore, Gunningham's formula (1.7) implies the gluing theorem (Theorem 1.3).

5. A TQFT formalism

This section discusses a TQFT formalism via the gluing theorem. Our approach is analogous to that in [3]. We refer to [1, 9] for Frobenius algebra and TQFT.

5.1. Extended gluing theorem. Recall that spin Hurwitz numbers are not defined for $(h, p) = (0, -)$ because the only theta characteristic on $\mathbb{P}^1$ is the even theta characteristic $\mathcal{O}(-1)$. In lieu of Lemma 2.2 and the formula (4.2), if we define

$$
H(0,-)_{\alpha^1,\ldots,\alpha^k} = \sum_{\lambda\in \mathrm{SP}^+(d)} 2c_\lambda^2 \prod_i \mathbf{f}_{\alpha^i}(\lambda) - \sum_{\lambda\in \mathrm{SP}^-(d)} c_\lambda^2 \prod_i \mathbf{f}_{\alpha^i}(\lambda), \tag{5.1}
$$

then the gluing theorem (4.1) extends to include the case $(h, p) = (0, -)$.

5.2. Functor. The (extended) gluing theorem naturally induces a functor between tensor categories,

$$\mathsf{SH} : 2\mathbf{Cob}^\pm \to \mathbf{Vect}.$$

Here **Vect** denotes the usual tensor category of complex vector spaces. The objects of the category $2\mathbf{Cob}^\pm$ are finite unions of oriented circles. The morphisms are given by pairs (D, p), where D is an oriented cobordism (modulo diffeomorphism relative to the boundary) between two objects and $p \in \mathbb{Z}_2$. We denote by

$$D^s_r(h, p)$$

the connected cobordism of genus h with parity p from a disjoint union of r circles to a disjoint union of s circles. The composition of morphisms, obtained by concatenation of cobordisms, respects the $\mathbb{Z}_2$-grading (or parity). The tensor structure on the category $2\mathbf{Cob}^\pm$ is given by disjoint union.

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We define $\mathsf{SH}(S^1) = \mathcal{F}$ to be the vector space with basis $\{v_\alpha : \alpha \in \mathrm{OP}(d)\}$ labelled by odd partitions $\alpha \in \mathrm{OP}(d)$ and let

$$\mathsf{SH}\Big(S^1 \coprod \cdots \coprod S^1\Big) = \mathcal{F} \otimes \cdots \otimes \mathcal{F}.$$

For connected cobordisms $D_r^s(h, p)$, we define a linear map

$$\mathsf{SH}(D_r^s(h, p)) : \mathcal{F}^{\otimes r} \to \mathcal{F}^{\otimes s} \quad \text{by } v_{\boldsymbol{\alpha}} \mapsto H(h, p)_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} v_{\boldsymbol{\beta}},$$

where $v_{\boldsymbol{\alpha}} = v_{\alpha^1} \otimes \cdots \otimes v_{\alpha^r}$ for $\boldsymbol{\alpha} = \alpha^1, \ldots, \alpha^r$. Taking tensor product, we extend this definition to disconnected cobordisms.

SH takes the identity morphism $D_1^1(0, +)$ to the identity map on $\mathcal{F}$,

$$v_\alpha \mapsto H(0, +)_\alpha^\beta v_\beta = v_\alpha,$$

by Lemma 5.1(b) below. Also, by the (extended) gluing theorem, SH takes the concatenation of cobordisms to the composition of linear maps (cf. [2, Proposition 4.1]). Therefore, SH is a well-defined functor. In particular, one obtains a two-dimensional TQFT, $2\mathbf{Cob}^+ \to \mathbf{Vect}$, by restricting to even cobordisms.

Recall that $\mathbf{1}$ denotes the partition $(1^d) \in \mathrm{SP}(d)$. The fact below follows from the same calculation of Hurwitz numbers because the parity of the maps with domain $\mathbb{P}^1$ is even.

Lemma 5.1. *For $\alpha, \beta \in \mathrm{OP}(d)$,*

(a) $H(0, +)^\alpha = \delta_{\mathbf{1}\alpha} v_{\mathbf{1}}$;
(b) $H(0, +)_\alpha^\beta = \delta_{\alpha\beta}$.

5.3. Frobenius algebra. The even cap $D^1(0, +)$ defines a unit $U : \mathbb{C} \to \mathcal{F}$ by $U(1) = H(0, +)^\alpha v_\alpha = v_{\mathbf{1}}$ (see Lemma 5.1(a)), while the even pair of pants $D_2^1(0, +)$ defines a multiplication $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ by

$$v_\alpha \otimes v_\beta \mapsto v_\alpha v_\beta = H(0, +)_{\alpha\beta}^\gamma v_\gamma.$$

By Lemma 2.3 and (4.2), the algebra $\mathcal{F}$ is isomorphic to the algebra $\mathcal{Z}_0^+$ in (2.8) with isomorphism $v_\alpha \mapsto u_\alpha$. The Frobenius algebra structure on $\mathcal{F} = \mathcal{Z}_0^+$ is given by the counit

$$T : \mathcal{F} \to \mathbb{C}, \quad \text{where } T(v_\alpha) = \mathsf{SH}(D_1(0, +))(v_\alpha) = H(0, +)_\alpha = \delta_{\mathbf{1}\alpha}/\vartheta_{\mathbf{1}}.$$

Using Lemma 2.2 and the structure constants $a_{\alpha\beta}^\gamma = H(0, +)_{\alpha\beta}^\gamma$, one can find, by hand, an idempotent basis $\{e_\lambda : \lambda \in \mathrm{SP}(d)\}$ $(e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda)$:

$$e_\lambda = \begin{cases} \displaystyle\sum_{\alpha \in \mathrm{OP}(d)} 2c_\lambda^2 \vartheta_\alpha f_\alpha(\lambda) v_\alpha & \text{if } \lambda \in \mathrm{SP}^+(d), \\ \displaystyle\sum_{\alpha \in \mathrm{OP}(d)} c_\lambda^2 \vartheta_\alpha f_\alpha(\lambda) v_\alpha & \text{if } \lambda \in \mathrm{SP}^-(d). \end{cases}$$

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The Frobenius algebra $\mathcal{F} = \mathcal{Z}_0^+$ is semisimple since $\mathcal{F} = \oplus\mathbb{C}e_\lambda$, where the $\mathbb{C}e_\lambda$ are one-dimensional Frobenius algebras with counit $e_\lambda \mapsto t_\lambda = T(e_\lambda)$. Since $\mathbf{f_1}(\lambda) = 1$,

$$t_\lambda = \begin{cases} 2c_\lambda^2 & \text{if } \lambda \in \mathrm{SP}^+(d), \\ c_\lambda^2 & \text{if } \lambda \in \mathrm{SP}^-(d). \end{cases}$$

Observe that the Frobenius algebra $\mathcal{F}$ has an involution given by

$$A := \mathsf{SH}(D_1^1(0,-)) : \mathcal{F} \to \mathcal{F}. \tag{5.2}$$

Remark 5.2. In [6], Gunningham constructed a fully extended two-dimensional spin TQFT, which is a functor from the 2-category of spin cobordisms to the category of superalgebras. The spin TQFT gives the formula (1.7) and hence the gluing theorem (4.1). In our case, on the other hand, we obtained from the gluing theorem a modified two-dimensional TQFT, which includes both odd and even spin Hurwitz numbers and whose underlying Frobenius algebra has an additional structure, the involution (5.2) induced by (5.1). To see the full spin TQFT, one may need more information than the underlying Frobenius algebra with an involution.

5.4. Dimension-zero GW invariants of Kähler surfaces. The semisimplicity, $\mathcal{F} = \oplus\mathbb{C}e_\lambda$, implies that e_λ is an eigenvector with eigenvalue t_λ^{-1} for the genus adding operator

$$G = \mathsf{SH}(D_1^1(1,+)) : \mathcal{F} \to \mathcal{F}.$$

One can also see, by simple calculation, that e_λ is an eigenvector with eigenvalue $(-1)^{\ell(\lambda)}$ for the involution A of $\mathcal{F}$.

Now, noting that $U(1) = v_{\mathbf{1}} = \sum e_\lambda$, $H(h,p) = \mathsf{SH}(D(h,p))(1)$ and

$$\mathsf{SH}(D(h,p)) = \mathsf{SH}(D_1(0,+) \circ D_1^1(h,p) \circ D^1(0,+)) = T \circ A^p \circ G^h \circ U,$$

we can write the dimension zero GW invariants of Kähler surfaces (1.1) succinctly as

$$GT_d^{h,p} = 2^{(1/2)(\chi(C)+\chi(D))} H(h,p) = 2^{(1/2)(\chi(C)+\chi(D))} \bigg(\sum_{\lambda \in \mathrm{SP}(d)} (-1)^{p\cdot\ell(\lambda)} t_\lambda^{\chi(D)/2} \bigg),$$

where $\chi(D) = 2-2h$ is the Euler characteristic of the smooth canonical divisor and $\chi(C)$ is the domain Euler characteristic.

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